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NPV, IRR, PI, PP, and DPP: A Unified View

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Abstract: This paper introduces a class of investment project's profitability metrics that includes the net present value criterion (which labels a project as weakly profitable if its NPV is nonnegative), the internal rate of return (IRR), the profitability index (PI), the payback period (PP) and its discounted counterpart (DPP) as special cases. An axiomatic characterization of this class, as well as of the mentioned conventional metrics within the class, is presented. This approach is useful at least in three respects. First, it suggests a unified interpretation for profitability metrics as measures of financial stability of a project with respect to a collection of scenarios of economic environment. Second, it shows that, with the exception of the NPV criterion, a profitability metric is necessarily incomplete (i.e., there are incomparable projects). In particular, this implies that any extension of the IRR to the space of all projects does not meet a set of reasonable conditions. A similar conclusion is valid for the other mentioned conventional metrics. For each of these metrics, we provide a complete characterization of pairs of compatible projects and describe the largest subset of projects to which the metric can be unambiguously extended. Third, it determines the conditions under which the use of one metric is superior to the others.

Keywords: Capital budgeting, Net present value, Internal rate of return, (Discounted) payback period, Profitability index

JEL classification: G11, G31

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NPV, IRR, PI, PP, and DPP: a unified view

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Abstract This paper introduces a class of investment project's profitability metrics that includes the net present value criterion (which labels a project as weakly profitable if its NPV is nonnegative), the internal rate of return (IRR), the profitability index (PI), the payback period (PP) and its discounted counterpart (DPP) as special cases. An axiomatic characterization of this class, as well as of the mentioned conventional metrics within the class, is presented. This approach is useful at least in three respects. First, it suggests a unified interpretation for profitability metrics as measures of financial stability of a project with respect to a collection of scenarios of economic environment. Second, it shows that, with the exception of the NPV criterion, a profitability metric is necessarily incomplete (i.e., there are incomparable projects). In particular, this implies that any extension of the IRR to the space of all projects does not meet a set of reasonable conditions. A similar conclusion is valid for the other mentioned conventional metrics. For each of these metrics, we provide a complete characterization of pairs of compatible projects and describe the largest subset of projects to which the metric can be unambiguously extended. Third, it determines the conditions under which the use of one metric is superior to the others.

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1. Introduction

The most common capital budgeting techniques include the net present value (NPV), internal rate of return (IRR), payback period (PP), discounted payback period (DPP), and profitability index (PI). Though the literature seems to agree that NPV outperforms the others as an investment criterion, being convenient numerical representations of various aspects of an investment project, the other metrics continue to be widely used in practice (Ryan and Ryan, 2002; Brounen et al., 2004). Moreover, there are situations when calculation of a particular metric is prescribed by the law.[1] This paper aims to provide a unified perspective on these five metrics as profitability measures.

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[1] One such situation is documented in Promislow (1997). The criminal code of Canada prohibits to lend money at an annual effective rate exceeding 60%. Similar restrictions are set out in the criminal codes of Japan and several states of the U.S. One, therefore, has to evaluate the IRR of a cash flow associated with a loan to verify its lawfulness.

The paper employs an axiomatic approach to characterize project's profitability metrics. The characterized metrics include, as a proper subset, a particularly nice class possessing a multi-utility representation with respect to a set of NPV criteria (the NPV criterion labels an investment project as weakly profitable iff its NPV is nonnegative). Elements of this class enjoy a unified interpretation as measures of financial stability of a project with respect to a set of scenarios of economic environment. We show that IRR, PP, DPP, and PI belong to this class. The multi-utility representation is a straightforward generalization of the one studied in Bronshtein and Akhmetova (2004) and helps to answer several questions of interest. For instance, the literature contains numerous efforts to modify the notion of IRR in order to be well defined for every project. To mention just a few, Arrow and Levhari (1969), Cantor and Lippman (1983), Promislow and Spring (1996), and Weber (2014) suggest unconditional solutions, whereas the balance function approach (Teichroew et al., 1965; Spring, 2012), the modified IRR (Lin, 1976; Beaves, 1988; Shull, 1992), and the average IRR (Magni, 2010, 2016) provide solutions conditional on exogenously given, respectively, reinvestment rate, reinvestment rate and cost of capital, and capital stream. We show that except a somewhat degenerate case that corresponds to the NPV criterion, a profitability metric is necessarily incomplete (i.e., there are incomparable projects). This implies that any extension of the IRR (as well as of any other profitability metric, including PI, PP, and DPP) to the space of all projects necessarily does not satisfy a set of natural axioms. In the case of IRR, a similar impossibility result was established in Promislow (1997). To overcome this problem, the literature suggests to reduce the space of projects to those for which the profitability metric is well defined (in the case of IRR, e.g., to the space of conventional/normal investments that have only one change of sign in their net cash flow streams). In this paper, we follow a different approach by allowing for incomparable projects. This approach suggests a natural extension of the IRR to a larger class of projects and provides a complete characterization of pairs of projects that are compatible in the sense of IRR. Similar results are provided for the other conventional metrics.

With the exception of the NPV criterion, the investment appraisal literature seems to be controversial with respect to conditions under which the use of one metric is superior to the others. Our analysis suggests that the choice of a particular metric should be determined by the source of uncertainty an investor faces. Namely, the NPV criterion should be used under complete certainty, IRR is preferable if the investor faces uncertain discount rate, PI should be chosen under the risk of reduction (in the form of an unknown scale factor) of future cash flow, whereas PP and DPP are superior under the risk of project truncation (say, for external environmental reasons). This also provides a clear interpretation for combinations of these metrics. Since there are other sources of uncertainty, the collection of conventional metrics (IRR, PI, PP, and DPP) cannot be considered as comprehensive. For instance, investment in the real sector may face uncertain intensity of the project implementation or risk of postponement; these sources of uncertainty induce new profitability metrics which, however, reduce, respectively, to IRR and PI in the case of exponential discounting.

We consider several proper subsets of interest of the space of investment projects, in particular, the set of conventional investments (i.e., projects in which a series of cash outflows is followed by a series of cash inflows) and the set of projects that require an initial cash outflow. For each of these subsets, we characterize those profitability metrics that make every pair of projects from the subset comparable.

Finally, the paper presents characterizations of the conventional profitability metrics. In particular, we show that a profitability metric is well defined for each investment project requiring an initial outlay if and only if it is consistent with PI. Furthermore, a profitability metric is well defined for investment operations with only two transactions and is stable under reduction (in the form of a scale factor) of future cash flow if and only if it is consistent with PI. IRR is known as an extension of the rate of return (the yield rate) defined over the set of investment operations with two transactions (an initial outlay and a final inflow). Though there are other extensions, e.g., the metrics introduced in Arrow and Levhari (1969) and Bronshtein and Skotnikov (2007), we show that the IRR is a unique one satisfying a set of reasonable conditions. A genuine counterpart of the IRR under nonexponential discounting is also presented and its axiomatic characterization is provided. Finally, a profitability metric is well defined for investment operations with only two transactions and is stable under truncation (that is, the ordering it induces is invariant with respect to the operation of project truncation) if and only if it is a refinement of the DPP. This refinement reduces to the conventional DPP for projects with continuous cash flows and coincides with the DPP obtained using linear interpolation of the cumulative discounted cash flow for projects with discrete cash flows. Note that it is a common practice to use linear interpolation to evaluate DPP (e.g., see Götze et al., 2015, p. 72). Our result, therefore, provides a formal justification for this practice.

Important contributions to the literature on axiomatic approach to profitability metrics include Promislow and Spring (1996), Promislow (1997), and Vilensky and Smolyak (1999). In particular, Promislow and Spring (1996) proposed a general measure-theoretic construct for the IRR-like profitability metrics. Promislow (1997) is, to our knowledge, the first formal impossibility result that shows that the IRR cannot be unambiguously extended to the space of all projects. By allowing for incomparable projects, the author provided various classes of profitability metrics, which are closely related to those we derive. Vilensky and Smolyak (1999) (the paper, which is unfortunately almost unknown in the field) presented a characterization of the IRR as well as of its extensions to nonexponential discounting and stochastic cash flows. An axiomatic approach to valuation of cash flow streams (Norberg, 1990; Promislow, 1994; Spring, 2012) and, more generally, utility streams (Chichilnisky, 1996; Neyman, 2023, to mention just a few) provides some significant related results.

The paper is organized as follows. Section 2 attempts to formalize the concept of profitability. It introduces the main object of our analysis, called a profitability ordering, by means of an axiomatic approach. Section 3 studies profitability orderings that are total (complete) being restricted to a given subset of interest. In particular, we characterize profitability orderings that are total on the set of conventional/normal investments. Section 4 shows that various standard capital budgeting metrics, including IRR, PI, PP, and DPP, are induced by profitability orderings. With the help of this result, for each of these metrics we characterize the largest subset of projects on which it is unambiguously defined. All proofs and auxiliary results are given in the Appendix.

2. A profitability ordering

We begin with basic definitions and notation. R_{++}, R_{+}, and R are the sets of positive, nonnegative, and all real numbers, respectively. $\overline{\mathrm{R}} := [-\infty, +\infty]$ and $\overline{\mathrm{R}}_{+} := [0, +\infty]$ are the extended real and nonnegative real numbers. We equip subsets of $\overline{\mathrm{R}}$ with the usual topology. For a topological space, the topological closure and interior operators are denoted by cl and int. The indicator function of a set S is denoted by I_{S}.

Let P be the space of right-continuous regulated real-valued functions on R_+, that is, $x \in \mathrm{P}$ if $x: \mathrm{R}_+ \to \mathrm{R}$ is right-continuous and possesses finite left limits $x(t-) := \lim_{\tau \to t-} x(\tau)$ for all $t \in \mathrm{R}_{++}$ and $x(+\infty) := \lim_{\tau \to +\infty} x(\tau)$. Being endowed with the supremum norm $\|x\| := \sup_{t \in \mathrm{R}_+} |x(t)|$, P becomes a Banach space (Monteiro et al., 2018, Theorem 4.2.1, Corollary 4.2.4). Set $\mathrm{P}_+ := \{x \in \mathrm{P} : \inf_{t \in \mathrm{R}_+} x(t) \geq 0\}$ and $\mathrm{P}_{++} := \{x \in \mathrm{P} : \inf_{t \in \mathrm{R}_+} x(t) > 0\}$. Note that P_+ is a closed convex cone and $\mathrm{P}_{++} = \operatorname{int} \mathrm{P}_+$. We write $x \geq y$ (resp. $x > y$) if $x - y \in \mathrm{P}_+$ (resp. $x - y \in \mathrm{P}_{++}$). For any $\tau, t \in \mathrm{R}_+$, put

$$1_\tau(t) := \begin{cases} 1, t \geq \tau \\ 0, t < \tau \end{cases}.$$

Note that $1_0 \in \mathrm{P}_{++}$ and, therefore, it is an order unit for the ordering cone P_+ in the vector space P (that is, for every $x \in \mathrm{P}$ there is $\lambda \in \mathrm{R}_{++}$ such that $\lambda 1_0 \geq x$). An element $x \in \mathrm{P}$ is interpreted as a project's *cumulative* cash flow so that $x(t)$ is the balance of the project – the difference between cumulative cash inflows and cash outflows – at time t.[2] P_+ is the set of projects with the property that the cumulative cash inflow all the time dominates the cumulative cash outflow. The project 1_τ is interpreted as receiving a money unit at time τ. It is known (Monteiro et al., 2018, p. 82) that P is the closure of the linear span of $\{1_\tau, \tau \in \mathrm{R}_+\}$ in the space of bounded functions on R_+ endowed with the topology induced by the supremum norm. Thus, P is a natural extension of the practically relevant space of investment projects with finitely many transactions.

The topological dual of P is denoted by P^*. We equip P^* with the weak* topology. The dual cone of a set $\mathrm{C} \subseteq \mathrm{P}$ is given by $\mathrm{C}^\circ := \{F \in \mathrm{P}^* : F(x) \geq 0 \; \forall x \in \mathrm{C}\}$, the dual cone of a set $\mathrm{K} \subseteq \mathrm{P}^*$ is defined in a similar fashion, $\mathrm{K}^\circ := \{x \in \mathrm{P} : F(x) \geq 0 \; \forall F \in \mathrm{K}\}$. The set of all additive (i.e., $F(x) + F(y) = F(x + y)$ for all $x, y \in \mathrm{P}$) and positive (i.e., $F(\mathrm{P}_+) \subseteq \mathrm{R}_+$) functionals $F: \mathrm{P} \to \mathrm{R}$ satisfying $F(1_0) = 1$ is denoted by $\mathcal{NPV}$. $\mathcal{NPV}$ is interpreted as the set of possible net present value functionals. A routine argument shows that every additive and positive functional is homogeneous and continuous, so that $\mathcal{NPV} = \{F \in \mathrm{P}_+^\circ : F(1_0) = 1\}$. Denote by $\mathcal{A}$ the set of all nonnegative and nonincreasing functions $\alpha: \mathrm{R}_+ \to \mathrm{R}_+$ satisfying $\alpha(0) = 1$. Using a result on the structure of the dual space P^* (Monteiro et al., 2018, Theorem 8.2.8) it can be shown that $F \in \mathcal{NPV}$ if and only if there exists $\alpha \in \mathcal{A}$ such that

$$F(x) = x(0) + \int_0^\infty \alpha \mathrm{d}x, \tag{1}$$

where the integral is the Kurzweil-Stieltjes integral (see Lemma 11 in the Appendix for details). As $\alpha(t) = F(1_t)$, i.e., $\alpha(t)$ is the present worth of receiving a money unit at time t, in what follows α will be called a *discount function*.[3] Identities $\alpha(t) = F(1_t)$ and (1) define a one-to-one correspondence between the sets $\mathcal{NPV}$ and $\mathcal{A}$. In what follows, this allows us to identify a

[2] In this paper, we identify an investment project with the cumulative cash flow it generates. We prefer to describe a project by means of the cumulative (rather than net) cash flow as this setup enables a uniform treatment of discrete- and continuous-time settings.
[3] Note that in most models involving discounting nonincreasingness of a discount function is an *assumption*, whereas in our model this is a *consequence*.

discount function with the NPV functional it induces. We use the notation $F^{(\alpha)}$ for an NPV functional whenever we want to emphasize that it is induced by a discount function α. The support of a discount function α is denoted by $\operatorname{supp}\{\alpha\} := \{t \in \mathrm{R}_+ : \alpha(t) > 0\}$. The discount function that corresponds to the extremely impatient case is denoted by $\chi(t) := \begin{cases} 1 \text{ if } t = 0 \\ 0 \text{ if } t > 0 \end{cases}$.

Projects profitabilities in this paper are ranked by means of a binary relation $\succeq$ on P. The statement $x \succeq y$ means that project x is at least as profitable as y. The symmetric and asymmetric parts of $\succeq$ are denoted by $\succ$ and $\sim$. The upper and strict upper contour sets of $\succeq$ at $x \in \mathrm{P}$ are $\mathrm{U}_{\succeq}(x) := \{y \in \mathrm{P} : y \succeq x\}$ and $\mathrm{U}_{\succ}(x) := \{y \in \mathrm{P} : y \succ x\}$. The lower and strict lower contour sets, $\mathrm{L}_{\succeq}(x)$ and $\mathrm{L}_{\succ}(x)$, are defined in a similar fashion. $\succeq$ is said to be a *profitability ordering* (*PO* for short) if the following four conditions hold.

Nontrivial preorder (*NT*): $\succeq$ is nontrivial (i.e., $\succeq \neq \mathrm{P} \times \mathrm{P}$), reflexive, and transitive.

Monotonicity (*M*): $x \geq y \ \& \ y \succeq z \ \Rightarrow \ x \succeq z$.

Upper semicontinuity (*USC*): for every $x \in \mathrm{P}$, $\mathrm{U}_{\succeq}(x)$ is closed.

Internality (*I*): for every $x \in \mathrm{P}$, the sets $\mathrm{L}_{\succeq}(x)$ and $\mathrm{U}_{\succeq}(x)$ are closed under addition.[4]

Axiom (NT) states that projects are comparable in a coherent way. In view of the results on nonextendability of the IRR-like profitability metrics to the space of all projects (Promislow, 1997; Vilensky and Smolyak, 1999), we do not require $\succeq$ to be total (complete). Axiom (M) ensures that higher cash flow provides higher profitability. Note that the combination of conditions (NT) and (M) holds if and only if $\succeq$ is nontrivial, transitive and $x \geq y \ \Rightarrow \ x \succeq y$. Axiom (USC) is a standard regularity condition, which assures that small perturbations of cash flows result in a minor perturbation of the ordering. Finally, axiom (I) relates profitability of a pool of projects with profitabilites of its components. In particular, it makes valid the following natural guidance: to guarantee the target level of profitability for a pool of projects it suffices to keep the target for each project in the pool. The axiom also implies $x \succeq y \ \Rightarrow \ x \succeq x + y \succeq y$, that is, union of a project with a more (resp. less) profitable one increases (resp. decreases) profitability of the union. This condition is appealing in practice, since it allows an investor to decompose a complex investment decision into separate evaluation of individual investment projects. Axiom (I) is closely related to the decomposition axiom used in Promislow (1997) to classify loans by their annual effective rate. A stronger form of the internality axiom was also used in Vilensky and Smolyak (1999, section 1.4) to characterize IRR.

The general structure of a PO is described in the following proposition.

Proposition 1.

Conditions (NT), (M), (USC), (I) are independent (i.e., any three of them do not imply the fourth). For a binary relation $\succeq$ on P the following statements are equivalent:

(a) $\succeq$ is a PO;

(b) there is a nonempty family $\mathcal{U} \subset 2^{\mathcal{NPV}}$ of nonempty subsets of $\mathcal{NPV}$ such that for all $z \in \mathrm{P}$ the set $\bigcap_{\mathrm{K} \in \mathcal{U} : z \notin \mathrm{K}^{\circ}} (\mathrm{P} \setminus \mathrm{K}^{\circ})$ is closed under addition and

[4] A set $\mathrm{C} \subseteq \mathrm{P}$ is said to be closed under addition if $\mathrm{C} + \mathrm{C} \subseteq \mathrm{C}$.

$$x \succeq y \iff I_{\mathrm{K}^\circ}(x) \geq I_{\mathrm{K}^\circ}(y) \text{ for all } \mathrm{K} \in \mathcal{U}. \tag{2}$$

Moreover, without loss of generality, elements of the family $\mathcal{U}$ in part (b) can be chosen closed (in the weak* topology) and convex.

Note that the right-hand side of the equivalence (2) can also be represented as $\{\mathrm{K} \in \mathcal{U} : x \in \mathrm{K}^\circ\} \supseteq \{\mathrm{K} \in \mathcal{U} : y \in \mathrm{K}^\circ\}$. In what follows, a family $\mathcal{U} \subset 2^{\mathcal{NPV}}$ satisfying the conditions of part (b) of Proposition 1 is called a *representation* of the PO $\succeq$. Clearly, a representation is nonunique. A particular way to choose it is $\mathcal{U} = \{(\mathrm{U}_{\succeq}(z))^\circ \cap \mathcal{NPV},\, z \in \mathrm{P}\}$.

Further facts about POs are collected in the next lemma.

Lemma 1.

A PO $\succeq$ enjoys the following properties.

1º. $\lambda x \sim x$ for all $x \in \mathrm{P}$ and $\lambda > 0$.

2º. $\succeq$ is not lower semicontinuous (i.e., it is not true that $\mathrm{L}_{\succeq}(x)$ is closed for every $x \in \mathrm{P}$).

3º. There are projects x and y such that $x > y$ and $x \sim y$.

4º. There are projects x and y such that $x \succ y$, whereas $x \sim x + y$ (similarly, there are x and y such that $x \succ y$, whereas $y \sim x + y$).

5º. If $1_0 \succ x$ and $1_0 \succ -x$, then x and $-x$ are incomparable.

6º. The intersection of a collection of POs is a PO.

Property 1º states that profitability takes no account of the investment size and hence is a relative measure. All known measures of profitability satisfy this property.

Upper and lower semicontinuity are desirable properties as cash flows contain future components which are measured with an error. However, unless $x \sim 1_0$, we have $0 \notin \mathrm{L}_{\succeq}(x)$, so that lower semicontinuity does not hold (property 2º).

Though $x \geq y \implies x \succeq y$, strictly higher cash flow does not necessarily imply strictly higher profitability (property 3º). If $x \succ y$, then one would expect $x \succ x + y \succ y$ (recall that, by (I), $x \succeq x + y \succeq y$). For instance, if two investment projects have the IRRs r and s such that $r < s$, then the IRR of their union, if it exists, falls strictly between r and s. Unfortunately, the strict inequalities cannot hold for every pair of comparable projects (property 4º).

It is widely recognized in the literature that ambiguity with the IRR is a consequence of change of the status of the investor from that of a lender to that of a borrower for mixture projects (Gronchi, 1986; Hazen, 2003; Hartman and Schafrick, 2004; Promislow, 2015, section 2.12; Magni, 2016). Property 5º shows that every profitability metric suffers from the same drawback. The problem is that pure investment and pure financing, no matter how they are defined, differ by sign and, therefore, by property 5º, are incompatible.

Finally, property 6º provides a way to aggregate multiple profitability criteria.

An example of a PO is given by an NPV criterion. A PO $\succeq$ is said to be an *NPV criterion* if it has a singleton representation, that is, there is $F \in \mathcal{NPV}$ such that $x \succeq y \iff I_{\{F\}^\circ}(x) \geq I_{\{F\}^\circ}(y)$. An NPV criterion partitions P into the sets of (nonstrictly) profitable and not profitable projects with, respectively, nonnegative and negative NPV.

The next proposition shows that an NPV criterion is the only total (complete) PO.

Proposition 2.

For a PO $\succeq$ the following statements are equivalent:

(a) $\succeq$ is an NPV criterion;

(b) $\succeq$ is total (complete);

(c) for every $x \in \mathrm{P}$, $\mathrm{L}_{\succ}(x)$ is closed under addition;

(d) for every $x \in \mathrm{P}$, $\mathrm{U}_{\succ}(x)$ is closed under addition.

According to Proposition 2, unless $\succeq$ is an NPV criterion, the union of projects with strictly lower (resp. higher) profitabilities than x does not necessarily produce strictly lower (resp. higher) profitability than x.

As already noted in the introduction, the literature contains numerous efforts to modify the notion of IRR in order to be well defined for every project. As we will show in section 4.1, the IRR can be identified with a utility representation of the restriction a particular PO. Proposition 2 shows that, except a somewhat degenerate case that corresponds to an NPV criterion, a PO is incomplete. Therefore, such efforts necessary result in a ranking that does not satisfy the set of axioms introduced above. A similar impossibility result was established in Promislow (1997). Proposition 2 shows that the same conclusion holds for every profitability metric.

Example 1.

The criminal code of Canada prohibits to lend money at an annual effective rate exceeding 60%. How to interpret this clause for loans whose cash flows have no IRR? This problem is studied in details in Promislow (1997).

Let $\mathrm{U} \subset \mathrm{P}$ be a set interpreted as the set of usurious (illegal) loans from lender perspective. The following conditions on U and $\mathrm{N} := \mathrm{P} \setminus \mathrm{U}$ (the set of nonusurious loans) seem to be reasonable.

1º. $x \geq y$ & $y \in \mathrm{U} \Rightarrow x \in \mathrm{U}$.

2º. N is open.

3º. The sets U and N are closed under addition.

4º. $-1_0 + 1.6^t 1_t \in \mathrm{U}$ for each $t > 0$; $-1_0 + a 1_t \in \mathrm{N}$, whenever $t > 0$ and $a < 1.6^t$.

Condition 1º states that a loan with higher lender cash flow than an usurious loan is usurious. According to 2º, a small perturbation of an nonusurious loan is nonusurious. By condition 3º, U and N are closed under the operation of union of loans. Finally, condition 4º states that simple loans with two transactions – the initial lending and final repayment – having an annual effective rate of (resp. less than) 60% are usurious (resp. nonusurious).

In order to characterize U define the binary relation $\succeq$ over P by $x \succeq y \Leftrightarrow I_{\mathrm{U}}(x) \geq I_{\mathrm{U}}(y)$. From conditions 1º–3º it follows that $\succeq$ is a total PO and, therefore, by Proposition 2, it is an NPV criterion. Condition 4º implies that the NPV is induced by the discount function $t \mapsto 1.6^{-t}$. The obtained result suggests to correct the statement of the criminal code and classify a loan with a lender cash flow $x \in \mathrm{P}$ as nonusurious (resp. usurious) if $F(x) < 0$ (resp. $F(x) \geq 0$), where F is the NPV functional induced by the discount function $t \mapsto 1.6^{-t}$. The suggested rule is consistent with the old one: if the lender cash flow x possesses the IRR and it is less than (resp. equals or exceeds) 60%, then $F(x) < 0$ (resp. $F(x) \geq 0$).

A PO $\succeq$ is said to be *symmetric* (*SPO* for short) if it is the intersection of a collection of NPV criteria, that is, there is a nonempty set $\mathcal{F} \subseteq \mathcal{NPV}$ such that

$$x \succeq y \iff I_{\{F\}^\circ}(x) \geq I_{\{F\}^\circ}(y) \text{ for all } F \in \mathcal{F}. \tag{3}$$

Note that, by property 6º in Lemma 1, the binary relation defined by (3) is indeed a PO. The right-hand side of the equivalence (3) can also be represented as $\{x\}^\circ \cap \mathcal{F} \supseteq \{y\}^\circ \cap \mathcal{F}$. The multi-utility representation (3) has a straightforward interpretation. Identify each element of $\mathcal{F}$ with a possible scenario of economic environment (the environment affects various economic factors, including interest rates, and, therefore, the discount function). Then, according to (3), $x \succeq y$ if and only if x is profitable (i.e., has nonnegative NPV) in a larger set of scenarios than y. The interpretation shows that an SPO measures financial stability of a project with respect to a set of scenarios. In the case of projects with finitely many transactions, two particular SPOs (with $\mathcal{F} = \mathcal{NPV}$ and $\mathcal{F}$ being the set of NPV induced by the family of exponential discount functions) were studied in Bronshtein and Akhmetova (2004). In what follows, we mainly exploit SPO due to its simple representation and particularly nice interpretation.

The set $\mathcal{F}$ in (3) is called a *representation* of the SPO $\succeq$ (we also say that $\mathcal{F}$ *represents* or *induces* $\succeq$). A representation of an SPO is in general nonunique. For instance, if $F, G \in \mathcal{NPV}$, $F \neq G$ and W is a dense subset of $(0,1)$, then $\{wF + (1-w)G,\ w \in \mathrm{W}\}$ and $\{wF + (1-w)G,\ w \in (0,1)\}$ represent the same SPO. In what follows, by *the* representation of an SPO $\succeq$ we mean the greatest subset of $\mathcal{NPV}$ representing $\succeq$, that is, $\bigcap\{F \in \mathcal{NPV} : I_{\{F\}^\circ}(x) \geq I_{\{F\}^\circ}(y)\}$, where the intersection is taken over all pairs $x, y \in \mathrm{P}$ such that $x \succeq y$.

The structure of an SPO with a closed and convex representation is described in the following example.

Example 2.

Given a nonempty closed (in the weak* topology) set $\mathcal{S} \subseteq \mathcal{NPV}$, let $\mathcal{F}$ be the closed convex hull of $\mathcal{S}$ and $\succeq$ be the SPO induced by $\mathcal{F}$. To motivate the idea behind $\succeq$ note that since $1_0 \in \mathrm{P}_{++}$, the set $\mathcal{NPV}$ is compact (Jameson, 1970, Theorem 3.8.6) and, therefore, so is $\mathcal{F}$. As $\mathcal{S}$ is closed, it contains the closure of the set of extreme points of $\mathcal{F}$ and, by the integral form of the Krein-Milman theorem, every $F \in \mathcal{F}$ can be treated as the expected NPV under a probability measure over $\mathcal{S}$. Thus, the SPO $\succeq$ induced by $\mathcal{F}$ can be interpreted as a measure of project's financial stability under (unknown) probabilistic uncertainty with respect to the set of scenarios $\mathcal{S}$.

One can show (see Lemma 12 in the Appendix) that

$$x \succeq y \iff \sup_{\lambda \in \mathrm{R}_+} \min_{F \in \mathcal{S}} F(x - \lambda y) \geq 0. \tag{4}$$

Moreover, if $\mathcal{S}$ is not necessarily closed, then $\min$ in (4) should be replaced by $\inf$. In particular, if $\mathcal{S}$ is finite, then $x \succeq y$ if and only if there exists $\lambda \in \mathrm{R}_+$ such that $F(x - \lambda y) \geq 0$ for all $F \in \mathcal{S}$. Or, in words, $x \succeq y$ if and only if the project y can be rescaled such that x has nonstrictly higher NPV than the rescaled y in every scenario from $\mathcal{S}$.

To illustrate, consider the SPO $\succeq$ induced by $\mathcal{NPV}$. From representation (4) we deduce that $x \succeq y$ if and only if for any $\varepsilon > 0$, there exists $\lambda \in \mathrm{R}_+$ such that $x + \varepsilon 1_0 \geq \lambda y$.

3. Completeness on a predetermined subset of projects

Though a PO is in general incomplete, one may require it to be total (complete) being restricted to a predetermined subset of interest $\mathrm{Q} \subseteq \mathrm{P}$. For instance, a generic net cash flow stream for investment in a stock is composed an initial outlay associated with buying the stock, dividends received during the holding period, and the gain that occurs when the stock is sold. Therefore, totality over the set of projects in which an initial capital outflow is followed by a series of cash inflows is a desirable property for a PO used to evaluate stock performance. To be more precise, given $\mathrm{Q} \subseteq \mathrm{P}$, a PO is said to be Q-*complete* if its restriction to Q is total (i.e., Q is a chain). Q-completeness seems to be a reasonable condition for a PO to be useful for evaluation projects from Q. At least when Q is second countable (e.g., this holds in the practically relevant case of discrete projects, i.e., if Q is a subset of the closure of the linear span of $\{1_\tau, \tau = 0,1,...\}$) the restriction admits an upper semicontinuous utility representation due to the Rader theorem (Rader, 1963).

Though being rather trivial, the following lemma is a useful tool in verifying Q-completeness. It shows that a PO is Q-complete if and only if elements of its representation can be totally preordered in a natural way.

Lemma 2.

Let $\mathrm{Q} \subseteq \mathrm{P}$, $\succeq$ be a PO with a representation $\mathcal{U}$ and $\sqsupseteq$ be the preorder over $\mathcal{U}$ defined by $\mathrm{K} \sqsupseteq \mathrm{L} \Leftrightarrow \mathrm{K}^\circ \cap \mathrm{Q} \supseteq \mathrm{L}^\circ \cap \mathrm{Q}$. Then $\succeq$ is Q-complete if and only if $\sqsupseteq$ is total.

Given $\mathrm{Q} \subseteq \mathrm{P}$ and $\mathcal{F} \subseteq \mathcal{NPV}$, a preorder $\succcurlyeq$ on $\mathcal{F}$ is said to be *induced* by Q if for any $F, G \in \mathcal{F}$, $F \succcurlyeq G \Leftrightarrow \{F\}^\circ \cap \mathrm{Q} \supseteq \{G\}^\circ \cap \mathrm{Q}$. The relations $\succ$ and $\sim$ are defined as usual. The relation $F \succcurlyeq G$ means that scenario F is more favorable than G for an investor considering projects from Q: each project $x \in \mathrm{Q}$ which is profitable under G (i.e., $G(x) \geq 0$) is also profitable under F. It follows from Lemma 2 that an SPO with a representation $\mathcal{F}$ is Q-complete if and only if $\succcurlyeq$ is total.

In the rest of this section, we describe the structure of Q-complete SPOs for Q comprising various types of investments. These results relate completeness over several notable subsets of projects and totality of the restriction of well-known partial orderings over $\mathcal{NPV}$. Put $\mathrm{Q}_1 := \{x \in \mathrm{P}: \ x(0) < 0 \text{ and } x \text{ is nondecreasing}\}$, $\mathrm{Q}_2 := \mathrm{Q}_1 \cup \{x \in \mathrm{P} \setminus \mathrm{P}_+: \ x(0) \leq 0 \text{ and there is } \tau \in \mathrm{R}_{++} \text{ such that } x \text{ is nonincreasing (resp. nondecreasing) on } [0,\tau) \text{ (resp. } [\tau,+\infty))\}$, $\mathrm{Q}_3 := \{x \in \mathrm{P} \setminus \mathrm{P}_+: \text{ there is } \tau \in \mathrm{R}_{++} \text{ such that } x \text{ is nonpositive on } [0,\tau) \text{ and nonnegative on } [\tau,+\infty)\}$, $\mathrm{Q}_4 := \{x \in \mathrm{P}: x(0) < 0\}$, and $\mathrm{Q}_5 := \{x \in \mathrm{P}: \text{ there is } \tau \in \mathrm{R}_{++} \text{ such that } x(\tau) < 0 \text{ and } x \text{ is nonpositive on } [0,\tau]\}$. The set Q_1 (Q_2) consists of conventional investments in which an initial cash outflow (a series of cash outflows) is followed by a series of cash inflows. The set Q_3 comprises investments whose cumulative cash flows have one change of sign (this class of projects is studied, e.g., in Norstrøm, 1972). Finally, Q_4 (Q_5) comprises all investments, i.e., the projects that require an initial cash outflow (outflows).

We endow $\mathcal{NPV}$ (or, equivalently, $\mathcal{A}$) with the three transitive binary relations. Let $F, G \in \mathcal{NPV}$ and let α and β be the discount functions associated with F and G, respectively. We write $F \succcurlyeq_1 G$ if $\alpha \geq \beta$ (pointwise). The relation $\succcurlyeq_1$ describes the strength of discounting. We write $F \succcurlyeq_2 G$ if $\operatorname{supp}\{\alpha\} = \operatorname{supp}\{\beta\}$ and the function $t \mapsto \alpha(t)/\beta(t)$ defined on $\operatorname{supp}\{\beta\}$ is nondecreasing. Provided that the discount functions are positive and differentiable, $F \succcurlyeq_2 G$ holds if and only if the instantaneous discount rate under β dominates that under α, $-(\ln\beta)' \geq -(\ln\alpha)'$. $\succcurlyeq_2$ is known as the patience ordering (e.g., see Quah and Strulovici, 2013, section II.C). Finally, we write $F \succcurlyeq_3 G$ if α and β are differentiable, $\alpha' < 0$, $\beta' < 0$, and the function $t \mapsto \alpha'(t)/\beta'(t)$ is nondecreasing. $\succcurlyeq_3$ is the relative decreasing impatience (or spread seeking) relation studied in Rohde (2009). Note that $\succcurlyeq_1$, $\succcurlyeq_2$, and the restriction of $\succcurlyeq_2 \cap \succcurlyeq_3$ to the subset of discount functions possessing negative derivative are partial orderings. Moreover, $\succcurlyeq_2 \subset \succcurlyeq_1$, whereas $\succcurlyeq_3 \not\subset \succcurlyeq_1$.

The next two lemmas describe the structure of Q_1- and Q_2-complete SPOs.

Lemma 3.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$. Put $Q_1' := \{-1_0 + a1_\tau,\ a \geq 1,\ \tau > 0\}$. The following conditions are equivalent:

(a) $\succeq$ is Q_1-complete;

(b) $\succeq$ is Q_1'-complete;

(c) the restriction of $\succcurlyeq_1$ to $\mathcal{F}$ is total.

Lemma 4.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$. Set $Q_2' := \{-1_t + a1_\tau,\ 0 \leq t < \tau,\ a > 0\}$ and $Q_2'' := \{-1_t + a1_\tau,\ 0 \leq t < \tau,\ a \geq 1\}$. The following conditions are equivalent:

(a) $\succeq$ is Q_2-complete;

(b) $\succeq$ is Q_2'-complete;

(c) the restriction of $\succcurlyeq_2$ to $\mathcal{F}$ is total.

If each NPV functional from $\mathcal{F}$ has a positive discount function, then (a)–(c) are also equivalent to

(d) $\succeq$ is Q_2''-complete.

Lemma 3 (resp. Lemma 4) states that an SPO with a representation $\mathcal{F}$ is Q_1-complete (resp. Q_2-complete) if and only if for any $F, G \in \mathcal{F}$, either $\alpha \leq \beta$ or $\alpha \geq \beta$ (resp. $\operatorname{supp}\{\alpha\} = \operatorname{supp}\{\beta\}$ and the function $t \mapsto \alpha(t)/\beta(t)$ defined on $\operatorname{supp}\{\beta\}$ is monotone), where α and β are the discount functions associated with F and G. It also shows that to check Q_1-completeness (resp. Q_2-completeness) it is sufficient to test it on the set of projects Q_1' (resp. Q_2') possessing only two transactions.

Our next result characterizes Q_3-complete SPOs.

Lemma 5.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$. Assume that for every $F \in \mathcal{F}$, the discount function associated with F has a negative derivative. Then the following conditions are equivalent:

(a) $\succeq$ is Q_3-complete;

(b) the restriction of $\succcurlyeq_2 \cap \succcurlyeq_3$ to $\mathcal{F}$ is total.

In order to formulate the next result we introduce the following notation. For $\alpha \in \mathcal{A} \setminus \{\chi\}$, denote by $H_\gamma^{(\alpha)}$, $\gamma \in [0, 1/\alpha(0+)]$ the NPV functional induced by the discount function $\gamma\alpha + (1-\gamma)\chi$. Set $H_\gamma^{(\chi)} := F^{(\chi)}$ for all $\gamma \in \mathrm{R}_+$. Note that $H_\gamma^{(\alpha)}(x) = x(0) + \gamma(F^{(\alpha)}(x) - x(0))$.

Lemma 6.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$. Set $Q_4' := \{-1_0 + a1_t + b1_\tau, (a,b,t,\tau) \in \mathrm{R}^2 \times \mathrm{R}^2_{++}\}$, $Q_5' := \{-1_s + a1_{s+t} + b1_{s+\tau}, (a,b,s,t,\tau) \in \mathrm{R}^2 \times \mathrm{R}_+ \times \mathrm{R}^2_{++}\}$. The following conditions are equivalent:

(a) $\succeq$ is Q_4-complete (resp. Q_5-complete);

(b) $\succeq$ is Q_4'-complete (resp. Q_5'-complete);

(c) there is $\alpha \in \mathcal{A}$ and $\Gamma \subseteq [0,1]$ (resp. $\Gamma \subseteq (0,1]$) such that $\mathcal{F} = \{H_\gamma^{(\alpha)}, \gamma \in \Gamma\}$.

As we will show in section 4.1, the IRR is a utility representation of the restriction of the SPO induced by the family of exponential discount functions, $\{t \mapsto e^{-\lambda t}, \lambda \in \mathrm{R}_+\}$. Clearly, the restriction of $\succcurlyeq_2$ and $\succcurlyeq_2 \cap \succcurlyeq_3$ to this family is total. Thus, by Lemmas 4 and 5, this SPO is Q_2- (and, hence, Q_1-) and Q_3-complete. This is a reformulation of well-known facts that a project whose either net or cumulative cash flow has one change of sign possesses the IRR (Norstrøm, 1972). Lemmas 3–5 extend these results by describing the general structure of Q_1-, Q_2-, and Q_3-complete SPOs. On the other hand, it is known that an investment project with at least three transactions may have no IRR. That is, the SPO induced by the family of exponential discount functions is not Q_4'-complete. One can use Lemma 6 to suggest a relevant profitability measure for projects from Q_4' (or, equivalently, Q_4). Namely, from Lemma 6 it follows that if an SPO $\succeq$ is Q_4-complete, then there is $F \in \mathcal{NPV}$ such that for any $x, y \in Q_4$, $PI^F(x) \geq PI^F(y) \Rightarrow x \succeq y$, where PI^F is the *profitability index* defined by $PI^F(x) := (F(x) - x(0))/(-x(0))$. A partial converse to this assertion is established in section 4.3. This suggests a profitability index as a natural profitability measure for projects from Q_4.

Set $Q_5'' := \{x \in \mathrm{P} : x(0) \leq 0\}$. From the proof it follows that Q_5-completeness in part (a) of Lemma 6 can be replaced by Q_5''-completeness without changing the result. Note that Q_4 (resp. Q_5'') is an open (resp. closed) half-space of P induced by the NPV functional $x \mapsto x(0)$. The statement "(a) $\Leftrightarrow$ (c)" of Lemma 6, therefore, can be generalized as follows.

Lemma 7.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$. For a given nonzero functional $G \in \mathrm{P}^*$, put $Q_G := \{x \in \mathrm{P} : G(x) < 0\}$, $Q_G' := \{x \in \mathrm{P} : G(x) \leq 0\}$. If $\succeq$ is Q_G-complete, then there is $F \in \mathcal{NPV}$ such that $\mathcal{F}$ lies in the linear span of $\{F, G\}$ in P^*. In particular, if $G \in \mathcal{NPV}$, then $\succeq$ is Q_G-complete (resp. Q_G'-complete) if and only if there are $F \in \mathcal{NPV}$ and $\mathrm{W} \subseteq [0,1]$ (resp. $\mathrm{W} \subseteq (0,1]$) such that $\mathcal{F} = \{wF + (1-w)G, w \in \mathrm{W}\}$.

By Lemma 7, given a nonzero functional $G \in \mathcal{NPV}$, if an SPO $\succeq$ is Q_G-complete, then there is $F \in \mathcal{NPV}$ such that for any $x, y \in \mathrm{Q}_G$, $RI_G^F(x) \geq RI_G^F(y) \;\Rightarrow\; x \succeq y$, where RI_G^F is the *ratio index* defined by $RI_G^F(x) := 1 - F(x)/G(x)$. A partial converse to this assertion is established in section 4.3. This shows that ratio indices RI_G^F, $F \in \mathcal{NPV}$ are natural profitability measures for projects from Q_G. More results on the indices PI^F and RI_G^F are obtained in section 4.3.

Given nonzero $G_1, ..., G_n \in \mathrm{P}^*$, a minor modification of the proof of Lemma 7 shows that if an SPO with a representation $\mathcal{F}$ is $\{G_1, ..., G_n\}^{\circ}$-complete, then there is $F \in \mathcal{NPV}$ such that $\mathcal{F}$ lies in the linear span of $\{F, G_1, ..., G_n\}$.

4. Profitability metrics

We proceed by describing the class of real maps defined on a subset of P that could be used for profitability measurement purposes. A real-valued function M defined on a nonempty set $\mathrm{Q} \subseteq \mathrm{P}$ is said to be a *profitability metric* if there exists an SPO $\succeq$ such that for any $x, y \in \mathrm{Q}$, $x \succeq y \;\Leftrightarrow\; M(x) \geq M(y)$. An SPO satisfying this property is said to be M*-consistent*. Put differently, a profitability metric is a utility representation of the restriction of an SPO.

Recall that the intersection of a collection of POs (SPOs) is a PO (SPO). Therefore, for any profitability metric M, the set of all M-consistent SPOs (treated as subsets of $\mathrm{P} \times \mathrm{P}$) contains the least element. This element describes the unambiguous part, which agrees with every M-consistent SPO. Given a profitability metric $M : \mathrm{Q} \to \mathrm{R}$, the greatest set $\mathrm{D} \supseteq \mathrm{Q}$ such that the least M-consistent SPO is D-complete is said to be the *natural domain* of M. The restriction of the least M-consistent SPO to the natural domain is said to be the *natural extension* of M. In what follows, a utility representation of the natural extension (if any) is also, rather loosely, referred to as the natural extension of M. The natural domain determines to what extent the profitability metric can be uniquely extended and this unique extension is referred to as the natural one.

To motivate the idea behind the introduced notions consider the IRR capital budgeting technique. It is well known that the IRR is well defined for conventional investment projects that have only one change of sign in their net cash flow streams. The investment appraisal literature addresses the question of what is the largest set of projects for which the IRR is unambiguously defined. If the IRR (considered as a real-valued function defined on the set of conventional projects) is a profitability metric, then the natural domain answers the question and the natural extension defines a unique extension of the IRR to the natural domain.

The natural domain need not exist. For instance, a constant function M on P_+ is a profitability metric with the least M-consistent SPO induced by $\mathcal{NPV}$. For every $x \notin \mathrm{P}_+ \cup (-\mathrm{P}_{++})$ and $\mathrm{D} \supseteq \mathrm{P}_+ \cup \{x, -x\}$, this SPO is $\mathrm{P}_+ \cup \{x\}$- and $\mathrm{P}_+ \cup \{-x\}$-complete, but not D-complete as x and $-x$ are incomparable by property 5° in Lemma 1. Thus, M has no natural domain. A simple sufficient condition for the existence of the natural domain is given in the following lemma.

Lemma 8.

Given a profitability metric $M : \mathrm{Q} \to \mathrm{R}$, $\mathrm{Q} \subseteq \mathrm{P}$, set $\mathcal{F} = \bigcap \{F \in \mathcal{NPV} : I_{\{F\}^\circ}(x) \ge I_{\{F\}^\circ}(y)\}$, where the intersection is taken over all pairs $x, y \in \mathrm{Q}$ such that $M(x) \ge M(y)$. The following statements hold.

(a) $\mathcal{F}$ is the representation of the least M-consistent SPO.

(b) If the preorder $\succcurlyeq$ over $\mathcal{F}$ induced by Q is antisymmetric (i.e., $F \succcurlyeq G$ & $G \succcurlyeq F \Rightarrow F = G$), then the natural domain D of M exists and admits the representation

$$\mathrm{D} = \bigcap \{x \in \mathrm{P} : I_{\{F\}^\circ}(x) \ge I_{\{G\}^\circ}(x)\}, \tag{5}$$

where the intersection is taken over all pairs $F, G \in \mathcal{F}$ such that $F \succcurlyeq G$.

Example 3.

To illustrate the introduced notions consider the function $\pi : \mathrm{Q}_1' \to [1, +\infty)$ (where Q_1' is defined in Lemma 3) given by $\pi(-1_0 + b1_\tau) := b$. $\pi - 1$ is just the undiscounted net benefit of a project from Q_1'. One can show (see Lemma 14 in the Appendix) that π is a profitability metric whose natural extension is the undiscounted profitability index. To be more precise, π-consistent SPOs are induced by $\{H_\gamma^{(1_0)}, \gamma \in \Gamma\}$, where $(0,1) \subseteq \Gamma \subseteq [0,1]$. The representation of the least π-consistent SPO is $\{H_\gamma^{(1_0)}, \gamma \in [0,1]\}$. The natural domain of π is $\mathrm{D} = \mathrm{P} \setminus \{x \in \mathrm{P} : x(0) \ge 0, x(+\infty) < 0\}$ and the natural extension of π is the total preorder over D with a utility representation $\bar{\pi} : \mathrm{D} \to \overline{\mathrm{R}}_+$ given by

$$\bar{\pi}(x) := \begin{cases} 0 & \text{if } x(0) < 0 \text{ and } x(+\infty) < 0 \\ (x(+\infty) - x(0))/(-x(0)) & \text{if } x(0) < 0 \text{ and } x(+\infty) \ge 0 \\ +\infty & \text{if } x(0) \ge 0 \text{ and } x(+\infty) \ge 0 \end{cases}. \tag{6}$$

Note that if $x(0) < 0$ and $x(+\infty) \ge 0$, then $\bar{\pi}(x) = PI^F(x)$, where F is the NPV functional induced by the discount function 1_0.

We proceed by showing that various standard capital budgeting metrics, in particular, IRR, PP/DPP (more accurately, an order-reversing transformation of PP/DPP), PI as well as several other ratio type indices, are profitability metrics. We describe their natural domains and natural extensions.

4.1. IRR

The purpose of this section is threefold. First, we introduce a generalization of the conventional notion of IRR to nonexponential families of discount functions. Second, we show that the IRR (as well as the generalization) is a profitability metric, find its natural domain and natural extension, describe the IRR-consistent SPOs, and provide their axiomatic characterization. Third, we show that the conventional IRR is a unique profitability metric whose restriction to Q_2'' (where Q_2'' is defined in Lemma 4) is the rate of return, i.e., the metric that sends each project $-1_t + b1_{t+\tau} \in \mathrm{Q}_2''$ to its yield rate, $(1/\tau)\ln b$.

The notion of the internal rate of return does not necessary assume exponential discounting and can formally be applied to a parametric family of discount functions indexed by a parameter interpreted as the discount rate. For instance, assuming power discounting (Harvey, 1986), one can define the internal rate of return of a project x as the value of the discount rate $\lambda \in \mathrm{R}_+$ under which $F_\lambda(x) = 0$, where F_λ is the NPV functional associated with the discount function $t \mapsto (1+t)^{-\lambda}$. The following definition introduces a generic parametric family of discount functions that produces a consistent notion of IRR.

A collection $\mathrm{A} := \{\alpha_\lambda, \lambda \in \mathrm{R}_+\} \subset \mathcal{A}$ of discount functions is said to be a *D-family* if the following two conditions hold: (I) each $\alpha_\lambda \in \mathrm{A}$ is positive; (II) for any $0 \le t < \tau$, the function $\lambda \mapsto \alpha_\lambda(\tau)/\alpha_\lambda(t)$ is strictly decreasing and onto $(0,1]$. In what follows, the NPV functional associated with a discount function $\alpha_\lambda \in \mathrm{A}$ is denoted by $F_\lambda^{(\mathrm{A})}$. Set $\mathcal{F}^{(\mathrm{A})} := \{F_\lambda^{(\mathrm{A})}, \lambda \in \mathrm{R}_+\}$. Condition (II) allows us to interpret parameter λ as the discount rate (or, more accurately, the degree of impatience). Indeed, in the most general sense, one can define the degree of impatience as a characteristic of time preference that, when increased, makes the earlier of any two timed outcomes more preferable. This is exactly what the strict decreasingness of $\lambda \mapsto \alpha_\lambda(\tau)/\alpha_\lambda(t)$ asserts: for any $t < \tau$ and $a,b \in \mathrm{R}_{++}$, $F_\lambda^{(\mathrm{A})}(a1_t) \ge F_\lambda^{(\mathrm{A})}(b1_\tau) \;\Rightarrow\; F_{\lambda'}^{(\mathrm{A})}(a1_t) > F_{\lambda'}^{(\mathrm{A})}(b1_\tau) \;\; \forall \lambda' > \lambda$. Provided that elements of A are differentiable, condition (II) implies that for any t, the instantaneous discount rate, $-(\ln \alpha_\lambda(t))'$, is a nondecreasing function of λ. The definition also implies that for any $t > 0$, the function $\lambda \mapsto \alpha_\lambda(t)$ is a strictly decreasing homeomorphism of R_+ onto $(0,1]$. Moreover, it can be shown that the function $\lambda \mapsto \alpha_\lambda(+\infty) := \lim_{t \to +\infty} \alpha_\lambda(t)$ is nonincreasing and continuous. In the special case when the function $(\lambda, t) \mapsto \alpha_\lambda(t)$ is continuously differentiable an analogue of a D-family governed by a real (rather than a nonnegative real) discount rate and its relation to the notion of IRR is studied in Vilensky and Smolyak (1999).

The family $\{\alpha^\lambda, \lambda \in \mathrm{R}_+\}$, where α is a strictly decreasing discount function, called a *power family*, serves as an example of a D-family. In particular, the exponential discounting family $\mathrm{E} := \{t \mapsto e^{-\lambda t}, \lambda \in \mathrm{R}_+\}$, constant sensitivity discounting families $\{t \mapsto \exp(-\lambda t^\beta), \lambda \in \mathrm{R}_+\}$, $\beta > 0$ (Ebert and Prelec, 2007) and generalized hyperbolic discounting families $\{t \mapsto (1+\beta t)^{-\lambda/\beta}, \lambda \in \mathrm{R}_+\}$, $\beta > 0$ (Loewenstein and Prelec, 1992) are power and, hence, D-families. We also note that the restriction of $\succcurlyeq_2$ to a D-family A is total, so that the SPO induced by $\mathcal{F}^{(\mathrm{A})}$ is Q_2-complete (Lemma 4).

Given a D-family A, it follows from a convergence theorem (Monteiro et al., 2018, Theorem 6.8.6) that for any $x \in \mathrm{P}$ the function $g_x^{(\mathrm{A})}(\lambda) := F_\lambda^{(\mathrm{A})}(x)$ is continuous on R_+. If it has one change of sign, the internal rate of return is defined as follows. A project x is said to *possess the IRR w.r.t.* A if there exists a number $IRR^{(\mathrm{A})}(x) \in \mathrm{R}_+$ such that $\operatorname{sgn} g_x^{(\mathrm{A})}(\lambda) = \operatorname{sgn}(IRR^{(\mathrm{A})}(x) - \lambda)$ for all $\lambda \in \mathrm{R}_+$. Put differently, x possesses the IRR w.r.t. A if $g_x^{(\mathrm{A})}$ has a unique root and moreover at this root the function changes sign from positive to negative. If $\mathrm{A} = \mathrm{E}$, this definition reduces to the conventional definition of the IRR. Denote by $\mathrm{Q}^{(\mathrm{A})} \subset \mathrm{P}$ the set of projects possessing the IRR w.r.t. A.

Clearly, $Q_2'' \subset Q^{(A)}$ for any D-family A. The restriction of $IRR^{(A)} : Q^{(A)} \to R_+$ to Q_2'', denoted by $RR^{(A)}$, is called the *rate of return w.r.t.* A. $RR^{(A)}$ sends each project $-1_t + b1_\tau \in Q_2''$ to the solution $\lambda \in R_+$ of the equation $\alpha_\lambda(t) = b\alpha_\lambda(\tau)$. For instance, if A is a power family $\{\alpha^\lambda, \lambda \in R_+\}$, then $RR^{(A)}(-1_t + b1_\tau) = (\ln\alpha(t) - \ln\alpha(\tau))^{-1} \ln b$.

Our next result shows that $RR^{(A)}$ and $IRR^{(A)}$ are profitability metrics.

Proposition 3.

Let A be a D-family. The following statements hold.

(a) $RR^{(A)}$ is a profitability metric.

(b) An SPO is $RR^{(A)}$-consistent if and only if it is induced by $\{F_\lambda^{(A)}, \lambda \in \Lambda\}$, where Λ is a dense subset of R_+.

(c) The least $RR^{(A)}$-consistent SPO is induced by $\mathcal{F}^{(A)}$.

(d) The natural domain of $RR^{(A)}$, denoted by $D^{(A)}$, consists of projects $x \in P$ such that the function $g_x^{(A)}$ is either nonnegative, or negative, or there is $\lambda \in R_+$ such that $g_x^{(A)}$ is nonnegative on $[0, \lambda]$ and negative on $(\lambda, +\infty)$.

(e) The natural extension of $RR^{(A)}$ is the total preorder over $D^{(A)}$ with a utility representation $\overline{RR}^{(A)} : D^{(A)} \to \overline{R}$ given by $\overline{RR}^{(A)}(x) := \sup\{\lambda \in R_+ : g_x^{(A)}(\lambda) \geq 0\}$ (with the convention $\sup\varnothing = -\infty$).

Moreover, statements (a)–(e) remain valid with $RR^{(A)}$ replaced by $IRR^{(A)}$.

Proposition 3 demonstrates that the notion of the rate of return w.r.t. A defined for investment operations with only two transactions Q_2'' admits a unique extension (satisfying several reasonable conditions) to $Q^{(A)}$. This extension is exactly $IRR^{(A)}$. The literature knows several nonequivalent metrics that reduce to the logarithmic rate of return $RR^{(E)}$ (or an order-preserving transformation of this value) being restricted to Q_2'': the conventional IRR, the metrics introduced in Arrow and Levhari (1969) and Bronshtein and Skotnikov (2007), to mention just a few. Proposition 3 shows that the conventional IRR is the only metric among them that can serve for profitability measurement purposes.

If the function $g_x^{(E)}$ has multiple roots, the literature suggests various modifications of IRR that reduce to the conventional IRR whenever $g_x^{(E)}$ has one change of sign. For instance, the minimal root is important as the asymptotic growth rate of a sequence of repeated projects (Cantor and Lippman, 1983). More involved selection procedures among the roots were proposed in Hartman and Schafrick (2004) and Weber (2014).[5] Proposition 3 shows that these modifications are not profitability metrics. The largest extension of the domain of $IRR^{(E)}$ is described in part (d). Unfortunately, from an economic viewpoint $D^{(E)}$ adds almost nothing to $Q^{(E)}$. Loosely speaking, $IRR^{(E)}$ (as well as $IRR^{(A)}$) does not possess an extension to a larger set preserving completeness.

The picture changes if we allow for an incomplete extension. Parts (b) and (c) describe all the extensions (not necessarily complete) of $RR^{(A)}$ and their common part. To illustrate their worth

[5] We also refer to Hazen (2003) for an interesting result that the choice of a particular root is in some sense immaterial.

relative to the conventional IRR, consider the projects $x = 1_0 - 2\cdot 1_1 + 1.1\cdot 1_2$, $y = -1_0 + 2\cdot 1_1 - 0.7\cdot 1_2$, $z = -1_0 + 2.7\cdot 1_1 - 1.8\cdot 1_2$. They are incomparable in the sense of conventional IRR: the IRR equation for x (i.e., $g_x^{(\mathrm{E})}(\lambda) = 0$) has no roots, the IRR equation for z has two roots, 0.18 and 0.41, whereas y possesses the IRR 0.44. However, $x \succ y \succ z$ for *every* $IRR^{(\mathrm{E})}$-consistent SPO $\succeq$. As another illustration, one can easily construct projects $x, y \in \mathrm{P}$ such that x and y possess the IRR w.r.t. E and $IRR^{(\mathrm{E})}(x) > IRR^{(\mathrm{E})}(y)$, whereas $x + y$ does not possess the IRR w.r.t. E. Thus, $x + y$ is incomparable with x and y in the sense of IRR. However, $x \succ x + y \succ y$ for *all* $IRR^{(\mathrm{E})}$-consistent SPO $\succeq$. Note that the fact that the SPO with the representation $\mathcal{F}^{(\mathrm{E})}$ can be considered as an extension of $IRR^{(\mathrm{E})}$ was observed in Bronshtein and Akhmetova (2004).[6]

Some authors argue (Gronchi, 1986; Promislow, 2015, section 2.12) that the root uniqueness condition in the form we use in the definition of the IRR is not sufficient to be relevantly used for decision-making. However, Proposition 3 shows that the definition of $IRR^{(\mathrm{E})}$ we adopt is meaningful (at least for profitability measurement purposes) and, moreover, admits further generalization. On the other hand, in order to extend the class of projects possessing the IRR, some authors argue (e.g., see Vilensky and Smolyak, 1999) to take into account roots of $g_x^{(\mathrm{E})}$ only in a reasonable range $[0, \lambda^*]$, where λ^* is the greatest feasible interest rate. Clearly, this modification is a profitability metric, a corresponding consistent SPO is induced by $\{F_\lambda^{(\mathrm{E})},\ \lambda \in [0, \lambda^*]\}$.

In order to formulate our next result we introduce the following definition. Put $\mathrm{S} := \mathrm{R}_{++}\mathrm{Q}_2'' = \{-a1_t + b1_\tau,\ 0 \le t < \tau,\ 0 < a \le b\}$; to simplify the notation we write $(a,b;t,\tau)$ for $-a1_t + b1_\tau \in \mathrm{S}$. A function $M : \mathrm{S} \to \mathrm{R}$ is said to be a *rate of return* if the following four conditions hold.

1°. $M(a,b;t,\tau) = M(\lambda a, \lambda b; t, \tau)$ for any $\lambda > 0$.

2°. $M(a,b;t,\tau) = M(b,c;\tau,\delta) \Rightarrow M(a,b;t,\tau) = M(a,c;t,\delta)$.

3°. M is strictly increasing in its second argument.

4°. For any $x \in \mathrm{S}$ and $0 \le t < \tau$, there are $0 < a \le b$ such that $M(a,b;t,\tau) = M(x)$.

The value $M(a,b;t,\tau)$ is interpreted as the yield rate (the rate of return) of the project $-a1_t + b1_\tau \in \mathrm{S}$. Condition 1° states that a rate of return takes no account of the investment size and hence is a relative measure. According to 2°, if rates of return over two subsequent periods are equal, the rate of return over the consolidated period will be the same. By 3°, a rate of return is an increasing function of the final outcome. Finally, according to condition 4°, delay can always be compensated by changing a money flow. An example of a rate of return is provided by the logarithmic rate of return, $M(a,b;t,\tau) = (\tau - t)^{-1}\ln(b/a)$.

Though the notion of the IRR w.r.t. a D-family seems to be intuitive, it is introduced ad hoc, just by analogy with the conventional IRR. Our next result shows that it is actually a genuine extension of a rate of return, which in turn is proved to be a profitability metric. Namely, a function $M : \mathrm{S} \to \mathrm{R}$ is a rate of return if and only if there are a D-family A and a strictly increasing

[6] Being applied to valuation of cash flow (or, more generally, utility) streams rather than profitability, multiple discount rates have gained increasing popularity in the recent literature (Chambers and Echenique, 2018; Drugeon et al., 2019).

function φ such that $M = \varphi \circ IRR^{(\mathrm{A})}$ on S (or, equivalently, $M(a,b;t,\tau) = \varphi \circ RR^{(\mathrm{A})}(1,b/a;t,\tau)$). Moreover, and this is the main result of this section, the IRR w.r.t. a D-family is, up to an order-preserving transformation, the only profitability metric whose restriction to Q_2'' satisfies two natural conditions – continuity and monotonicity.

Proposition 4.

For an SPO $\succeq$, the following conditions are equivalent:

(a) there exists a D-family A such that $\succeq$ is $RR^{(\mathrm{A})}$-consistent (equivalently, $IRR^{(\mathrm{A})}$-consistent);

(b) there exists a rate of return M such that $\succeq$ is M-consistent;

(c) the restriction of $\succeq$ to Q_2'' is a lower semicontinuous (in the subspace topology) total preorder and $-1_t + a1_\tau \succ -1_t + b1_\tau$, whenever $0 \le t < \tau$ and $a > b \ge 1$.

An interesting consequence of Proposition 4 is that a rate of return is nondecreasing in its third argument and nonincreasing in the fourth argument, that is, delay is undesirable. This follows from the definition of $RR^{(\mathrm{A})}$ and nonincreasingness of a discount function. Note that conditions 1º–4º do not contain any explicit assumption on how a rate of return depends on time.

We proceed by characterizing SPOs consistent with the conventional $IRR^{(\mathrm{E})}$. For any $x \in \mathrm{P}$ and $\tau > 0$, put $x^{(+\tau)}(t) := \begin{cases} 0 & \text{if } t < \tau \\ x(t-\tau) & \text{if } t \ge \tau \end{cases}$. That is, $x^{(+\tau)}$ is the project x postponed until τ. A binary relation $\succeq$ over a set $\mathrm{Q} \subseteq \mathrm{P}$ (with $\sim$ being the symmetric part of $\succeq$) is said to be *stationary* if $x \sim x^{(+\tau)}$, whenever $\tau > 0$ and $x, x^{(+\tau)} \in \mathrm{Q}$. The condition states that postponement of a project does not affect profitability. Note that if a project $x \in \mathrm{P}$ possesses the IRR w.r.t. E, then so is $x^{(+\tau)}$ and $IRR^{(\mathrm{E})}(x) = IRR^{(\mathrm{E})}(x^{(+\tau)})$. Our next result characterizes $IRR^{(\mathrm{E})}$-consistent SPOs by means of stationarity and monotonicity conditions. In particular, it shows that $IRR^{(\mathrm{E})}$ is, up to an order-preserving transformation, the only profitability metric whose restriction to Q_2'' is stationarity and monotone. This characterization could be predicted in view of Proposition 4 and a well-known fact that multiplicative discounting reduces to exponential discounting under stationarity (Fishburn and Rubinstein, 1982, Theorem 2).

Proposition 5.

For an SPO $\succeq$, the following conditions are equivalent:

(a) $\succeq$ is $RR^{(\mathrm{E})}$-consistent (equivalently, $IRR^{(\mathrm{E})}$-consistent);

(b) the restriction of $\succeq$ to Q_2'' is stationary and there exists $t > 0$ such that $-1_0 + a1_t \succ -1_0 + b1_t$ for any $a > b \ge 1$.

By Proposition 3 (part (b)), there is $\beth_2$ (the cardinality of the set of all dense subsets of R_+) $IRR^{(\mathrm{A})}$-consistent SPOs, each of which we consider as a possible extension of $IRR^{(\mathrm{A})}$. We proceed by showing that these extensions are essentially unique, namely, they coincide on a large class of projects, called regular. We introduce the following definition. Given $\mathcal{F} \subseteq \mathcal{NPV}$, a project $x \in \mathrm{P}$ is said to be *regular w.r.t.* $\mathcal{F}$ if $\{x\}^\circ \cap \mathcal{F}$ is a regular closed set (i.e., is equal to the closure of its interior) in the subspace topology on $\mathcal{F}$. Denote by $\mathcal{R}(\mathcal{F})$ the set of all projects that are regular w.r.t. $\mathcal{F}$. The following lemma motivates the definition.

Lemma 9.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$, $\mathcal{F}'$ be a dense subset of $\mathcal{F}$, and $\succeq'$ be the SPO induced by $\mathcal{F}'$. Then the restrictions of $\succeq$ and $\succeq'$ to $\mathcal{R}(\mathcal{F})$ coincide.

To illustrate Lemma 9, assume that in the notation of the lemma $\mathcal{F}$ is open in the subspace topology on $\{F \in \mathrm{P}^* : F(1_0) = 1\}$ and convex, then $\mathcal{R}(\mathcal{F}) = \mathrm{P}$, so that $\succeq = \succeq'$. This can be established with the help of the fact that if a convex subset C of a topological vector space has a nonempty interior, then $\mathrm{cl}(\mathrm{C}) = \mathrm{cl}(\mathrm{int}(\mathrm{C}))$ (Aliprantis and Border, 2006, Lemma 5.28). We omit the details.

Given a D-family A, it can be shown that the map $\lambda \mapsto F_\lambda^{(\mathrm{A})}$ is a homeomorphism between R_+ and $\mathcal{F}^{(\mathrm{A})}$ endowed with the subspace topology (see Lemma 13 in the Appendix). Thus, Lemma 9 implies that the structure of the dense subset Λ of R_+ in part (b) of Proposition 3 is immaterial, provided that we restrict ourselves to regular projects. So a project $x \in \mathrm{P}$ is regular w.r.t. $\mathcal{F}^{(\mathrm{A})}$ iff $\mathrm{A}(x) := \{\lambda \in \mathrm{R}_+ : g_x^{(\mathrm{A})}(\lambda) \ge 0\}$ is regular closed in R_+. To illustrate, consider a project $x \in \mathrm{P}$ such that the set of solutions $\{\lambda\}$ of the IRR equation, $g_x^{(\mathrm{A})}(\lambda) = 0$, is finite. For instance, this condition holds if A is a power family and x is nonzero and contains finitely many transactions, i.e., lies in the linear span of $\{1_\tau, \tau \in \mathrm{R}_+\}$ (Tossavainen, 2006). Since $g_x^{(\mathrm{A})}$ is continuous on R_+, $\mathrm{A}(x)$ is a union of finitely many closed intervals. Therefore, x is regular w.r.t. $\mathcal{F}^{(\mathrm{A})}$ if and only if $\mathrm{A}(x)$ has no isolated points, that is, $g_x^{(\mathrm{A})}$ has no zero local maxima. In particular, if a project possesses a nonzero IRR w.r.t. A, then it is regular.

A simple sufficient condition of regularity w.r.t. $\mathcal{F}^{(\mathrm{A})}$ is given in the following lemma.

Lemma 10.

Let A be a D-family. Given a project $x \in \mathrm{P}$, assume that $g_x^{(\mathrm{A})}$ is differentiable on R_{++}. If (a) $x(0) \ne 0$, $x(+\infty) \ne 0$, and (b) there is no $\lambda \in \mathrm{R}_{++}$ such that $g_x^{(\mathrm{A})}(\lambda) = (g_x^{(\mathrm{A})})'(\lambda) = 0$, then x is regular w.r.t. $\mathcal{F}^{(\mathrm{A})}$.

It can be shown that if A is a power family $\{\alpha^\lambda, \lambda \in \mathrm{R}_+\}$ generated by a differentiable discount function α with $\lim_{t \to +\infty} \alpha(t) = 0$ (in particular, if $\mathrm{A} = \mathrm{E}$), then for any $x \in \mathrm{P}$, the function $g_x^{(\mathrm{A})}$ is differentiable on R_{++} so that Lemma 10 is applicable in this case. To motivate the conditions of Lemma 10 note that if $x \in \mathrm{Q}^{(\mathrm{A})}$ does not satisfy (a) or (b), then every neighborhood of x contains a project that does not possess the IRR w.r.t. A. Indeed, if $\lim_{\lambda \to +\infty} g_x^{(\mathrm{A})}(\lambda) = x(0) = 0$ (resp. $g_x^{(\mathrm{A})}(0) = x(+\infty) = 0$), then $x + \varepsilon 1_0 \notin \mathrm{Q}^{(\mathrm{A})}$ (resp. $x - \varepsilon 1_0 \notin \mathrm{Q}^{(\mathrm{A})}$) for any $\varepsilon > 0$. Now assume that $(g_x^{(\mathrm{A})})'(IRR^{(\mathrm{A})}(x)) = 0$ and pick $y \in \mathrm{P}$ such that $g_y^{(\mathrm{A})}(IRR^{(\mathrm{A})}(x)) = 0$ and $(g_y^{(\mathrm{A})})'(IRR^{(\mathrm{A})}(x)) > 0$. Then $x + \varepsilon y \notin \mathrm{Q}^{(\mathrm{A})}$ for any $\varepsilon > 0$ as $g_{x+\varepsilon y}^{(\mathrm{A})}(IRR^{(\mathrm{A})}(x)) = 0$ and

$(g^{(\mathrm{A})}_{x+\varepsilon y})'(IRR^{(\mathrm{A})}(x))>0$. As a partial converse, we have that if the function $(z,\lambda)\mapsto g^{(\mathrm{A})}_{z}(\lambda)$ defined on $\mathrm{P}\times\mathrm{R}_+$ is continuously Fréchet differentiable and a project x satisfies conditions (a) and (b) of Lemma 10, then using the implicit function theorem one can show that $IRR^{(\mathrm{A})}$ is well defined in a neighborhood of x.

Example 1 (cont.).

As noted in Promislow (1997) the partition of P into the set of usurious $\mathrm{U}=\{F\}^{\circ}$ (where F is the NPV induced by the discount function $t\mapsto 1.6^{-t}$) and nonusurious $\mathrm{N}=\mathrm{P}\setminus\mathrm{U}$ loans suffers from the following drawback. If the threshold interest rate decreases (resp. increases), say from 60% to 50% (resp. 70%), then one would expect the loans which were usurious (nonusurious) at the old rate should remain such. However, this is not the case of the proposed solution. The drawback can be overcome only at the price of incompleteness. We suggest to use the level sets of the least $RR^{(\mathrm{E})}$-consistent SPO to describe the sets of usurious and nonusurios loans. To be more precise, given the threshold logarithmic rate $r\in\mathrm{R}_+$, define the set of usurious U_r (resp. nonusurious N_r) loans as $\mathrm{U}_{\succeq}(-1_0+e^{r}1_1)$ (resp. $\mathrm{L}_{\succ}(-1_0+e^{r}1_1)$), where $\succeq$ is the least $RR^{(\mathrm{E})}$-consistent SPO. That is, $\mathrm{U}_r:=\{x\in\mathrm{P}:F^{(\mathrm{E})}_{\lambda}(x)\geq 0\ \forall\lambda\in[0,r]\}$, $\mathrm{N}_r:=\{x\in\mathrm{P}:F^{(\mathrm{E})}_{\lambda}(x)<0\ \forall\lambda\in(r,+\infty)\}\setminus\mathrm{U}_r$. This definition generalizes the statement of the criminal code: indeed, if x possesses the IRR w.r.t. E, then $x\in\mathrm{N}_r$ (resp. $x\in\mathrm{U}_r$) if and only if $IRR^{(\mathrm{E})}(x)<r$ (resp. $IRR^{(\mathrm{E})}(x)\geq r$). Moreover, whenever $r'>r$, we have $\mathrm{U}_{r'}\subset\mathrm{U}_r$ and $\mathrm{N}_r\subset\mathrm{N}_{r'}$ as desired. However, we lose completeness as $\mathrm{U}_r\cup\mathrm{N}_r\neq\mathrm{P}$.

4.2. PP and DPP

In this section, we show that an order-reversing transformation of the payback period, as well as of its discounted counterpart, is a profitability metric. We find its natural domain, the natural extension and describe the corresponding consistent SPOs.

For any $x\in\mathrm{P}$ and $\tau\in\mathrm{R}_+$, denote by $x_{\leq\tau}(t):=\begin{cases}x(t) \text{ if } t\leq\tau\\ x(\tau) \text{ if } t>\tau\end{cases}$ the project truncated at τ.

Notice that $x_{\leq\tau}\in\mathrm{P}$. Given $\alpha\in\mathcal{A}$, set $G^{(\alpha)}_{\tau}(x):=F^{(\alpha)}(x_{\leq\tau})=x(0)+\int_0^{\infty}\alpha\mathrm{d}(x_{\leq\tau})=x(0)+\int_0^{\tau}\alpha\mathrm{d}x$. Note that $G^{(\alpha)}_{\tau}\in\mathcal{NPV}$. Indeed, $G^{(\alpha)}_{\tau}$ is the NPV functional induced by the discount function $t\mapsto\begin{cases}\alpha(t) \text{ if } t\leq\tau\\ 0 \quad\ \text{if } t>\tau\end{cases}$. The function $\tau\mapsto G^{(\alpha)}_{\tau}(x)$ represents the cumulative discounted cash flow associated with x. If it has one change of sign over R_{++}, the discounted payback period is defined as follows. A project $x\in\mathrm{P}$ is said to *possess the DPP w.r.t.* α if there exists a number $DPP^{(\alpha)}(x)\in\mathrm{R}_{++}$ such that $G^{(\alpha)}_{\tau}(x)<0$ (resp. $G^{(\alpha)}_{\tau}(x)\geq 0$) for any $\tau\in(0,DPP^{(\alpha)}(x))$ (resp. $\tau\in[DPP^{(\alpha)}(x),+\infty)$). For every $x\in\mathrm{P}$ the function $\tau\mapsto G^{(\alpha)}_{\tau}(x)$ belongs to P (Monteiro et al., 2018, Corollary 6.5.5). In particular, if x possesses the DPP w.r.t. α, then $x(0)=G^{(\alpha)}_{0}(x)=\lim_{\tau\to 0+}G^{(\alpha)}_{\tau}(x)\leq 0$ (i.e., $x\in\mathrm{Q}''_5$). The conventional notions of PP and DPP correspond to $\alpha=1_0$ and $\alpha(t)=e^{-\lambda t}$, $\lambda\in\mathrm{R}_{++}$, respectively. Let $\mathrm{Q}^{(\alpha)}\subset\mathrm{P}$ be the set of projects possessing the DPP w.r.t. α. Note that $\mathrm{Q}^{(\alpha)} \neq \varnothing$ unless $\alpha = \chi$. We also notice that $\mathrm{Q}_1' \subset \mathrm{Q}^{(\alpha)}$ if and only if $\alpha = 1_0$, i.e., $DPP^{(\alpha)}$ is the conventional (undiscounted) PP.

The reciprocals of PP and DPP are known to be crude estimates of the IRR (Gordon, 1955; Sarnat and Levy, 1969; Bhandari, 2009). Our next result shows that $RDPP^{(\alpha)}(x) := 1/DPP^{(\alpha)}(x)$ is a profitability metric.

Proposition 6.

Let $\alpha \in \mathcal{A} \setminus \{\chi\}$. The following statements hold.

(a) $RDPP^{(\alpha)} : \mathrm{Q}^{(\alpha)} \to \mathrm{R}_{++}$ is a profitability metric.

(b) An SPO is $RDPP^{(\alpha)}$-consistent if and only if it is induced by $\{G_\tau^{(\alpha)}, \tau \in \mathrm{T}\} \cup \{H_\gamma^{(\alpha)}, \gamma \in \Gamma\}$, where T is a dense subset of $\operatorname{int}(\operatorname{supp}\{\alpha\})$ and $\Gamma \subseteq [1, 1/\alpha(0+)]$.

(c) The least $RDPP^{(\alpha)}$-consistent SPO is induced by $\{G_\tau^{(\alpha)}, \tau \in \operatorname{int}(\operatorname{supp}\{\alpha\})\} \cup \{H_\gamma^{(\alpha)}, \gamma \in [1, 1/\alpha(0+)]\}$.

(d) The natural domain of $RDPP^{(\alpha)}$ is $\mathrm{D}^{(\alpha)} := \mathrm{Q}_-^{(\alpha)} \cup \mathrm{Q}^{(\alpha)} \cup \mathrm{Q}_+^{(\alpha)}$, where $\mathrm{Q}_-^{(\alpha)}$ is the set of projects $x \in \mathrm{P}$ such that $G_\tau^{(\alpha)}(x) < 0$ for all $\tau \in \mathrm{R}_{++}$ and $\mathrm{Q}_+^{(\alpha)}$ is the set of projects $x \in \mathrm{P}$ such that $H_{1/\alpha(0+)}^{(\alpha)}(x) \geq 0$ and $G_\tau^{(\alpha)}(x) \geq 0$ for all $\tau \in \mathrm{R}_{++}$.

(e) The natural extension of $RDPP^{(\alpha)}$ is the total preorder over $\mathrm{D}^{(\alpha)}$ with a utility representation $\overline{RDPP}^{(\alpha)} : \mathrm{D}^{(\alpha)} \to \overline{\mathrm{R}}$ given by

$$\overline{RDPP}^{(\alpha)}(x) := \begin{cases} \sup\{\gamma \in [\alpha(0+), 1] : H_{1/\gamma}^{(\alpha)}(x) \geq 0\} - 1 & \text{if } x \in \mathrm{Q}_-^{(\alpha)} \\ RDPP^{(\alpha)}(x) & \text{if } x \in \mathrm{Q}^{(\alpha)} \\ +\infty & \text{if } x \in \mathrm{Q}_+^{(\alpha)} \end{cases} \tag{7}$$

(with the convention $\sup\varnothing = -\infty$).

Note that if $\alpha(0+) = 1$, then $\overline{RDPP}^{(\alpha)}$, defined in (7), reduces to

$$\overline{RDPP}^{(\alpha)}(x) = \begin{cases} -\infty & \text{if } x \in \mathrm{Q}_-^{(\alpha)} \text{ and } F^{(\alpha)}(x) < 0 \\ 0 & \text{if } x \in \mathrm{Q}_-^{(\alpha)} \text{ and } F^{(\alpha)}(x) = 0 \\ RDPP^{(\alpha)}(x) & \text{if } x \in \mathrm{Q}^{(\alpha)} \\ +\infty & \text{if } x \in \mathrm{Q}_+^{(\alpha)} \end{cases}.$$

In the case when the function $\tau \mapsto G_\tau^{(\alpha)}(x)$ has multiple changes of sign, some authors suggest to define the DPP as the minimum time t (if any) such that $G_\tau^{(\alpha)}(x) \geq 0$ for all $\tau \geq t$ (e.g., see Hajdasiński, 1993). Though this definition seems to be intuitive from an economic viewpoint, Proposition 6 shows that an order-reversing transformation of the DPP defined this way is not a profitability metric and, therefore, contrary to the claim (Hajdasiński, 1993, p. 184), unable to serve for profitability measurement purposes.

We proceed by considering a refinement of the adopted definition of DPP. An essential property of the payback period that motivates the definition is stability under truncation. An SPO is said to be *stable under truncation* if $x \succeq y \;\Rightarrow\; x_{\leq\tau} \succeq y_{\leq\tau}$ for any $\tau \geq 0$. The condition states that termination of projects (say, for external environmental reasons) do not result in a major perturbation of the profitability ordering. Given $\alpha \in \mathcal{A}$, denote by $G_{\tau,\lambda}^{(\alpha)}$, $\tau > 0$, $\lambda \in [0,1]$ the NPV functional induced by the discount function

$$t \mapsto \begin{cases} \alpha(t) & \text{if } t < \tau \\ \lambda\alpha(t) & \text{if } t = \tau \\ 0 & \text{if } t > \tau \end{cases}.$$

Note that $G_{\tau,\lambda}^{(\alpha)} = \lambda G_{\tau}^{(\alpha)} + (1-\lambda)\lim_{t\to\tau-} G_t^{(\alpha)}$. The next result characterizes Q_1'-complete SPOs that are stable under truncation.

Proposition 7.

Let $\succeq$ be an SPO with a representation $\mathcal{F}$. The following conditions are equivalent:

(a) $\succeq$ is Q_1'-complete and stable under truncation;

(b) there exists $\alpha \in \mathcal{A}$ such that $\{G_{\tau}^{(\alpha)}, \tau \in \{0\} \cup \mathrm{int}(\mathrm{supp}\{\alpha\})\} \subseteq \mathcal{F} \subseteq \{F^{(\chi)}, F^{(\alpha)}\} \cup \{G_{t,\lambda}^{(\alpha)}, (t,\lambda) \in (\mathrm{supp}\{\alpha\} \setminus \{0\}) \times [0,1]\}$.

Proposition 7 suggests the following refinement of the conventional notion of DPP. A project $x \in \mathrm{P}$ is said to *possess the refined DPP w.r.t.* α if $x(0) < 0$ and there exists a number $\tau^{(\alpha)}(x) \in \mathrm{R}_{++}$ such that $G_{t,0}^{(\alpha)}(x) < 0$ and $G_t^{(\alpha)}(x) < 0$ for all $t \in (0, \tau^{(\alpha)}(x))$ and $G_t^{(\alpha)}(x) \geq 0$ for all $t \in [\tau^{(\alpha)}(x), +\infty)$. If x possesses the refined DPP w.r.t. α, then denote by $\lambda^{(\alpha)}(x)$ the least solution $\lambda \in [0,1]$ of the equation $G_{\tau^{(\alpha)}(x),\lambda}^{(\alpha)}(x) = 0$. Let $\succeq$ be the SPO induced by $\{F^{(\chi)}, F^{(\alpha)}\} \cup \{G_{t,\lambda}^{(\alpha)}, (t,\lambda) \in (\mathrm{supp}\{\alpha\} \setminus \{0\}) \times [0,1]\}$. Then, provided that projects x and y possess the refined DPP w.r.t. α,

$$x \succeq y \iff (\tau^{(\alpha)}(y), \lambda^{(\alpha)}(y)) \geq_{\mathrm{lex}} (\tau^{(\alpha)}(x), \lambda^{(\alpha)}(x)), \tag{8}$$

where $\geq_{\mathrm{lex}}$ is the lexicographic order. Clearly, if x possesses the refined DPP w.r.t. α, it also possesses the DPP w.r.t. α and $DPP^{(\alpha)}(x) = \tau^{(\alpha)}(x)$. If $x(0) < 0$ and the function x is continuous, the converse is also true.

Real-world investment projects are discrete, i.e., lie in the closure of the linear span of $\{1_\tau, \tau = 0,1,...\}$. It is a common practice to use linear interpolation of the cumulative discounted cash flow to evaluate DPP of a discrete project (e.g., see Götze et al., 2015, p. 72). For a discrete project x, the DPP obtained via interpolation is given by $DPP_*^{(\alpha)}(x) := \tau^{(\alpha)}(x) - 1 + \lambda^{(\alpha)}(x)$. Note that the restriction of the ordering (8) to the set of discrete projects possessing the DPP w.r.t. α coincides with the ordering induced by $1/DPP_*^{(\alpha)}$. This observation provides a formal justification for the linear interpolation practice.

4.3. PI and other ratio type indices

In this section, we show that the profitability index PI^F (as well as the ratio index RI_G^F) introduced in section 3 is a profitability metric. We find its natural domain and the natural extension, describe the corresponding consistent SPOs and provide their axiomatic characterization.

Let F and G $(\neq F)$ be NPV functionals. A project $x \in \mathrm{P}$ is said to *possess the ratio index* (*RI*) *w.r.t.* F *and* G if $F(x) \geq 0$ and $G(x) < 0$; in this case the index is defined as $RI_G^F(x) = 1 - F(x)/G(x)$. RI_G^F comprises several popular profitability measures. For instance, the undiscounted profitability index (the return on investment), $x \mapsto (x(+\infty) - x(0))/(-x(0))$, considered in Example 3, corresponds to $F = F^{(1_0)}$ and $G = F^{(\chi)}$. The conventional discounted profitability index PI^F corresponds to RI_G^F with $G = F^{(\chi)}$. If $x \in \mathrm{Q}_2$, i.e., x is a conventional investment in which a series of cash outflows is followed after some time $\tau \in \mathrm{R}_+$ by a series of cash inflows, then $RI_G^F(x)$ is the discounted benefit-cost ratio provided that $F = F^{(\alpha)}$ and $G = G_\tau^{(\alpha)}$.

Denote by $\mathrm{Q}_G^F \subset \mathrm{P}$ the set of projects possessing the RI w.r.t. F and G. Note that $\mathrm{Q}_G^F \neq \varnothing$ for any distinct $F, G \in \mathcal{NPV}$. The next result shows that $RI_G^F : \mathrm{Q}_G^F \to [1, +\infty)$ is a profitability metric.

Proposition 8.

Given $F, G \in \mathcal{NPV}$, $F \neq G$, put $\widetilde{\mathrm{W}} := \{w \in \mathrm{R} : wF + (1-w)G \in \mathcal{NPV}\}$, $\widetilde{F} := G + \sup \widetilde{\mathrm{W}}(F - G)$, $\widetilde{G} := G + \inf \widetilde{\mathrm{W}}(F - G)$.[7] The following statements hold.

(a) $RI_G^F : \mathrm{Q}_G^F \to [1, +\infty)$ is a profitability metric.

(b) An SPO is RI_G^F-consistent if and only if it is induced by $\{wF + (1-w)G,\ w \in \mathrm{W}\}$, $(0,1) \subseteq \mathrm{W} \subseteq \widetilde{\mathrm{W}}$.

(c) The least RI_G^F-consistent SPO is induced by $\{w\widetilde{F} + (1-w)\widetilde{G},\ w \in [0,1]\}$.

(d) The natural domain of RI_G^F is given by $\mathrm{D}_G^F = \mathrm{P} \setminus \{x \in \mathrm{P} : \widetilde{F}(x) < 0,\ \widetilde{G}(x) \geq 0\}$.

(e) The natural extension of RI_G^F is the total preorder on D_G^F with a utility representation $\overline{RI}_G^F : \mathrm{D}_G^F \to \overline{\mathrm{R}}_+$ given by

$$\overline{RI}_G^F(x) := \begin{cases} 0 & \text{if } \widetilde{F}(x) < 0 \text{ and } \widetilde{G}(x) < 0 \\ RI_{\widetilde{G}}^{\widetilde{F}}(x) & \text{if } \widetilde{F}(x) \geq 0 \text{ and } \widetilde{G}(x) < 0 \\ +\infty & \text{if } \widetilde{F}(x) \geq 0 \text{ and } \widetilde{G}(x) \geq 0 \end{cases}.$$

Let F be an NPV functional induced by a discount function α satisfying $\alpha(0+) = 1$. It follows from Proposition 8 that the natural domain of $PI^F : \{x \in \mathrm{P} : F(x) \geq 0,\ x(0) < 0\} \to [1, +\infty)$ is $\mathrm{P} \setminus \{x \in \mathrm{P} : F(x) < 0,\ x(0) \geq 0\}$ and the natural extension is

$$\overline{PI}^F(x) := \begin{cases} 0 & \text{if } F(x) < 0 \text{ and } x(0) < 0 \\ PI^F(x) & \text{if } F(x) \geq 0 \text{ and } x(0) < 0 \\ +\infty & \text{if } F(x) \geq 0 \text{ and } x(0) \geq 0 \end{cases}.$$

Unfortunately, from an economic viewpoint $\overline{PI}^F$ adds almost nothing to PI^F. Loosely speaking, PI^F does not possess an extension to a larger set.

[7] Since the set $\mathcal{NPV}$ is compact, it is routine to verify that $\widetilde{\mathrm{W}}$ is bounded and $\widetilde{F}, \widetilde{G} \in \mathcal{NPV}$.

In order to characterize PI^F we introduce the following definition. Given $x \in \mathrm{P}$ and $\gamma \in [0,1]$, denote by $x_\gamma(t) := x(0) + \gamma(x(t) - x(0))$ the project whose future cash flow $x - x(0)1_0$ is reduced by the scale factor γ. Note that, by construction, $F^{(\alpha)}(x_\gamma) = H_\gamma^{(\alpha)}(x)$, where the functional $H_\gamma^{(\alpha)}$ is defined in section 3. An SPO is said to be *stable under reduction* if $x \succeq y \Rightarrow x_\gamma \succeq y_\gamma$ for any $\gamma \in (0,1)$. The condition states that reduction of projects' future cash flows do not result in a major perturbation of the profitability ordering. An SPO is said to be *monotone* if $-1_0 + a1_t \succ -1_0$, $t > 0 \Rightarrow -1_0 + b1_t \succ -1_0 + a1_t$ for any $b > a$.

Proposition 9.

For an SPO $\succeq$, the following conditions are equivalent:

(a) either $\succeq$ is the NPV criterion induced by $F^{(\chi)}$ or $\succeq$ is monotone and PI^F-consistent for some $F \in \mathcal{NPV} \setminus \{F^{(\chi)}\}$;

(b) $\succeq$ is Q_4-complete and monotone;

(c) $\succeq$ is Q_1'-complete, stable under reduction, and monotone;

(d) there are a discount function α and a set $(0,1) \subseteq \Gamma \subseteq [0,1]$ such that $\{H_\gamma^{(\alpha)}, \gamma \in \Gamma\}$ represents $\succeq$.

Proposition 9 provides two characterizations of monotone PI^F-consistent SPOs. First, it shows that an incomplete monotone SPO is PI^F-consistent for some $F \in \mathcal{NPV}$ if and only if it is Q_4-complete (a similar assertion is valid with regard to a RI_G^F-consistent SPO and Q_G-completeness, where Q_G is defined in Lemma 7). Second, under monotonicity and Q_1'-completeness, PI^F-consistent SPOs are exactly those that are incomplete and stable under reduction of a future cash flow. As a corollary of Proposition 9, we get that an SPO $\succeq$ is π-consistent, where π is defined in Example 3, if and only if $\succeq$ is Q_4-complete and $-1_0 + b1_t \succ -1_0 + a1_t$ for any $t > 0$ and $b > a \geq 1$.

4.4. Discussion

An investor faces multiple sources of uncertainty: uncertain discount rate and uncertain project's cash flows, to mention just a few. Investment in the real sector may also suffer from the risk of project truncation (for external environmental reasons), postponement, or uncertain intensity of the project implementation. The definition of an SPO $\succeq$ with a representation $\mathcal{F}$ suggests that $\succeq$ can be treated as a measure of project's financial stability under a specific source of uncertainty determined by $\mathcal{F}$.

1. The structure of the set representing an NPV criterion – a singleton – suggests that the criterion should be used under complete certainty.

2. The structure of the set representing an $IRR^{(\mathrm{A})}$-consistent SPO (Proposition 3) suggests that the IRR can be considered as a measure of project's financial stability under uncertain discount rate.

3. The structure of a representation of a $RDPP^{(\alpha)}$-consistent SPO (Proposition 6) suggests that the DPP can be considered as a measure of project's financial stability under truncation. Recall that the value $G_\tau^{(\alpha)}(x)$ can be interpreted as the NPV of the truncated project $x_{\le\tau}$.

4. The structure of the set representing a RI_G^F-consistent SPO, $\{w\widetilde{G}+(1-w)\widetilde{F},\ w\in \mathrm{W}\}$, $\mathrm{W}\subseteq[0,1]$ (Proposition 8), suggests that the ratio index RI_G^F can be considered as a measure of project's financial stability under probabilistic uncertainty with respect to two scenarios. Namely, one can interpret $\widetilde{F}$ and $\widetilde{G}$ as the NPV functionals associated with the optimistic and pessimistic scenarios, respectively. Then $w\widetilde{F}+(1-w)\widetilde{G}$, $w\in[0,1]$ is the expected NPV provided that the probability of the optimistic scenario is w.

The structure of a PI^F-consistent SPO suggests that the profitability index can be thought a measure of project's financial stability under uncertain future cash flow, $x-x(0)1_0$, in the form of its proportional (to some scale factor $\gamma\in[0,1]$) reduction. This interpretation follows directly from the identity $H_\gamma^{(\alpha)}(x)=F^{(\alpha)}(x_\gamma)$.[8] Recall that $H_\gamma^{(\alpha)}$, $\gamma\in[0,1]$ is the NPV functional associated with the discount function $\gamma\alpha+(1-\gamma)\chi$. Here $\gamma\in[0,1]$ can be interpreted as the short term discount factor or present bias parameter (Laibson, 1997). Therefore, the profitability index is also a measure of project's financial stability under uncertain short term discount factor.

5. The following SPO may serve as a measure of financial stability under uncertain intensity of project implementation. For given $\alpha\in\mathcal{A}$ and $\lambda\in \mathrm{R}_{++}$, set $U_\lambda^{(\alpha)}(x):=x(0)+\int_0^\infty \alpha(t)\mathrm{d}x(\lambda t)$. $U_\lambda^{(\alpha)}$ is the value of a project implemented with the intensity λ. Changing the variable in the integral, we get that $U_\lambda^{(\alpha)}$ is the NPV functional associated with the discount function $t\mapsto\alpha(t/\lambda)$. Therefore, the SPO induced by $\{U_\lambda^{(\alpha)},\ \lambda\in\Lambda\subseteq \mathrm{R}_{++}\}$ can be used if the investor faces uncertain intensity of the project implementation. Note that if $\alpha(t)=e^{-\lambda_0 t}$, $\lambda_0\in \mathrm{R}_{++}$, then $\{U_\lambda^{(\alpha)},\ \lambda\in\Lambda\}\subset\mathcal{F}^{(\mathrm{E})}$, so that the resulting profitability metric is consonant with the IRR.

6. The following SPO may serve as a measure of financial stability under risk of postponement of the project implementation. For given $\alpha\in\mathcal{A}$ and $\tau\in \mathrm{R}_+$, denote by $V_\tau^{(\alpha)}(x):=x(0)+\int_\tau^\infty \alpha(t)\mathrm{d}x(t-\tau)$ the value of the project x whose implementation (with the exception of the initial transaction, $x(0)$) is postponed until τ. Changing the variable in the integral, we get that $V_\tau^{(\alpha)}$ is the NPV functional associated with the discount function $t\mapsto\begin{cases}1 & \text{if } t=0\\ \alpha(t+\tau) & \text{if } t>0\end{cases}$. Therefore, the SPO induced by $\{V_\tau^{(\alpha)},\ \tau\in \mathrm{R}_+\}$ can be used if the investor faces the risk of project postponement. Note that if $\alpha(t)=e^{-\lambda t}$, $\lambda\in \mathrm{R}_{++}$, then $\{V_\tau^{(\alpha)},\ \tau\in \mathrm{R}_+\}=\{H_\gamma^{(\alpha)},\ \gamma\in(0,1]\}$, so that the resulting profitability metric is the conventional PI.

[8] The identity $(1/\gamma)H_\gamma^{(\alpha)}(x)=F^{(\alpha)}\big((1/\gamma)x(0)1_0+(x-x(0)1_0)\big)$, $\gamma\in(0,1]$ demonstrates that the profitability index can equivalently be interpreted as a measure of project's financial stability under uncertain initial outflow $x(0)$.

The literature seems to be controversial with respect to conditions under which the use of one profitability metric is superior to other. The interpretation above suggests that the choice of a particular metric should be determined by the source of uncertainty the investor faces. Some authors argue to use multiple criteria (say, the combination of IRR and DPP) to choose between projects. In view of property 6° of Lemma 1, such a multiple criterion can be constructed by uniting the sets of NPV functionals representing the partial criteria.

5. Conclusion

This paper provides an axiomatic characterization of a project's profitability ranking. We adopt axioms similar to those used in Promislow (1997) and Vilensky and Smolyak (1999), but in contrast to the latter paper, allow for incomparable projects. This results in a class of orderings that includes the ones induced by conventional capital budgeting metrics, in particular, by the NPV criterion, IRR, PI, PP, and DPP.

The project space P we deal with covers investment projects with bounded deterministic cash flows. Theoretical financial models operate unbounded and/or stochastic cash flows, so that other types of project spaces are of interest. Note that all the obtained results that do not explicitly rely on the structure of an NPV functional (namely, Propositions 1, 2, 8 and Lemmas 1, 2, 7, 8, 9, 12) remain valid for an ordered Hausdorff locally convex topological vector space with an order unit. That is, if P is a real Hausdorff locally convex topological vector space, $\mathrm{P}_+ \subset \mathrm{P}$ is a pointed closed convex cone with a nonempty interior $\mathrm{P}_{++} = \operatorname{int} \mathrm{P}_+$, and $\mathcal{NPV} = \{F \in \mathrm{P}_+^\circ : F(e) = 1\}$, where $e \in \mathrm{P}_{++}$ is a distinguished element called an order unit. Notice that the existence of an order unit implies that the cone of nonnegative projects P_+ is generating, i.e., every $x \in \mathrm{P}$ can be represented in the form $x = x_+ - x_-$, $x_+, x_- \in \mathrm{P}_+$. Such a representation is vital for x to be interpreted as a cash flow as, by definition, cash flow is the net of cash inflows and outflows.

We close with a discussion of two open problems.

1. The paper mainly exploits SPO (rather than PO) due to its simple representation and nice interpretation. Therefore, it would be desirable to provide its separate axiomatic characterization.
2. A slightly more intuitive relation than SPO can be introduced as follows. Given a nonempty set $\mathcal{F} \subseteq \mathcal{NPV}$, define the preorder $\gtrsim$ over P by

$$x \gtrsim y \iff \operatorname{sgn} F(x) \geq \operatorname{sgn} F(y) \text{ for all } F \in \mathcal{F}.$$

The relation $\gtrsim$ seems to be a little bit more relevant for profitability measurement purposes than the SPO $\succeq$ induced by $\mathcal{F}$. First, $x \succ 0 \succ -x$ (where $\succ$ is the asymmetric part of $\gtrsim$) for every $x \in \mathrm{P}_{++}$, whereas for $\succeq$ we have a counterintuitive $0 \succeq x$ for all $x \in \mathrm{P}$. Second, in contrast to $\succeq$, $\gtrsim$ satisfies the skew symmetry condition, $x \gtrsim y \Rightarrow -y \gtrsim -x$. Various types of projects imply the existence of two sides, whose cash flows differ by sign (e.g., the borrower and lender sides of a loan). The skew symmetry condition asserts that the two sides rank projects' profitabilities in the reversed order. Though the preorders $\gtrsim$ and $\succeq$ are "essentially the same" from a practical viewpoint (the closure of an upper contour set of $\gtrsim$ coincides with the corresponding upper contour set of $\succeq$), it would be desirable to present an axiomatic foundation for $\gtrsim$ and exploit it to study completeness on a predetermined subset of projects and profitability metrics in the manner of sections 3 and 4.

6. Appendix. Auxiliary results and proofs

Lemma 11.

$F: \mathrm{P} \to \mathrm{R}$ is a net present value functional if and only if representation (1) holds for some $\alpha \in \mathcal{A}$.

Proof.

Let $F \in \mathcal{NPV}$. A routine argument shows that every additive and positive functional is homogeneous and continuous. Therefore, there exists a function of bounded variation $\alpha: \mathrm{R}_+ \to \mathrm{R}$ such that $F(x) = \alpha(0)x(0) + \int_0^\infty \alpha \mathrm{d}x$ (Monteiro et al., 2018, Theorem 8.2.8). The function α is nonnegative: indeed, for any $t \in \mathrm{R}_+$, we have $1_t \in \mathrm{P}_+$ and, therefore, $\alpha(t) = F(1_t) \geq 0$. α is nonincreasing: for any $t < \tau$, we have $\alpha(t) - \alpha(\tau) = F(1_t) - F(1_\tau) = F(1_t - 1_\tau) \geq 0$ as $1_t - 1_\tau \in \mathrm{P}_+$. Clearly, $\alpha(0) = F(1_0) = 1$, so that $\alpha \in \mathcal{A}$.

Now assume that Eq. (1) holds for some $\alpha \in \mathcal{A}$. Clearly, $F \in \mathrm{P}^*$ and $F(1_0) = 1$, so that we only have to prove that F is positive. Pick $x \in \mathrm{P}_+$ and note that for any $\varepsilon > 0$, there is a step-function $y = \sum_{k=1}^n c_k 1_{t_k} \in \mathrm{P}$, $c_1,\ldots,c_n \in \mathrm{R}$, $0 \leq t_1 < \ldots < t_n$ such that $\|x - y\| < \varepsilon$ (Monteiro et al., 2018, p. 82). The constants $c_1,\ldots,c_n$ can be chosen such that $y \in \mathrm{P}_+$, i.e., $c_1 + \ldots + c_k \geq 0$, $k = 1,\ldots,n$: indeed, the step-function $y_+(t) := \max\{y(t), 0\}$ satisfies $y_+ \in \mathrm{P}_+$ and $\|x - y_+\| < \varepsilon$. As $\alpha \in \mathcal{A}$, we have

$$F(y) = \sum_{k=1}^n c_k \alpha(t_k) = \alpha(t_n)(c_1 + \ldots + c_n) + \sum_{k=1}^{n-1} (\alpha(t_k) - \alpha(t_{k+1}))(c_1 + \ldots + c_k) \geq 0.$$

Since F is continuous, this proves that $F(x) \geq 0$. ■

Proof of Lemma 1.

1°. (NT), (I), and (USC) imply $\lambda x \succeq x$, $\lambda > 0$. This holds for all $x \in \mathrm{P}$ and $\lambda > 0$ if and only if $\lambda x \sim x$.

2°. Assume by way of contradiction that $\succeq$ is lower semicontinuous. By property 1°, $x \sim \lambda x$ for any $x \in \mathrm{P}$ and $\lambda > 0$. Tending $\lambda \to 0$ and using upper and lower semicontinuity, we obtain $x \sim 0$, which contradicts nontriviality of $\succeq$.

3°. By property 1°, $2 \cdot 1_0 \sim 1_0$, whereas $2 \cdot 1_0 > 1_0$.

4°. Since 1_0 is an order unit and $\succeq$ is nontrivial, property 1° and (M) imply $2 \cdot 1_0 \succ -1_0$. Now assume by way of contradiction that $\succeq$ satisfies $x \succ y \Rightarrow x \succ x + y$. Applying this implication to the inequality $2 \cdot 1_0 \succ -1_0$, we arrive to a contradiction: $2 \cdot 1_0 \succ 2 \cdot 1_0 - 1_0 = 1_0$. The remaining statement can be established in a similar fashion.

5°. Assume by way of contradiction that there is $x \in \mathrm{P}$ such that $1_0 \succ x$ and $1_0 \succ -x$, whereas $x \succeq -x$. As $\mathrm{L}_{\succeq}(x)$ is closed under addition and $x, -x \in \mathrm{L}_{\succeq}(x)$, we arrive to a contradiction: $x \succeq x - x = 0 \sim 1_0$.

6º. Straightforward. ■

Proof of Proposition 1.

To show independence of (NT), (M), (USC), (I) we provide four examples of binary relations on P that satisfy three of the conditions while violating the fourth. Pick $F \in \mathcal{NPV}$ and $G \in \mathrm{P}^* \setminus \mathrm{P}_+^\circ$. The binary relation $\succeq$ defined by $x \succeq y \Leftrightarrow \max\{I_{\{F\}^\circ}(x), I_{\{F\}^\circ}(y)\} = 1$ satisfies all the conditions except (NT). The binary relation given by $x \succeq y \Leftrightarrow I_{\{G\}^\circ}(x) \geq I_{\{G\}^\circ}(y)$ meets all the conditions except (M). The binary relation defined by $x \succeq y \Leftrightarrow I_{\{F\}^\circ}(-y) \geq I_{\{F\}^\circ}(-x)$ satisfies all the conditions except (USC). Finally, the binary relation given by $x \succeq y \Leftrightarrow F(x) \geq F(y)$ meets all the conditions except (I).

(a)$\Rightarrow$(b). Let $\succeq$ be a PO. From (USC), (I), and property 1º in Lemma 1 it follows that for any $z \in \mathrm{P}$, $\mathrm{U}_\succeq(z)$ is a closed convex cone. Set $\mathcal{U} = \{(\mathrm{U}_\succeq(z))^\circ \cap \mathcal{NPV}, z \in \mathrm{P}\}$. By (M), $((\mathrm{U}_\succeq(z))^\circ \cap \mathcal{NPV})$ is a base for the cone $((\mathrm{U}_\succeq(z))^\circ$. Thus, $((\mathrm{U}_\succeq(z))^\circ \cap \mathcal{NPV})^\circ = (\mathrm{U}_\succeq(z))^{\circ\circ} = \mathrm{U}_\succeq(z)$, where the second equality follows from the bipolar theorem (Aliprantis and Border, 2006, Theorem 5.103). By (NT), $x \succeq y \Leftrightarrow \{z \in \mathrm{P} : x \in \mathrm{U}_\succeq(z)\} \supseteq \{z \in \mathrm{P} : y \in \mathrm{U}_\succeq(z)\} \Leftrightarrow \{\mathrm{K} \in \mathcal{U} : x \in \mathrm{K}^\circ\} \supseteq \{\mathrm{K} \in \mathcal{U} : y \in \mathrm{K}^\circ\} \Leftrightarrow I_{\mathrm{K}^\circ}(x) \geq I_{\mathrm{K}^\circ}(y)$ for all $\mathrm{K} \in \mathcal{U}$. From (I) it follows that the set $\mathrm{L}_\succeq(z) = \{x \in \mathrm{P} : z \succeq x\} = \bigcap_{\mathrm{K} \in \mathcal{U} : z \notin \mathrm{K}^\circ} (\mathrm{P} \setminus \mathrm{K}^\circ)$ is closed under addition.

(b)$\Rightarrow$(a). It is straightforward to verify that the binary relation $\succeq$ defined in part (b) is a PO.

In order to show that elements of the family $\mathcal{U}$ in part (b) can be chosen closed and convex, note that $\succeq$ depends on $\mathrm{K} \in \mathcal{U}$ only through K°. For each $\mathrm{K} \in \mathcal{U}$, set $\overline{\mathrm{K}} := \mathrm{K}^{\circ\circ} \cap \mathcal{NPV}$. We have $\overline{\mathrm{K}}^\circ = (\mathrm{K}^{\circ\circ})^\circ = (\mathrm{K}^\circ)^{\circ\circ} = \mathrm{K}^\circ$, where the first equality follows from the fact that $\overline{\mathrm{K}}$ is a base for the cone $\mathrm{K}^{\circ\circ}$ and the last equality comes from the bipolar theorem. Thus, replacing each $\mathrm{K} \in \mathcal{U}$ with $\overline{\mathrm{K}}$ in the representation produces the same PO. $\overline{\mathrm{K}}$ is closed and convex as the intersection of the closed and convex sets $\mathrm{K}^{\circ\circ}$, P_+°, and $\{F \in \mathrm{P}^* : F(1_0) = 1\}$. ■

Proof of Proposition 2.

(a)$\Rightarrow$(b), (a)$\Rightarrow$(c), (a)$\Rightarrow$(d). Straightforward.

(b)$\Rightarrow$(a). Set $\mathrm{U} := \{z \in \mathrm{P} : z \sim 1_0\}$. Property 1º, (M), and (NT) imply that $\mathrm{U} = \mathrm{U}_\succeq(1_0)$, so that, by (I) and (USC), U is a closed convex cone. Denote $\mathrm{L} := \mathrm{P} \setminus \mathrm{U}$. Since $\succeq$ is nontrivial, $\mathrm{L} \neq \varnothing$. Pick $x, y \in \mathrm{L}$. As $\succeq$ is total, without loss of generality we may assume that $x \succeq y$. Combining this with $x \succeq x$ and using (I), we get $x \succeq x + y$, and, therefore, $x + y \in \mathrm{L}$. This proves that L is an open convex cone. By a separating hyperplane theorem (Aliprantis and Border, 2006, Lemma 5.66, Theorem 5.67), there is a nonzero $F \in \mathrm{P}^*$ such that $F(x) < 0 \leq F(y)$ for all $x \in \mathrm{L}$ and $y \in \mathrm{U}$. Condition (M) implies $\mathrm{P}_+ \subseteq \mathrm{U}$, so that F can be chosen such that $F \in \mathcal{NPV}$. Since $\mathrm{P} = \mathrm{L} \cup \mathrm{U}$, we have $\mathrm{U} = \{F\}^\circ$. For each $z \in \mathrm{L}$, $\mathrm{U}_\succeq(z)$ is a closed convex cone containing U as a proper subset. As U is a closed half-space, $\mathrm{U}_{\succeq}(z) = \mathrm{P}$. Thus, L is an equivalence class w.r.t. $\sim$, so that $x \succeq y \iff I_{\{F\}^\circ}(x) \geq I_{\{F\}^\circ}(y)$.

(c)$\Rightarrow$(a). Reproducing the beginning of the proof "(b)$\Rightarrow$(a)", we get that $\mathrm{U} := \{z \in \mathrm{P} : z \sim 1_0\} = \mathrm{U}_{\succeq}(1_0)$ is a closed convex cone and $\mathrm{L} := \mathrm{P} \setminus \mathrm{U} \neq \varnothing$. Condition (c) implies that L is an open convex cone. The rest of the proof reproduces the corresponding part of that of "(b)$\Rightarrow$(a)".

(d)$\Rightarrow$(a). Set $\mathrm{L} := \{z \in \mathrm{P} : z \sim -1_0\}$ and $\mathrm{U} := \mathrm{U}_{\succ}(-1_0)$. From property 1°, (M), and (NT) it follows that $\mathrm{U} = \mathrm{P} \setminus \mathrm{L}$. Nontriviality of $\succeq$, property 1°, and axiom (I) (resp. condition (d)) imply that L (resp. U) is a nonempty convex cone. By a separating hyperplane theorem, there is a nonzero $F \in \mathrm{P}^*$ such that $F(x) \leq 0 \leq F(y)$ for all $x \in \mathrm{L}$ and $y \in \mathrm{U}$. As $\mathrm{P}_+ \subseteq \mathrm{U}$, F can be chosen such that $F \in \mathcal{NPV}$. Set $\mathrm{H}_+ := \{x \in \mathrm{P} : F(x) > 0\}$ and $\mathrm{H}_- := \{x \in \mathrm{P} : F(x) < 0\}$ and note that $\mathrm{H}_+ \subseteq \mathrm{U}$ and $\mathrm{H}_- \subseteq \mathrm{L}$. For each $x \in \mathrm{H}_+$, $\mathrm{L}_{\succeq}(x)$ is a convex cone containing $\mathrm{L} \cup \{x\}$, i.e., $\mathrm{R}_+ x + \mathrm{L} \subseteq \mathrm{L}_{\succeq}(x)$. As $\mathrm{R}_+ x + \mathrm{L} \supseteq \mathrm{R}_+ x + \mathrm{H}_- = \mathrm{P}$, we get $\mathrm{L}_{\succeq}(x) = \mathrm{P}$. This proves that $\mathrm{H}_+ \subseteq \mathrm{U}_{\succeq}(1_0)$. We have $\{F\}^\circ = \mathrm{cl}(\mathrm{H}_+) \subseteq \mathrm{U}_{\succeq}(1_0) \subseteq \mathrm{U} \subseteq \{F\}^\circ$, where the first inclusion follows from the fact that $\mathrm{U}_{\succeq}(1_0)$ is closed. Thus, $\mathrm{U} = \mathrm{U}_{\succeq}(1_0) = \{F\}^\circ$. As $\mathrm{U}_{\succeq}(1_0) = \{z \in \mathrm{P} : z \sim 1_0\}$, i.e., U is an equivalence class w.r.t. $\sim$, we are done. ■

Lemma 12.

Given a nonempty set $\mathcal{S} \subseteq \mathcal{NPV}$, let $\mathcal{F}$ be the closed (in the weak* topology) convex hull of $\mathcal{S}$ and $\succeq$ be the SPO induced by $\mathcal{F}$. The following conditions are equivalent:

(a) $x \succeq y$;

(b) $\sup_{\lambda \in \mathrm{R}_+} \inf_{F \in \mathcal{S}} F(x - \lambda y) \geq 0$;

(c) $x \in \mathrm{cl}(\mathcal{S}^\circ + (\mathrm{R}_+ y))$.

In particular, if the set $\mathcal{S}^\circ + (\mathrm{R}_+ y)$ is closed (which holds, e.g., if $\mathcal{S}$ is finite[9]), then (a)–(c) are also equivalent to

(d) there exists $\lambda \in \mathrm{R}_+$ such that $F(x - \lambda y) \geq 0$ for all $F \in \mathcal{S}$.

Proof.

Note that condition (b) is equivalent to the following one which we refer to as (b)': for any $\varepsilon > 0$, there exists $\lambda \in \mathrm{R}_+$ such that $F(x - \lambda y) + \varepsilon \geq 0$ for all $F \in \mathcal{S}$. Condition (c) is equivalent to the following one which we refer to as (c)': for any neighborhood of zero O in P, there exists $(\lambda, z) \in \mathrm{R}_+ \times O$ such that $F(x + z - \lambda y) \geq 0$ for all $F \in \mathcal{S}$.

(b)'$\Rightarrow$(c)'. Pick an open neighborhood of zero O in P. Since O is absorbing, $\varepsilon 1_0 \in O$ for some $\varepsilon > 0$. By (b)', there exists $\lambda \in \mathrm{R}_+$ such that $F(x - \lambda y) + \varepsilon \geq 0$ for all $F \in \mathcal{S}$. Thus, (c)' holds with that λ and $z = \varepsilon 1_0$.

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[9] If $\mathcal{S}$ finite, then for any $y \in \mathrm{P}$, the cone $\mathcal{S}^\circ + (\mathrm{R}_+ y)$ is polyhedral (Luan and Yen, 2020, Theorem 2.11) and, therefore, closed.

(c)'$\Rightarrow$(b)'. Pick $\varepsilon > 0$ and put $O_\varepsilon := \varepsilon 1_0 - \mathrm{P}_{++}$. As O_ε is an open neighborhood of zero, condition (c)' implies that there exists $(\lambda, z) \in \mathrm{R}_+ \times O_\varepsilon$ such that $F(x+z-\lambda y) \geq 0$ for all $F \in \mathcal{S}$. Note that $s \in \mathrm{P}_{++}$ if and only if $F(s) > 0$ for all $F \in \mathcal{NPV}$ (Aliprantis and Tourky, 2007, Lemma 2.17). As $\varepsilon 1_0 - z \in \mathrm{P}_{++}$, we have $\varepsilon = F(\varepsilon 1_0) > F(z)$ for all $F \in \mathcal{NPV}$. Thus, $F(x-\lambda y) + \varepsilon > F(x+z-\lambda y) \geq 0$ for all $F \in \mathcal{S}$.

(a)$\Leftrightarrow$(c). Since $1_0 \in \mathrm{P}_{++}$, the set $\mathcal{NPV}$ is compact (Jameson, 1970, Theorem 3.8.6) and, therefore, so is $\mathcal{F}$. $\mathcal{F}$ constitutes a compact base for the cone $\mathrm{R}_+\mathcal{F}$ generated by $\mathcal{F}$, so that $\mathrm{R}_+\mathcal{F}$ is closed (Jameson, 1970, Theorem 3.8.3). Thus, $\mathrm{R}_+\mathcal{F}$ is the closed convex conical hull of $\mathcal{S}$ (i.e., the smallest closed convex cone containing $\mathcal{S}$) and $\mathcal{S}^{\circ\circ} = \mathrm{R}_+\mathcal{F}$ by the bipolar theorem. We have

$$\mathrm{U}_{\succeq}(y) = (\mathcal{F} \cap \{y\}^\circ)^\circ = ((\mathrm{R}_+\mathcal{F}) \cap (\mathrm{R}_+ y)^\circ)^\circ = (\mathcal{S}^{\circ\circ} \cap (\mathrm{R}_+ y)^\circ)^\circ = (\mathcal{S}^\circ + (\mathrm{R}_+ y))^{\circ\circ} = \mathrm{cl}(\mathcal{S}^\circ + (\mathrm{R}_+ y)),$$

where the first equality comes from the definition of the SPO induced by $\mathcal{F}$, whereas the remaining equalities follow from the properties of the duality operation (e.g., see Messerschmidt, 2015, Lemma 2.1) and the fact that the initial and the weak topologies on P have the same collection of closed convex sets (Aliprantis and Border, 2006, Theorem 5.98).

(c)$\Leftrightarrow$(d). Trivial. ■

Proof of Lemma 2.

Assume that $\succeq$ is Q-complete. To show that $\sqsupseteq$ is total, pick $\mathrm{K}, \mathrm{L} \in \mathcal{U}$ and assume by way of contradiction that neither $\mathrm{K} \sqsupseteq \mathrm{L}$ nor $\mathrm{L} \sqsupseteq \mathrm{K}$, i.e., there are $x, y \in \mathrm{Q}$ such that $x \in \mathrm{L}^\circ$, $x \notin \mathrm{K}^\circ$, $y \in \mathrm{K}^\circ$, and $y \notin \mathrm{L}^\circ$. This implies that x and y are incomparable w.r.t. $\succeq$, which is a contradiction. The same argument works in the other direction. ■

Proof of Lemma 3.

(a)$\Rightarrow$(b). Trivial.

(b)$\Rightarrow$(c). Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_1'. By condition (b) and Lemma 2, $\succcurlyeq$ is total, so that it is sufficient to verify that for any $F, G \in \mathcal{F}$, $F \succcurlyeq G \;\Rightarrow\; F \succcurlyeq_1 G$. Pick $F, G \in \mathcal{F}$ and denote by α and β the discount functions associated with F and G. Without loss of generality, we may assume that $F \succcurlyeq G$. If $\beta = \chi$, then trivially $\alpha \geq \beta$. Otherwise, pick $t \in \mathrm{supp}\{\beta\} \setminus \{0\}$ and set $x = -1_0 + (1/\beta(t))1_t$. Then $x \in \mathrm{Q}_1'$ and $G(x) = 0$, so that by the definition of $\succcurlyeq$ we have $0 \leq F(x) = -1 + (1/\beta(t))\alpha(t)$ as desired.

(c)$\Rightarrow$(a). Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_1. In view of Lemma 2, it is sufficient to show that for any $F, G \in \mathcal{F}$, $F \succcurlyeq_1 G \;\Rightarrow\; F \succcurlyeq G$. Pick $F, G \in \mathcal{F}$ and denote by α and β the discount functions associated with F and G. Assume that $F \succcurlyeq_1 G$ and pick $x \in \mathrm{Q}_1$ such that $G(x) \geq 0$. We have to show that $F(x) \geq 0$. Since x is nondecreasing and $\alpha \geq \beta$, we have $F(x) = x(0) + \int_0^\infty \alpha \mathrm{d}x \geq x(0) + \int_0^\infty \beta \mathrm{d}x = G(x) \geq 0$. ■

Proof of Lemma 4.

(a)$\Rightarrow$(b). Trivial.

(b)$\Rightarrow$(c). Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_2'. By condition (b) and Lemma 2, $\succcurlyeq$ is total, so that it is sufficient to verify that for any $F,G\in\mathcal{F}$, $F\succcurlyeq G \;\Rightarrow\; F\succcurlyeq_2 G$. Pick $F,G\in\mathcal{F}$ and denote by α and β the discount functions associated with F and G.

First, we show that $\operatorname{supp}\{\alpha\}=\operatorname{supp}\{\beta\}$. Pick $\tau>0$ and assume by way of contradiction that $\alpha(\tau)=0$, while $\beta(\tau)>0$. Put $x=-1_0+(1/\beta(\tau))1_\tau$ and $y=-1_\tau+c1_t$, where $\tau<t$ and $c\in(0,\beta(\tau)/\beta(t))$ (with the convention $\beta(\tau)/0=+\infty$). Then $x,y\in\mathrm{Q}_2'$, $F(x)<0$, $F(y)=0$, while $G(x)=0$, $G(y)<0$, so that x and y are incomparable, which is a contradiction. This proves that $\operatorname{supp}\{\beta\}\subseteq\operatorname{supp}\{\alpha\}$. The reverse inclusion can be shown by a similar argument.

If $\operatorname{supp}\{\beta\}=\{0\}$, then $\mathcal{F}$ is a singleton and (c) trivially holds. Thus, it remains to consider the case $\operatorname{supp}\{\beta\}\neq\{0\}$. Without loss of generality, we may assume that $F\succcurlyeq G$. Pick $0\le t<\tau\in\operatorname{supp}\{\beta\}$ and set $x=-1_t+(\beta(t)/\beta(\tau))1_\tau$. Then $x\in\mathrm{Q}_2'$ and $G(x)=0$, so that by the definition of $\succcurlyeq$ we have $0\le F(x)=-\alpha(t)+(\beta(t)/\beta(\tau))\alpha(\tau)$ as desired.

(c)$\Rightarrow$(a). Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_2. In view of Lemma 2, it is sufficient to verify that for any $F,G\in\mathcal{F}$, $F\succcurlyeq_2 G \;\Rightarrow\; F\succcurlyeq G$. Pick $F,G\in\mathcal{F}$ and let α and β be the discount functions associated with F and G. Assume that $F\succcurlyeq_2 G$ and pick $x\in\mathrm{Q}_2$ such that $G(x)\ge 0$. We have to show that $F(x)\ge 0$. If $x\in\mathrm{Q}_1$, the result follows from the fact that $\succcurlyeq_2\subset\succcurlyeq_1$ and Lemma 3. Now assume that $x\in\mathrm{Q}_2\setminus\mathrm{Q}_1$, i.e., $x(0)\le 0$ and there is $\tau\in\mathrm{R}_{++}$ such that x is nonincreasing (resp. nondecreasing) on $[0,\tau)$ (resp. $[\tau,+\infty)$). If $\tau\notin\operatorname{supp}\{\beta\}$, the inequality $G(x)\ge 0$ implies that the restriction of x to $\operatorname{supp}\{\beta\}$ is identically 0 and, as $\operatorname{supp}\{\alpha\}=\operatorname{supp}\{\beta\}$, we have $F(x)=G(x)=0$. Now assume that $\tau\in\operatorname{supp}\{\beta\}$ and set $\Delta x(\tau):=x(\tau)-x(\tau-)$, $\tilde{x}:=x-\Delta x(\tau)1_\tau$. We have

$$F(x)=x(0)+\int_0^\infty \alpha\mathrm{d}x = x(0)+\int_0^\infty \alpha\mathrm{d}(\tilde{x}+\Delta x(\tau)1_\tau) = x(0)+\int_0^\tau \alpha\mathrm{d}\tilde{x}+\int_\tau^\infty \alpha\mathrm{d}\tilde{x}+\alpha(\tau)\Delta x(\tau)$$

$$=x(0)+\frac{\alpha(\tau)}{\beta(\tau)}\left(\int_0^\tau \frac{\beta(\tau)}{\alpha(\tau)}\alpha\mathrm{d}\tilde{x}+\int_\tau^\infty \frac{\beta(\tau)}{\alpha(\tau)}\alpha\mathrm{d}\tilde{x}+\beta(\tau)\Delta x(\tau)\right)$$

$$\ge x(0)+\frac{\alpha(\tau)}{\beta(\tau)}\left(\int_0^\tau \beta\mathrm{d}\tilde{x}+\int_\tau^\infty \beta\mathrm{d}\tilde{x}+\beta(\tau)\Delta x(\tau)\right)=x(0)+\frac{\alpha(\tau)}{\beta(\tau)}(G(x)-x(0))\ge G(x)\ge 0.$$

Here the first inequality stems from the facts that $t\mapsto\alpha(t)/\beta(t)$ is nondecreasing on $\operatorname{supp}\{\beta\}$ (as $F\succcurlyeq_2 G$), $\tilde{x}$ is nonincreasing on $[0,\tau]$ and nondecreasing on $[\tau,+\infty)$. The second inequality follows from $G(x)-x(0)\ge 0$ (as $G(x)\ge 0$ and $x(0)\le 0$).

(b)$\Rightarrow$(d). Trivial.

(d)$\Rightarrow$(c). The proof of "(b)$\Rightarrow$(c)" remains valid with Q_2' replaced by Q_2'', provided that each NPV functional from $\mathcal{F}$ has positive discount function. ■

Proof of Lemma 5.

(a)$\Rightarrow$(b). Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_3. By condition (a) and Lemma 2, $\succcurlyeq$ is total, so that it is sufficient to verify that for any $F,G\in\mathcal{F}$, $F\succcurlyeq G \Rightarrow F\succcurlyeq_2 G$ & $F\succcurlyeq_3 G$. Pick $F,G\in\mathcal{F}$ and let α and β be the discount functions associated with F and G. Without loss of generality, we may assume that $F\succcurlyeq G$. Since $\mathrm{Q}_2''\subset\mathrm{Q}_3$, from Lemma 4 we conclude that $F\succcurlyeq_2 G$. Pick $0\le t_1<t_2<t_3<t_4$ and set $x=-1_{t_1}+1_{t_2}+a(1_{t_3}-1_{t_4})$, where $a=\dfrac{\beta(t_1)-\beta(t_2)}{\beta(t_3)-\beta(t_4)}>0$. Then $x\in\mathrm{Q}_3$ and $G(x)=0$, so that by the definition of $\succcurlyeq$ we must have $F(x)\ge 0$. The last inequality implies $\dfrac{\alpha(t_1)-\alpha(t_2)}{\beta(t_1)-\beta(t_2)}\le\dfrac{\alpha(t_3)-\alpha(t_4)}{\beta(t_3)-\beta(t_4)}$. Since $\alpha'<0$, $\beta'<0$, we deduce that the function α'/β' is nondecreasing, i.e., $F\succcurlyeq_3 G$.

(b)$\Rightarrow$(a). Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_3. In view of Lemma 2, it is sufficient to verify that for any $F,G\in\mathcal{F}$, $F\succcurlyeq_2 G$ & $F\succcurlyeq_3 G \Rightarrow F\succcurlyeq G$. Pick $F,G\in\mathcal{F}$ and let α and β be the discount functions associated with F and G.

Assume that $F\succcurlyeq_2 G$ and $F\succcurlyeq_3 G$ and pick $x\in\mathrm{Q}_3$ such that $G(x)\ge 0$. Let $\tau\in\mathrm{R}_{++}$ be such that x is nonpositive on $[0,\tau)$ and nonnegative on $[\tau,+\infty)$. We have to show that $F(x)\ge 0$. Set $\tilde{x}(t):=\begin{cases}0 & \text{if } t=\tau\\ x(t) & \text{otherwise}\end{cases}$. Using integration by parts and the substitution theorem (Monteiro et al., 2018, Theorem 6.4.2, Corollary 6.6.2), we have

$$F(x)=x(0)+\int_0^\infty \alpha\mathrm{d}x = x(0)+\int_0^\infty \alpha\mathrm{d}\tilde{x}=\alpha(+\infty)\tilde{x}(+\infty)-\int_0^\infty \tilde{x}\mathrm{d}\alpha=$$

$$=\alpha(+\infty)\tilde{x}(+\infty)-\int_0^\infty \tilde{x}(t)\alpha'(t)\mathrm{d}t=\alpha(+\infty)\tilde{x}(+\infty)-\int_0^\tau \tilde{x}(t)\alpha'(t)\mathrm{d}t-\int_\tau^\infty \tilde{x}(t)\alpha'(t)\mathrm{d}t$$

$$\ge\frac{\alpha(\tau)}{\beta(\tau)}\beta(+\infty)\tilde{x}(+\infty)-\int_0^\tau \tilde{x}(t)\frac{\alpha'(t)}{\beta'(t)}\beta'(t)\mathrm{d}t-\int_\tau^\infty \tilde{x}(t)\frac{\alpha'(t)}{\beta'(t)}\beta'(t)\mathrm{d}t$$

$$\ge\frac{\alpha'(\tau)}{\beta'(\tau)}\beta(+\infty)\tilde{x}(+\infty)-\int_0^\tau \tilde{x}(t)\frac{\alpha'(\tau)}{\beta'(\tau)}\beta'(t)\mathrm{d}t-\int_\tau^\infty \tilde{x}(t)\frac{\alpha'(\tau)}{\beta'(\tau)}\beta'(t)\mathrm{d}t=\frac{\alpha'(\tau)}{\beta'(\tau)}G(x)\ge 0.$$

Here the first inequality follows from $\alpha(+\infty)\ge(\alpha(\tau)/\beta(\tau))\beta(+\infty)$ (as $F\succcurlyeq_2 G$). The second one stems from the following facts: $\alpha'<0$ and $\beta'<0$ (by assumption), $\alpha(\tau)/\beta(\tau)\ge\alpha'(\tau)/\beta'(\tau)$ (as $F\succcurlyeq_2 G$), $\tilde{x}(+\infty)\ge 0$ (as $x\in\mathrm{Q}_3$), α'/β' is nondecreasing (as $F\succcurlyeq_3 G$), $\tilde{x}$ is nonpositive on $[0,\tau]$ and nonnegative on $[\tau,+\infty)$ (as $x\in\mathrm{Q}_3$). ■

Proof of Lemma 6.

(a)$\Rightarrow$(b). Trivial.

(b)$\Rightarrow$(c). Let $\succeq$ be Q_4'-complete. Set $\alpha(t):=\sup_{F\in\mathcal{F}}F(1_t)$. Clearly, $\alpha\in\mathcal{A}$. Pick $F,G\in\mathcal{F}$ and denote by β and δ the discount functions associated with F and G. Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_4'. By Lemma 2, $\succcurlyeq$ is total. Without loss of generality, we may assume that $F\succcurlyeq G$. As $\mathrm{Q}_1'\subset\mathrm{Q}_4'$, $\succeq$ is Q_1'-complete, and, therefore, $\beta\ge\delta$ pointwise (Lemma 3). Pick $0<t<\tau$ and set $x=-1_0+a1_t+b1_\tau$. By the definition of $\succcurlyeq$ we must have $F(x)\ge 0$ for every a

and b satisfying $G(x)=0$. This condition implies $\det\begin{pmatrix}\beta(t) & \beta(\tau)\\ \delta(t) & \delta(\tau)\end{pmatrix}=0$. Therefore, there is a constant $\gamma\in[0,1]$ such that $\delta=\gamma\beta+(1-\gamma)\chi$. If $\succeq$ is Q_5'-complete, then, as $Q_2'\subset Q_5'$, $\text{supp}\{\beta\}=\text{supp}\{\delta\}$ (Lemma 4), so that $\gamma\in(0,1]$. Therefore, $\mathcal{F}=\{H_\gamma^{(\alpha)},\gamma\in\Gamma\}$ for some $\Gamma\subseteq[0,1]$ (resp. $\Gamma\subseteq(0,1]$), whenever $\succeq$ is Q_4'-complete (resp. Q_5'-complete).

(c)$\Rightarrow$(a). We shall prove only Q_5-completeness, Q_4-completeness can be established in a similar way. Let $\succcurlyeq$ be the preorder over $\mathcal{F}$ induced by Q_5. It is sufficient to prove that for any $\gamma,\sigma\in\Gamma$, $\gamma\geq\sigma \;\Rightarrow\; H_\gamma^{(\alpha)}\succcurlyeq H_\sigma^{(\alpha)}$; this implies that $\succcurlyeq$ is total and, therefore, $\succeq$ is Q_5-complete. Assume that $\gamma\geq\sigma$ and there is $x\in Q_5$ such that $H_\sigma^{(\alpha)}(x)=x(0)+\sigma(F^{(\alpha)}(x)-x(0))\geq 0$. Since $x(0)\leq 0$ (as $x\in Q_5$) and $\sigma>0$ (as $\Gamma\subseteq(0,1]$), this implies $F^{(\alpha)}(x)-x(0)\geq 0$. Thus, $H_\gamma^{(\alpha)}(x)=x(0)+\gamma(F^{(\alpha)}(x)-x(0))\geq x(0)+\sigma(F^{(\alpha)}(x)-x(0))=H_\sigma^{(\alpha)}(x)\geq 0$. ■

Proof of Lemma 7.

Let $\succeq$ be Q_G-complete. Assume that there are $F_1,F_2\in\mathcal{F}$ such that each of which is linearly independent with G (otherwise, the statement holds trivially). Let $\succcurlyeq$ be the total preorder over $\mathcal{F}$ induced by Q_G. Without loss of generality, we may assume that $F_2\succcurlyeq F_1$.

Pick $x\in P$ such that $G(x)=F_1(x)=0$. As F_1 is linearly independent with G, there is $y^*\in Q_G$ such that $F_1(y^*)\geq 0$. For any $\lambda>0$, we have $G(\lambda y^*\pm x)<0$, $F_1(\lambda y^*\pm x)\geq 0$, and, therefore, $F_2(\lambda y^*\pm x)\geq 0$. Since the last inequality holds for all $\lambda>0$, we conclude that $F_2(x)=0$. Thus, the intersection of the kernels of G and F_1 lies in the kernel of F_2, so that $F_2=aG+bF_1$ for some scalars a and b (Aliprantis and Border, 2006, Lemma 5.91).

Now assume that $G\in\mathcal{NPV}$. As $F_1,F_2\in\mathcal{NPV}$, we have $a+b=1$. Setting $H:=F_1-G$, we get $F_2=G+bH$. As $y^*\in Q_G$ and $F_1(y^*)\geq 0$, in order to satisfy $F_2\succcurlyeq F_1$, we must have $F_2(y^*)\geq 0$. This implies $b>0$, so that $\mathcal{F}=\{G+bH, b\in B\}$ for some $B\subseteq R_+$. Recall that $V(z)>0$ for all nonzero $V\in P_+^\circ$ and $z\in P_{++}$ (Aliprantis and Tourky, 2007, Lemma 2.17). As $H(1_0)=0$, we get $H\notin P_+^\circ$ and, therefore, there is $x\in P_+$ such that $H(x)<0$. This shows that B is bounded. Set $F:=G+(\sup B)H$. Since the set $\mathcal{NPV}$ is closed, $F\in\mathcal{NPV}$ and the result follows.

Now assume that $G\in\mathcal{NPV}$ and $\succeq$ is Q_G'-complete. Since $\succeq$ is also Q_G'-complete, we have to show that $G\notin\mathcal{F}$ provided that $\mathcal{F}$ is not a singleton. Assume by way of contradiction that $G\in\mathcal{F}$ and there is $\widetilde{G}\in\mathcal{F}$ such that $\widetilde{G}\neq G$. Pick $x\in P$ (resp. $y\in P$) such that $G(x)<0$, $\widetilde{G}(x)\geq 0$ (resp. $G(y)=0$, $\widetilde{G}(y)<0$), then it is not true that $G\succcurlyeq\widetilde{G}$ (resp. $\widetilde{G}\succcurlyeq G$), a contradiction with totality of $\succcurlyeq$.

To prove the converse assume that $\mathcal{F}=\{wF+(1-w)G, w\in W\}$, $F\in\mathcal{NPV}$, $W\subseteq[0,1]$ (resp. $W\subseteq(0,1]$) represents $\succeq$. A minor modification of the proof "(c)$\Rightarrow$(a)" in Lemma 6 shows that for any $w_1,w_2\in W$, $w_1\geq w_2 \;\Rightarrow\; w_1F+(1-w_1)G\succcurlyeq w_2F+(1-w_2)G$, where $\succcurlyeq$ is the preorder over $\mathcal{F}$ induced by Q_G (resp. Q'_G). This proves that $\succcurlyeq$ is total and, therefore, $\succeq$ is Q_G-complete (resp. Q'_G-complete). ■

Proof of Lemma 8.

Denote by $\succeq$ the SPO with a representation $\mathcal{F}$.

(a). Let $\succeq'$ be an M-consistent SPO and let $\mathcal{F}'$ be a representation of $\succeq'$. By construction, for any $x, y \in Q$, $M(x) \geq M(y) \Rightarrow x \succeq y$. On the other hand, $\mathcal{F}' \subseteq \mathcal{F}$, so that $x \succeq y \Rightarrow x \succeq' y \Rightarrow M(x) \geq M(y)$. Thus, $\succeq$ is the least M-consistent SPO.

(b). As $\succeq$ is M-consistent and, hence, Q-complete, $\succcurlyeq$ is total. Let $\succcurlyeq_1$ be the preorder over $\mathcal{F}$ induced by D. By construction, $\succcurlyeq \subseteq \succcurlyeq_1$. On the other hand, as $Q \subseteq D$, we have $\succcurlyeq_1 \subseteq \succcurlyeq$. Thus, $\succcurlyeq_1 = \succcurlyeq$ and $\succcurlyeq_1$ is total. This proves that $\succcurlyeq_1$ is D-complete.

We have only to show that if C is a proper superset of D, then $\succeq$ is not C-complete. As C is a proper superset of D, there are $F, G \in \mathcal{F}$ and $x \notin C \setminus D$ such that $F \succcurlyeq G$, $F(x) < 0$, and $G(x) \geq 0$. The last two inequalities show that it is not true that $F \succcurlyeq_2 G$. As $\succcurlyeq$ is antisymmetric, we also have $F \succ G$. On the other hand, as $Q \subseteq C$, we get $\succcurlyeq_2 \subseteq \succcurlyeq$. Since $F \succ G$ (as $\succcurlyeq$ is antisymmetric), this proves that it is not true that $G \succcurlyeq_2 F$. This shows that F and G are incomparable with respect to $\succcurlyeq_2$ and, therefore, $\succeq$ is not C-complete. ■

Proof of Proposition 3.

(b). Let $\succeq$ be an SPO with a representation $\mathcal{F}$.

Assume that $\succeq$ is $RR^{(A)}$-consistent. Pick $F \in \mathcal{F}$ and denote by α the discount function associated with F. Pick $0 < t < \tau$ and set $x_\lambda := -1_0 + (1/\alpha_\lambda(t))1_t$ and $y_\lambda := -1_t + (\alpha_\lambda(t)/\alpha_\lambda(\tau))1_\tau$, $\lambda \in R_+$. Then $x_\lambda, y_\lambda \in Q''_2$ and $RR^{(A)}(x_\lambda) = \lambda = RR^{(A)}(y_\lambda)$. Since $\succeq$ is $RR^{(A)}$-consistent, we must have $I_{\{F\}^\circ}(x_\lambda) = I_{\{F\}^\circ}(y_\lambda)$. The last equality holds for any $\lambda \in R_+$ if and only if there is $\lambda^* \in R_+$ such that $\alpha(t) = \alpha_{\lambda^*}(t)$ and $\alpha(\tau) = \alpha_{\lambda^*}(\tau)$. Therefore, $\mathcal{F} = \{F_\lambda^{(A)}, \lambda \in \Lambda\}$, $\Lambda \subseteq R_+$. Λ is dense in R_+ (if $R_+ \setminus \Lambda$ contained a proper interval, than it would contradict $RR^{(A)}$-consistency).

To prove the converse assume that $\mathcal{F} = \{F_\lambda^{(A)}, \lambda \in \Lambda\}$, where Λ is dense in R_+. Clearly, if $x \in Q''_2$, then $\{F \in \mathcal{F} : F(x) \geq 0\} = \{F_\lambda^{(A)}, \lambda \in [0, RR^{(A)}(x)] \cap \Lambda\}$. Therefore, if $x, y \in Q''_2$, then $x \succeq y \Leftrightarrow RR^{(A)}(x) \geq RR^{(A)}(y)$.

(a), (c). These follow from part (b).

(d). The total preorder $\succcurlyeq$ over $\mathcal{F}^{(A)}$ induced by Q''_2 is given by $F_\lambda^{(A)} \succcurlyeq F_{\lambda'}^{(A)} \Leftrightarrow \lambda \leq \lambda'$. Clearly, $\succcurlyeq$ is antisymmetric, so that we can use representation (5) for the natural domain. From (5) it follows that $x \in D^{(A)}$ if and only if for any $0 \leq \lambda \leq \lambda'$, $g_x^{(A)}(\lambda') \geq 0 \Rightarrow g_x^{(A)}(\lambda) \geq 0$. That is, the natural domain of $RR^{(A)}$ consists of projects $x \in P$ such that $g_x^{(A)}$ is either nonnegative, or negative, or there is $\lambda \in R_+$ such that $g_x^{(A)}$ is nonnegative on $[0, \lambda]$ and negative on $(\lambda, +\infty)$.

(e). Let $\succeq$ be the SPO induced by $\mathcal{F}^{(\mathrm{A})}$. It is straightforward to verify that $\overline{RR}^{(\mathrm{A})}$ is a utility representation for the restriction of $\succeq$ to $\mathrm{D}^{(\mathrm{A})}$.

Clearly, $IRR^{(\mathrm{A})}$ is the restriction of $\overline{RR}^{(\mathrm{A})}$ to $\mathrm{Q}^{(\mathrm{A})}$, so that statements (a)–(e) remain valid with $RR^{(\mathrm{A})}$ replaced by $IRR^{(\mathrm{A})}$. ■

Proof of Proposition 4.

(a)$\Rightarrow$(b). Given a D-family A, it is straightforward to verify that the function $M:\mathrm{S}\to\mathrm{R}$ defined by $M(a,b;t,\tau):=RR^{(\mathrm{A})}(1,b/a;t,\tau)$ is a rate of return.

(b)$\Rightarrow$(a). Define the function $\varphi:\mathrm{R}_+\to\mathrm{R}$ by $\varphi(z):=M(1,e^z;0,1)$ and set $\widetilde{M}:=\varphi^{-1}\circ M$. From conditions 1°, 3°, and 4° it follows that $\widetilde{M}$ is well defined and maps S onto R_+. The function φ^{-1} is strictly increasing, so that an SPO is M-consistent if and only if it is $\widetilde{M}$-consistent. Let us show that there is a D-family A such that $\widetilde{M}(a,b;t,\tau)=RR^{(\mathrm{A})}(1,b/a;t,\tau)$.

Let $J:\mathrm{R}_+\times\{(t,\tau)\in\mathrm{R}^2_+ : t<\tau\}\to[1,+\infty)$ be the inverse of $(b;t,\tau)\mapsto\widetilde{M}(1,b;t,\tau)$ with respect to the first argument, that is, $\widetilde{M}(1,b;t,\tau)=\lambda \Leftrightarrow J(\lambda;t,\tau)=b$. By conditions 1°, 3°, and 4°, J is well defined and for any $0\le t<\tau$, $J(\cdot;t,\tau)$ is strictly increasing and onto $[1,+\infty)$.

Condition 2° implies

$$J(\lambda;t,\tau)J(\lambda;\tau,\delta)=J(\lambda;t,\delta). \tag{9}$$

Extend the domain of J to R^3_+ by setting $J(\lambda;t,t):=1$ and $J(\lambda;\tau,t):=1/J(\lambda;t,\tau)$ for $0\le t<\tau$. Then the Sincov functional equation (9) holds for all $(\lambda,t,\tau,\delta)\in\mathrm{R}^4_+$. Its general solution is $J(\lambda;t,\tau)=f(\lambda,t)/f(\lambda,\tau)$ for some function $f:\mathrm{R}^2_+\to\mathrm{R}_{++}$ (Aczél, 1966, p. 223). As $J(\lambda;t,\tau)\ge 1$ for all $0\le t<\tau$, f is nonincreasing in the second argument. Setting $\alpha_\lambda(t):=f(\lambda,t)/f(\lambda,0)$, $\lambda\in\mathrm{R}_+$, we have $\alpha_\lambda(0)=1$. Moreover, the function $\lambda\mapsto\alpha_\lambda(t)/\alpha_\lambda(\tau)=J(\lambda;t,\tau)$ is strictly increasing and onto $[1,+\infty)$. Therefore, α_λ is a discount function and $\mathrm{A}:=\{\alpha_\lambda,\ \lambda\in\mathrm{R}_+\}$ is a D-family. Comparing definitions of J and $RR^{(\mathrm{A})}$, we conclude that $\widetilde{M}(a,b;t,\tau)=RR^{(\mathrm{A})}(1,b/a;t,\tau)$.

(a)$\Rightarrow$(c). Straightforward.

(c)$\Rightarrow$(a). Let $\mathcal{F}$ be a representation of $\succeq$. By Proposition 3 (part (b)), it is sufficient to show that there is a D-family A such that $\mathcal{F}=\{F^{(\mathrm{A})}_\lambda,\ \lambda\in\Lambda\}$, where Λ is a dense subset of R_+.

First, we show that each NPV functional from $\mathcal{F}$ has positive discount function. Assume by way of contradiction that there is $F\in\mathcal{F}$ satisfying $F(1_t)=0$ for some $t>0$. By assumption, $-1_t+a1_\tau\succ-1_t+1_\tau$ for any $\tau>t$ and $a>1$. This implies that there is a functional $G\in\mathcal{F}$ satisfying $G(1_t)>G(1_\tau)$. Set $x=-1_t+1_\tau$ and $y=-1_0+b1_t$, $b\ge1$. Then $x,y\in\mathrm{Q}''_2$, $F(x)=0$, $G(x)<0$, whereas $F(y)<0$, $G(y)\ge0$ for sufficiently large b, so that projects x and y are incomparable, which is a contradiction.

Let α and β be the discount functions associated with some distinct $F,G\in\mathcal{F}$. Set $\gamma(t):=\alpha(t)/\beta(t)$. As $\succeq$ is Q''_2-complete, by Lemma 4, γ is monotone. Without loss of generality, we may assume that γ is nondecreasing. Let us show that it is actually strictly increasing. By contradiction, assume that there are $t_1<t_2$ such that $\gamma(t_1)=\gamma(t_2)$. Since $\alpha\neq\beta$, there is τ such

that $\gamma(\tau) > 1$. Set $x = -1_0 + (1/\alpha(\tau))1_\tau$ and $y = -1_{t_1} + (\beta(t_1)/\beta(t_2))1_{t_2}$. Then $F(x) = F(y) = G(y) = 0$ and $G(x) < 0$, so that $y \succ x$. Note that $F(-1_{t_1} + b1_{t_2}) < 0$ for any $b < \beta(t_1)/\beta(t_2)$, so that the set $\mathrm{U}_\succ(x) \cap \mathrm{Q}_2''$ does not contain a neighborhood of y in Q_2''. As $\mathrm{U}_\succ(x) = \mathrm{P} \setminus \mathrm{L}_\succeq(x)$, this is a contradiction with the lower semicontinuity of the restriction of $\succeq$ to Q_2''. This proves that $\alpha(\tau)/\alpha(t) > \beta(\tau)/\beta(t)$ for all $t < \tau$, in particular, $\alpha(\tau) > \beta(\tau)$ for all $\tau > 0$.

The map $F \mapsto -\ln F(1_1)$ defines a bijection between $\mathcal{F}$ and a subset Λ of R_+. Thus, we can write $\mathcal{F} = \{F_\lambda, \lambda \in \Lambda \subseteq \mathrm{R}_+\}$, where $F_\lambda \in \mathcal{F}$ is the NPV functional satisfying $-\ln F_\lambda(1_1) = \lambda$. In what follows the discount function associated with F_λ is denoted by α_λ. By construction, for any $t < \tau$, the function $\lambda \mapsto \alpha_\lambda(\tau)/\alpha_\lambda(t)$ from Λ into $(0,1]$ is strictly decreasing. The condition $-1_t + a1_\tau \succ -1_t + b1_\tau$, $t < \tau$, $a > b \geq 1$ implies that for any $t < \tau$, the image of the function $\lambda \mapsto \alpha_\lambda(\tau)/\alpha_\lambda(t)$ is dense in $(0,1]$. In particular, Λ is dense in R_+ (as the image of the dense subset $\{\alpha_\lambda(1), \lambda \in \Lambda\}$ of $(0,1]$ under the continuous map $z \mapsto -\ln z$).

Let us prove that $\{\alpha_\lambda, \lambda \in \Lambda\}$ can be complemented to a D-family. For every $\lambda \in \mathrm{R}_+ \setminus \Lambda$ set $\alpha_\lambda(t) := \sup_{c \in \Lambda : c > \lambda} \alpha_c(t)$. By construction, α_λ is a positive discount function. Let us show that $\mathrm{A} := \{\alpha_\lambda, \lambda \in \mathrm{R}_+\}$ is a D-family. Given $0 \leq t < \tau$, define the function $\phi : \mathrm{R}_+ \to \mathrm{R}_{++}$ by $\phi(\lambda) := \alpha_\lambda(\tau)/\alpha_\lambda(t)$.

First, we prove that ϕ is strictly decreasing. As Λ is dense in R_+ and the functions $\lambda \mapsto \alpha_\lambda(t)$ and $\lambda \mapsto \alpha_\lambda(\tau)$ on Λ are decreasing, we have

$$\phi(\lambda) = \frac{\alpha_\lambda(\tau)}{\alpha_\lambda(t)} = \frac{\sup_{c \in \Lambda : c > \lambda} \alpha_c(\tau)}{\sup_{c \in \Lambda : c > \lambda} \alpha_c(t)} = \frac{\lim_{c \to \lambda+, c \in \Lambda} \alpha_c(\tau)}{\lim_{c \to \lambda+, c \in \Lambda} \alpha_c(t)} = \lim_{c \to \lambda+, c \in \Lambda} \frac{\alpha_c(\tau)}{\alpha_c(t)} = \lim_{c \to \lambda+, c \in \Lambda} \phi(c),\ \lambda \in \mathrm{R}_+ \setminus \Lambda.$$

Pick $0 \leq \lambda_1 < \lambda_2$. Since Λ is dense in R_+, there are $\lambda_1', \lambda_2' \in \Lambda$ such that $\lambda_1 < \lambda_1' < \lambda_2' < \lambda_2$. We have $\phi(\lambda_1) = \lim_{c \to \lambda_1+, c \in \Lambda} \phi(c) \geq \phi(\lambda_1')$, $\phi(\lambda_2) = \lim_{c \to \lambda_2+, c \in \Lambda} \phi(c) \leq \phi(\lambda_2')$, and, therefore, $\phi(\lambda_1) \geq \phi(\lambda_1') > \phi(\lambda_2') \geq \phi(\lambda_2)$.

To complete the proof we have to show that ϕ is onto $(0,1]$. Assume by way of contradiction that $\phi(\mathrm{R}_+) \neq (0,1]$. Then, since ϕ is monotone, $(0,1] \setminus \phi(\mathrm{R}_+)$ contains an interval of positive length, which is a contradiction with density of $\phi(\Lambda)$ in $(0,1]$. ■

Proof of Proposition 5.

(a)$\Rightarrow$(b). Trivial.

(b)$\Rightarrow$(a). Let $\mathcal{F}$ be a representation of $\succeq$. In view of Proposition 3 (part (b)), we have to show that $\mathcal{F} = \{F_\lambda^{(\mathrm{E})}, \lambda \in \Lambda\}$, where Λ is a dense subset of R_+. Pick $F \in \mathcal{F}$ and denote by α the discount function associated with F. Note that if $x \in \mathrm{Q}_2''$, then so is $x^{(+\tau)}$.

First, we prove that α is positive. Assume by way of contradiction that $\alpha(\tau) = 0$ for some $\tau > 0$. Consider a project $x = -1_0 + a1_\tau \in \mathrm{Q}_2''$. Then $F(x) < 0$, whereas $F(x^{(+\tau)}) = 0$, a contradiction to stationary.

Pick $t>0$ and consider the project $y=-1_0+c1_t\in \mathrm{Q}_2''$ for some $c\ge 1$. If $c=1/\alpha(t)$, then $F(y)=-1+c\alpha(t)=0$ and, by stationarity, we must have

$$-\alpha(\tau)+(1/\alpha(t))\alpha(t+\tau)=-\alpha(\tau)+c\alpha(t+\tau)=F(-1_\tau+c1_{t+\tau})=F(y^{(+\tau)})\ge 0 \qquad (10)$$

for any $\tau>0$. We consider two cases.

Case 1: $\alpha(t)<1$. Setting $c\in[1,1/\alpha(t))$, a similar argument used to obtain (10) yields $-\alpha(\tau)+c\alpha(t+\tau)<0$. Tending $c\to 1/\alpha(t)-$ in the last inequality and combining the result with (10), we obtain the Cauchy functional equation $\alpha(t+\tau)=\alpha(t)\alpha(\tau)$, $(t,\tau)\in\mathrm{R}_{++}^2$. Its general positive nonincreasing solution is given by $\alpha(t)=e^{-\lambda t}$, $\lambda\in\mathrm{R}_+$ (Aczél, 1966, p. 38).

Case 2: $\alpha(t)=1$. Using (10) with $\tau=t$, we get $\alpha(2t)=1$. Iterating this result, we conclude that $\alpha=1_0$.

Thus, there exists $\Lambda\subseteq\mathrm{R}_+$ such that $\mathcal{F}=\{F_\lambda^{(\mathrm{E})},\lambda\in\Lambda\}$. Pick $a>b\ge 1$. By assumption, there exists $t>0$ such that $-1_0+a1_t\succ -1_0+b1_t$. Therefore, there is $\lambda\in\Lambda$ such that $\lambda\in(t^{-1}\ln b,t^{-1}\ln a]$. This proves that Λ is dense in R_+. ■

Proof of Lemma 9.

Given $\mathcal{S}\in\mathrm{P}^*$ and $z\in\mathrm{P}$, set $\mathcal{S}(z):=\{z\}^\circ\cap\mathcal{S}$.

We shall prove a slightly stronger result: if $x\in\mathcal{R}(\mathcal{F})$, then $\mathrm{U}_{\succeq}(x)=\mathrm{U}_{\succeq'}(x)$. During the proof cl and int are the topological closure and interior operators in $\mathcal{F}$. Clearly, $\mathrm{U}_{\succeq}(x)\subseteq\mathrm{U}_{\succeq'}(x)$. To prove the converse, we have to show that for any $y\in\mathrm{P}$, $\mathcal{F}'(x)\subseteq\mathcal{F}'(y)\;\Rightarrow\;\mathcal{F}(x)\subseteq\mathcal{F}(y)$. Note that $\mathcal{F}'(x)\subseteq\mathcal{F}'(y)$ implies $\mathrm{cl}(\mathcal{F}(x)\cap\mathcal{F}')=\mathrm{cl}(\mathcal{F}'(x))\subseteq\mathrm{cl}(\mathcal{F}'(y))=\mathrm{cl}(\mathcal{F}(y)\cap\mathcal{F}')\subseteq\mathcal{F}(y)$, where the last inclusion follows from the fact that $\mathcal{F}(y)$ is closed in $\mathcal{F}$ (as the intersection of the closed half-space $\{y\}^\circ$ and $\mathcal{F}$). Therefore, it is sufficient to prove that $\mathcal{F}(x)\subseteq\mathrm{cl}(\mathcal{F}(x)\cap\mathcal{F}')$, which readily follows from regularity of x. Indeed, let O be an open (in $\mathcal{F}$) neighborhood of a point from $\mathrm{int}\,\mathcal{F}(x)$. Since $O\cap\mathrm{int}\,\mathcal{F}(x)$ is a nonempty open set and $\mathcal{F}'$ is dense in $\mathcal{F}$, $O\cap\mathrm{int}\,\mathcal{F}(x)$ intersects $\mathcal{F}'$ and, therefore, also intersects $\mathcal{F}(x)\cap\mathcal{F}'$ (as $\mathrm{int}\,\mathcal{F}(x)\subseteq\mathcal{F}(x)$). Thus, every neighborhood of a point from $\mathrm{int}\,\mathcal{F}(x)$ intersects $\mathcal{F}(x)\cap\mathcal{F}'$, so that $\mathrm{int}\,\mathcal{F}(x)\subseteq\mathrm{cl}(\mathcal{F}(x)\cap\mathcal{F}')$. Taking the closure of the both sides, we get $\mathcal{F}(x)=\mathrm{cl}(\mathrm{int}\,\mathcal{F}(x))\subseteq\mathrm{cl}(\mathcal{F}(x)\cap\mathcal{F}')$ due to regularity of x. ■

Lemma 13.

Given a D-family A, the map $h(\lambda):=F_\lambda^{(\mathrm{A})}$ is a homeomorphism between R_+ and $\mathcal{F}^{(\mathrm{A})}$ endowed with the subspace topology.

Proof.

Clearly, h is a bijection. Pick $\lambda^*\in\mathrm{R}_+$.

Consider a convergent sequence $\lambda_n\to\lambda^*$. As for any $x\in\mathrm{P}$ the function $\lambda\mapsto F_\lambda^{(\mathrm{A})}(x)$ is continuous, $F_{\lambda_n}^{(\mathrm{A})}\to F_{\lambda^*}^{(\mathrm{A})}$ pointwise, so that h is continuous at λ^*.

In order to prove that h^{-1} is continuous at $F_{\lambda^*}^{(\mathrm{A})}$, pick $t > 0$ and recall that by the definition of a D-family, the function $\lambda \mapsto \alpha_\lambda(t)$ is a homeomorphism of R_+ onto $(0,1]$. In particular, as its inverse is continuous, for any $\varepsilon > 0$, there is $\delta > 0$ such that $\left|\alpha_\lambda(t) - \alpha_{\lambda^*}(t)\right| < \delta \;\Rightarrow\; \left|\lambda - \lambda^*\right| < \varepsilon$. As $\{F_\lambda^{(\mathrm{A})} : \left|F_\lambda^{(\mathrm{A})}(1_t) - F_{\lambda^*}^{(\mathrm{A})}(1_t)\right| < \delta\} = \{F_\lambda^{(\mathrm{A})} : \left|\alpha_\lambda(t) - \alpha_{\lambda^*}(t)\right| < \delta\}$ is an open neighborhood of $F_{\lambda^*}^{(\mathrm{A})}$ in $\mathcal{F}^{(\mathrm{A})}$, we are done. ■

Proof of Lemma 10.

To simplify the notation, we write g instead of $g_x^{(\mathrm{A})}$ during the proof. Since g is continuous on R_+, $g(0) = x(+\infty) \neq 0$, and $\lim_{\lambda \to +\infty} g(\lambda) = x(0) \neq 0$, there is a compact interval $\mathrm{I} \subset \mathrm{R}_{++}$ that contains all the roots of g (if any). We claim that g has a finite number of roots. Indeed, assume by way of contradiction that the set $\Lambda := \{\lambda \in \mathrm{I} : g(\lambda) = 0\}$ is infinite. Then, by the Bolzano–Weierstrass theorem, Λ has an accumulation point $\lambda^* \in \mathrm{I}$. Let $\{\lambda_n\} \subset \Lambda \setminus \{\lambda^*\}$ converge to λ^*. Then $g(\lambda^*) = \lim_{n\to\infty} g(\lambda_n) = 0$ and $g'(\lambda^*) = \lim_{n\to\infty} (g(\lambda_n) - g(\lambda^*))/(\lambda_n - \lambda^*) = 0$, which contradicts condition (b). Therefore, $\mathrm{A}(x)$ is a union of finitely many pairwise disjoint closed intervals. By condition (b), all the intervals are proper, so that $\mathrm{A}(x)$ is regular closed. ■

Proof of Proposition 6.

(b). Let $\succeq$ be an SPO with a representation $\mathcal{F}$.

Assume that $\succeq$ is $RDPP^{(\alpha)}$-consistent. Pick $F \in \mathcal{F}$ and denote by β be the discount function associated with F. Pick $\tau > 0$.

Claim 1: if $\alpha(\tau) = 0$, then $\beta(\tau) = 0$. Since $\alpha \neq \chi$, there is $t \in (0,\tau)$ such that $\alpha(t) > 0$. Set $x = -1_0 + (1/\alpha(t))1_t$. Then $DPP^{(\alpha)}(x) = DPP^{(\alpha)}(x + c1_\tau) = t$ for any $c \in \mathrm{R}$. Thus, $x \sim x + c1_\tau$ and we must have $I_{\{F\}^\circ}(x) = I_{\{F\}^\circ}(x + c1_\tau)$. The last equality holds for all $c \in \mathrm{R}$ if and only if $\beta(\tau) = 0$.

Claim 2: if $\beta(\tau) > 0$, then $\beta(\tau) \geq \alpha(\tau)$. Indeed, assume by way of contradiction that $\alpha(\tau) > \beta(\tau)$ and consider the projects $x = -1_0 + (1/\alpha(\tau))1_\tau$, $y = -1_0 + (1/\beta(\tau))1_\tau$. Then $DPP^{(\alpha)}(x) = DPP^{(\alpha)}(y) = \tau$, while $F(x) < 0$ and $F(y) = 0$ so that it is not true that $x \sim y$; a contradiction with $RDPP^{(\alpha)}$-consistency.

Claim 3: if $\beta(\tau) > 0$, then there is $\lambda \geq 1$ such that $\beta(t) = \lambda\alpha(t)$ for all $t \in (0,\tau]$. Indeed, pick $t \in (0,\tau)$. By claims 1 and 2, $\beta(\tau) \geq \alpha(\tau) > 0$ and $\beta(t) \geq \alpha(t) > 0$. Consider the projects $x = -1_0 + (1/\alpha(\tau))1_\tau$, $y = -1_0 + (1/\alpha(t) - \varepsilon)1_t + \varepsilon\,\alpha(t)/\alpha(\tau)1_\tau$. Then $DPP^{(\alpha)}(x) = DPP^{(\alpha)}(y) = \tau$ for any $\varepsilon > 0$. Thus, $x \sim y$ and we must have $I_{\{F\}^\circ}(x) = I_{\{F\}^\circ}(y)$. In view of claim 2, $I_{\{F\}^\circ}(x) = 1$. The equality $I_{\{F\}^\circ}(y) = 1$ holds for all $\varepsilon > 0$ if and only if $\beta(t)/\alpha(t) \leq \beta(\tau)/\alpha(\tau)$. By considering the projects $x' = -1_0 + (1/\alpha(t))1_t$ and $y' = -1_0 + (1/\alpha(t) + \varepsilon)1_t - \varepsilon\,\alpha(t)/\alpha(\tau)1_\tau$ with $\varepsilon > 0$, in the same manner we arrive to $\beta(t)/\alpha(t) \geq \beta(\tau)/\alpha(\tau)$.

Claim 4: if $\beta(\tau)=\lambda\alpha(\tau)>0$, $\lambda>1$, then $\beta(t)=\lambda\alpha(t)$ for any $t\in\mathrm{R}_{++}$. Pick $t\in\mathrm{R}_{++}$. If $t\in(0,\tau]$, the statement follows from claim 3. Now assume that $t>\tau$. If $\alpha(t)=0$, the statement follows from claim 1. If $\alpha(t)>0$, then, by claim 3, either $\beta(t)=\lambda\alpha(t)$ or $\beta(t)=0$. Assume that $\beta(t)=0$ and consider the projects $x=-1_0+(1/\beta(\tau))1_\tau+c1_t$ and $y=-1_0+c1_t$. Then $DPP^{(\alpha)}(x)=DPP^{(\alpha)}(y)=t$ for sufficiently large c, whereas $F(x)=0$ and $F(y)<0$ for any c; a contradiction with $RDPP^{(\alpha)}$-consistency. This proves that $\beta(t)=\lambda\alpha(t)$.

Claim 5: if $\beta(\tau)=\alpha(\tau)>0$, then either $\beta=\alpha$ or there is $c\in[\tau,+\infty)$ such that $\beta(t)=\begin{cases}\alpha(t) \text{ if } t\le c\\ 0 \quad \text{if } t>c\end{cases}$. From claims 2 and 3 it follows that either $\beta=\alpha$, or $\beta(t)=\begin{cases}\alpha(t) \text{ if } t\le c\\ 0 \quad \text{if } t>c\end{cases}$, $c\in[\tau,+\infty)$, or $\beta(t)=\begin{cases}\alpha(t) \text{ if } t<c\\ 0 \quad \text{if } t\ge c\end{cases}$, $c\in(\tau,+\infty)$. Unless $\alpha(t)=0$, the latter opportunity contradicts $RDPP^{(\alpha)}$-consistency: consider projects x and y such that $DPP^{(\alpha)}(x)=DPP^{(\alpha)}(y)=c$ and the function $t\mapsto G_t^{(\alpha)}(x)$ (resp. $t\mapsto G_t^{(\alpha)}(y)$) is continuous at c (resp. $\lim_{t\to c-}G_t^{(\alpha)}(y)<0$), then $F(x)=0$, but $F(y)<0$.

Claim 6: $\beta\ne\chi$. Since $\alpha\ne\chi$, there is τ' such that $\alpha(\tau')>0$. Assume by way of contradiction that $\beta=\chi$ and consider the projects $x(t)=(1_{\tau'}(t)-1)t+c1_{\tau'}(t)$ and $y=-1_0+c1_{\tau'}$. Then $DPP^{(\alpha)}(x)=DPP^{(\alpha)}(y)=\tau'$ for sufficiently large c, whereas $F(x)=0$ and $F(y)<0$ for any c.

Combining claims 1–6 we get that $\mathcal{F}=\{G_\tau^{(\alpha)},\tau\in\mathrm{T}\}\cup\{H_\gamma^{(\alpha)},\gamma\in\Gamma\}$ for some $\mathrm{T}\subseteq\mathrm{R}_{++}$ and $\Gamma\subseteq[1,1/\alpha(0+)]$. Without loss of generality, we may assume that $\mathrm{T}\subseteq\operatorname{int}(\operatorname{supp}\{\alpha\})$. Clearly, T must be dense in $\operatorname{int}(\operatorname{supp}\{\alpha\})$ in order to $\succeq$ be $RDPP^{(\alpha)}$-consistent.

To prove the converse, assume that $\mathcal{F}=\{G_\tau^{(\alpha)},\tau\in\mathrm{T}\}\cup\{H_\gamma^{(\alpha)},\gamma\in\Gamma\}$, where T is a dense subset of $\operatorname{int}(\operatorname{supp}\{\alpha\})$ and $\Gamma\subseteq[1,1/\alpha(0+)]$. Clearly, if $x\in\mathrm{Q}^{(\alpha)}$, then $\{F\in\mathcal{F}:F(x)\ge 0\}=\{G_\tau^{(\alpha)},\tau\in[DPP^{(\alpha)}(x),+\infty)\cap\mathrm{T}\}\cup\{H_\gamma^{(\alpha)},\gamma\in\Gamma\}$. Therefore, if $x,y\in\mathrm{Q}^{(\alpha)}$, then $x\succeq y \iff DPP^{(\alpha)}(x)\le DPP^{(\alpha)}(y) \iff RDPP^{(\alpha)}(x)\ge RDPP^{(\alpha)}(y)$.

(a), (c). These follow from part (b).

(d). Let $\succeq$ be the least $RDPP^{(\alpha)}$-consistent SPO. By part (c) it is induced by $\mathcal{F}=\{G_\tau^{(\alpha)},\tau\in\operatorname{int}(\operatorname{supp}\{\alpha\})\}\cup\{H_\gamma^{(\alpha)},\gamma\in[1,1/\alpha(0+)]\}$. Note that unless $\alpha(0+)=1$, the preorder over $\mathcal{F}$ induced by $\mathrm{Q}^{(\alpha)}$ is not antisymmetric so that we cannot use Lemma 8 to derive the natural domain.

Pick $x\in\mathrm{P}$. We consider three cases.

Case 1: $x\in\mathrm{Q}_-^{(\alpha)}$. In this case, the restriction of $\succeq$ to $\mathrm{Q}^{(\alpha)}\cup\{x\}$ is total.

Case 2: there are $t,\tau\in\mathrm{R}_{++}$ such that $G_t^{(\alpha)}\ge 0$ and $G_\tau^{(\alpha)}<0$. Then one can show that in order to the restriction of $\succeq$ to $\mathrm{Q}^{(\alpha)}\cup\{x\}$ be total we must have $x\in\mathrm{Q}^{(\alpha)}$.

Case 3: $G_\tau^{(\alpha)}(x)\ge 0$ for all $\tau\in\mathrm{R}_{++}$. Then in order to the restriction of $\succeq$ to $\mathrm{Q}^{(\alpha)}\cup\{x\}$ be total we must have $H_\gamma^{(\alpha)}(x)\ge 0$ for any $\gamma\in[1,1/\alpha(0+)]$, or, equivalently, $x\in\mathrm{Q}_+^{(\alpha)}$.

It is straightforward to verify that the restriction of $\succeq$ to $Q_{-}^{(\alpha)} \cup Q^{(\alpha)} \cup Q_{+}^{(\alpha)}$ is total.

(e). Let $\succeq$ be the SPO induced by $\{G_{\tau}^{(\alpha)}, \tau \in \operatorname{int}(\operatorname{supp}\{\alpha\})\} \cup \{H_{\gamma}^{(\alpha)}, \gamma \in [1, 1/\alpha(0+)]\}$. It is routine to verify that $\overline{RDPP}^{(\alpha)}$ is a utility representation for the restriction of $\succeq$ to $D^{(\alpha)}$. ■

Proof of Proposition 7.

(a)$\Rightarrow$(b). First, we show that if $x = -1_0 + a1_{\tau} \in Q_1'$, $y = -1_0 + b1_t \in Q_1'$, $0 < \tau < t$, and $x \succ -1_0$, then $x \succ y$ (during the proof, we refer this implication as (*)). Indeed, if it is not true that $x \succ y$, then, by Q_1'-completeness, $y \succeq x$. Applying the truncation operation to the inequality $y \succeq x$, we arrive to a contradiction, $-1_0 = y_{\leq\tau} \succeq x_{\leq\tau} = x$.

Set $\alpha(t) := \sup_{F \in \mathcal{F}} F(1_t)$. Clearly, $\alpha \in \mathcal{A}$. Let β be the discount function associated with some $G \in \mathcal{F}$. Pick $\tau > 0$ and assume that $\alpha(\tau) > \beta(\tau) > 0$. Setting $x = -1_0 + a1_{\tau}$ with $a \in (1/\alpha(\tau), 1/\beta(\tau))$, we have $x \succ -1_0$ and $G(x) < 0$. Then, for any $t > \tau$ and $b \geq 1$, (*) implies $G(-1_0 + b1_t) < 0$. This proves that $\beta(t) = 0$ for all $t > \tau$. Thus, $\mathcal{F} \subseteq \{F^{(\chi)}, F^{(\alpha)}\} \cup \{G_{t,\lambda}^{(\alpha)}, (t,\lambda) \in (\operatorname{supp}\{\alpha\} \setminus \{0\}) \times [0,1]\}$. By (*), the set $T := \{t :$ there is $\lambda \in [0,1]$ such that $G_{t,\lambda}^{(\alpha)} \in \mathcal{F}\}$ is dense in $\operatorname{supp}\{\alpha\} \setminus \{0\}$.

We claim that $G_{\tau}^{(\alpha)} = G_{\tau,1}^{(\alpha)} \in \mathcal{F}$ for any $\tau \in \operatorname{int}(\operatorname{supp}\{\alpha\})$. Indeed, pick $\tau \in \operatorname{int}(\operatorname{supp}\{\alpha\})$ and consider a project x such that the function $t \mapsto G_t^{(\alpha)}(x)$ is continuous except at τ, $G_{\tau}^{(\alpha)}(x) = 0$, $F^{(\alpha)}(x) < 0$, and $G_t^{(\alpha)}(x) < 0$ for all $t \neq \tau$. Assume by way of contradiction that $G_{\tau}^{(\alpha)} \notin \mathcal{F}$. Then, as $F(x) < 0$ for any $F \in \{F^{(\chi)}, F^{(\alpha)}\} \cup \{G_{t,\lambda}^{(\alpha)}, (t,\lambda) \in R_{++} \times [0,1]\} \setminus \{G_{\tau}^{(\alpha)}\}$, we have $x \sim -1_0$. By stability under truncation, $x_{\leq\tau} \sim (-1_0)_{\leq\tau} = -1_0$. Since T is dense in $\operatorname{supp}\{\alpha\} \setminus \{0\}$, there is $t \in T$ and $\lambda \in [0,1]$ such that $t > \tau$ and $G_{t,\lambda}^{(\alpha)} \in \mathcal{F}$. We have $G_{t,\lambda}^{(\alpha)}(x_{\leq\tau}) = G_{\tau}^{(\alpha)}(x) = 0$, which is a contradiction to $x_{\leq\tau} \sim -1_0$. By considering a project x such that $\operatorname{sgn} x(t) = -\operatorname{sgn} t$ for any $t \in R_{+}$, in a similar manner, we can show that $F^{(\chi)} = G_0^{(\alpha)} \in \mathcal{F}$.

(b)$\Rightarrow$(a). Clearly, $\mathcal{F}$ is totally ordered by $\succcurlyeq_1$, so that $\succeq$ is Q_1'-complete (Lemma 3). To establish stability under truncation, it is sufficient to prove that for any $F \in \mathcal{F}$ and $\tau \in R_{+}$, the NPV functional $x \mapsto F(x_{\leq\tau})$ also belongs to $\mathcal{F}$. Pick $x \in P$ and $\tau \in R_{+}$. We have $F^{(\chi)}(x_{\leq\tau}) = F^{(\chi)}(x)$; $F^{(\alpha)}(x_{\leq\tau}) = G_{\tau}^{(\alpha)}(x)$ if $\tau \in \{0\} \cup \operatorname{int}(\operatorname{supp}\{\alpha\})$; $F^{(\alpha)}(x_{\leq\tau}) = F^{(\alpha)}(x)$ if $\tau \notin \{0\} \cup \operatorname{int}(\operatorname{supp}\{\alpha\})$; $G_{t,\lambda}^{(\alpha)}(x_{\leq\tau}) = G_{t,\lambda}^{(\alpha)}(x)$ if $0 < t \leq \tau$; $G_{t,\lambda}^{(\alpha)}(x_{\leq\tau}) = G_{\tau}^{(\alpha)}(x)$ if $t > \tau$ (note that the conditions $G_{t,\lambda}^{(\alpha)} \in \mathcal{F}$ and $t > \tau$ imply that $\tau \in \{0\} \cup \operatorname{int}(\operatorname{supp}\{\alpha\})$). ■

Proof of Proposition 8.

(b). Let $\succeq$ be an SPO with a representation $\mathcal{F}$.

Assume that $\succeq$ is RI_G^F-consistent. Pick $H \in \mathcal{F}$ and $x \in Q_G^F$. Consider $y \in P$ such that $F(y) = G(y) = 0$. For any real λ, we have $x + \lambda y \in Q_G^F$ and $RI_G^F(x) = RI_G^F(x + \lambda y)$. By RI_G^F-

consistency, we must have $I_{\{H\}^\circ}(x) = I_{\{H\}^\circ}(x+\lambda y)$. The last equality holds for all real λ if and only if $H(y)=0$. This implies that $H = aG + bF$ for some scalars a and b (Aliprantis and Border, 2006, Lemma 5.91). As $H \in \mathcal{F}$, we have $a+b=1$. Therefore, $\mathcal{F} = \{wF + (1-w)G,\ w \in \mathrm{W}\}$, $\mathrm{W} \subseteq \widetilde{\mathrm{W}}$. $\mathrm{W} \cap (0,1)$ is dense in $(0,1)$ (indeed, if $(0,1)\setminus \mathrm{W}$ contained a proper interval, than it would contradict RI_G^F-consistency). Density of $\mathrm{W}\cap(0,1)$ in $(0,1)$ implies that $\mathcal{F}$ and $\{wF + (1-w)G,\ w \in \mathrm{W} \cup (0,1)\}$ represent the same SPO.

To prove the converse, assume that $\mathcal{F} = \{wF + (1-w)G,\ w \in \mathrm{W}\}$, $(0,1) \subseteq \mathrm{W} \subseteq \widetilde{\mathrm{W}}$. If $x \in \mathrm{Q}_G^F$, then $\{x\}^\circ \cap \mathcal{F} = \left\{wF + (1-w)G,\ w \in [1/RI_G^F(x), +\infty) \cap \mathrm{W}\right\}$. Therefore, provided that $x, y \in \mathrm{Q}_G^F$, $x \succeq y \iff RI_G^F(x) \geq RI_G^F(y)$.

(a), (c). These follow from part (b).

(d). Let $\succeq$ be the least RI_G^F-consistent SPO. By part (c) it is induced by $\{w\widetilde{F} + (1-w)\widetilde{G},\ w \in [0,1]\}$. It is straightforward to verify that the restriction of $\succeq$ to D_G^F is total. On the other hand, if $x \notin \mathrm{D}_G^F$, i.e., $\widetilde{F}(x) < 0$ and $\widetilde{G}(x) \geq 0$, then x and $y \in \mathrm{Q}_G^F$ are incomparable (recall that $\widetilde{F}(y) \geq 0$ and $\widetilde{G}(y) < 0$ for all $y \in \mathrm{Q}_G^F$).

(e). Let $\succeq$ be the SPO induced by $\{w\widetilde{F} + (1-w)\widetilde{G},\ w \in [0,1]\}$. It is straightforward to verify that $\overline{RI}_G^F$ is a utility representation for the restriction of $\succeq$ to D_G^F. ■

Lemma 14.

For the function $\pi : \mathrm{Q}_1' \to [1, +\infty)$ defined by $\pi(-1_0 + b1_\tau) := b$, the following statements hold.

(a) π is a profitability metric.

(b) An SPO is π-consistent if and only if it is induced by $\{H_\gamma^{(1_0)},\ \gamma \in \Gamma\}$, where $(0,1) \subseteq \Gamma \subseteq [0,1]$.

(c) The least π-consistent SPO is induced by $\{H_\gamma^{(1_0)},\ \gamma \in [0,1]\}$.

(d) The natural domain of π is $\mathrm{D} = \mathrm{P} \setminus \{x \in \mathrm{P} : x(0) \geq 0, x(+\infty) < 0\}$.

(e) The natural extension of π is the total preorder over D with a utility representation $\overline{\pi} : \mathrm{D} \to \overline{\mathrm{R}}_+$ given by (6).

Proof.

It is sufficient to prove part (b). The rest then follows from Proposition 8 with $G = F^{(\chi)}$ and $H = F^{(1_0)}$.

Let $\succeq$ be a π-consistent SPO and $\mathcal{F} \subseteq \mathcal{NPV}$ represent $\succeq$. Pick $F \in \mathcal{F}$ and denote by α the discount function associated with F. Pick $t, \tau \in \mathrm{R}_{++}$. As $\pi(-1_0 + b1_\tau) = \pi(-1_0 + b1_t)$, $b \geq 1$, by π-consistency, we have $I_{\mathrm{R}_+}(-1 + b\alpha(\tau)) = I_{\{F\}^\circ}(-1_0 + b1_\tau) = I_{\{F\}^\circ}(-1_0 + b1_t) = I_{\mathrm{R}_+}(-1 + b\alpha(t))$. This equality holds for all $b \geq 1$ if and only if $\alpha(\tau) = \alpha(t)$. Therefore, $\mathcal{F} = \{H_\gamma^{(1_0)},\ \gamma \in \Gamma\}$, $\Gamma \subseteq [0,1]$. $\Gamma \cap (0,1)$ is dense in $(0,1)$ (indeed, if $(0,1) \setminus \Gamma$ contained a proper interval, than it would contradict

π-consistency). Density of $\Gamma \cap (0,1)$ in $(0,1)$ implies that $\mathcal{F}$ and $\{H_\gamma^{(1_0)}, \gamma \in \Gamma \cup (0,1)\}$ represent the same SPO.

It is straightforward to verify that the SPO induced by $\{H_\gamma^{(1_0)}, \gamma \in \Gamma\}$, $(0,1) \subseteq \Gamma \subseteq [0,1]$ is π-consistent. ■

Proof of Proposition 9.

Let $\mathcal{F}$ be a representation of $\succeq$.

(a)$\Rightarrow$(d). If $\succeq$ is the NPV criterion induced by $F^{(\chi)}$, then (d) holds with $\alpha = \chi$. Now assume that $\succeq$ is monotone and PI^F-consistent for some $F \in \mathcal{NPV} \setminus \{F^{(\chi)}\}$. From part (b) of Proposition 8 with $G = F^{(\chi)}$ it follows that there are $\alpha \in \mathcal{A} \setminus \{\chi\}$ and $\Gamma \subseteq [0, 1/\alpha(0+)]$ such that $\mathcal{F} = \{H_\gamma^{(\alpha)}, \gamma \in \Gamma\}$. Without loss of generality, α and Γ can be chosen such that $\sup\Gamma = 1$. As $\alpha \neq \chi$, from monotonicity it follows that for any $\gamma \in \Gamma \setminus \{0\}$ and $\varepsilon > 0$, $\Gamma \cap (\gamma - \varepsilon, \gamma) \neq \varnothing$. Thus, Γ is dense in $(0,1)$. This implies that $\mathcal{F}$ and $\{H_\gamma^{(\alpha)}, \gamma \in \Gamma \cup (0,1)\}$ represent the same SPO.

(b)$\Rightarrow$(d). By Lemma 6, there are $\alpha \in \mathcal{A}$ and $\Gamma \subseteq [0,1]$ such that $\mathcal{F} = \{H_\gamma^{(\alpha)}, \gamma \in \Gamma\}$. As it is shown in the proof of the “(a)$\Rightarrow$(d)” part, monotonicity implies that α and Γ can be chosen such that $(0,1) \subseteq \Gamma \subseteq [0,1]$.

(c)$\Rightarrow$(d). Set $\alpha(t) := \sup\limits_{H \in \mathcal{F}} H(1_t)$. If $\alpha = \chi$, then (d) trivially holds with that α. We assume that $\alpha \neq \chi$ in what follows. Denote by F the NPV functional induced by the discount function α.

Let β be the discount function associated with some $G \in \mathcal{F}$. Note that $\alpha \geq \beta$. Pick $\tau > 0$ such that $\alpha(\tau) > 0$. We claim that $\beta(t)/\alpha(t) = \beta(\tau)/\alpha(\tau)$ for any $0 < t < \tau$. If $\beta(t) = 0$ or $\beta(\tau) = 1$ (and, therefore, $\alpha(\tau) = 1$), the statement holds trivially. Assume that $\beta(t) > 0$ and $\beta(\tau) < 1$ and put $x = -1_0 + (1/\beta(t))1_t$ and $y = -1_0 + a1_\tau$, $a \in [1, 1/\beta(\tau))$ (with the convention $1/0 = +\infty$). Then $G(x) = 0$, $G(y) < 0$, and, therefore, by Q_1'-completeness, $x \succ y$. Since $F(x_\gamma) < 0$ for any $\gamma \in (0, \beta(t)/\alpha(t))$, by stability under reduction, we must also have $F(y_\gamma) < 0$. The latter inequality holds for all $a \in [1, 1/\beta(\tau))$ and $\gamma \in (0, \beta(t)/\alpha(t))$ if and only if $\beta(t)/\alpha(t) \leq \beta(\tau)/\alpha(\tau)$. If $\beta(t) \neq 1$, the inverse inequality, $\beta(t)/\alpha(t) \geq \beta(\tau)/\alpha(\tau)$, can be derived in the same manner by considering the projects $x' = -1_0 + (1/\beta(\tau))1_\tau$ and $y' = -1_0 + b1_t$, $b \in [1, 1/\beta(t))$. Finally, if $\beta(t) = 1$, combining the inequalities $\beta(t)/\alpha(t) \leq \beta(\tau)/\alpha(\tau)$ and $\alpha \geq \beta$, we also arrive to $\beta(t)/\alpha(t) = \beta(\tau)/\alpha(\tau)$.

Thus, $\mathcal{F} = \{H_\gamma^{(\alpha)}, \gamma \in \Gamma\}$ for some $\Gamma \subseteq [0,1]$. As it is shown in the proof of the “(a)$\Rightarrow$(d)” part, monotonicity implies that $(0,1) \subseteq \Gamma \subseteq [0,1]$.

(d)$\Rightarrow$(a), (d)$\Rightarrow$(b), (d)$\Rightarrow$(c). Trivial. ■

7. References

Aczél J., 1966. Lectures on functional equations and their applications. New York.

Aliprantis C.D., Border K.C., 2006. Infinite dimensional analysis: a hitchhiker's guide. Springer, Berlin.

Aliprantis C.D., Tourky R., 2007. Cones and duality. American Mathematical Society.

Arrow K.J., Levhari D., 1969. Uniqueness of the internal rate of return with variable life of investment. The Economic Journal. Vol. 79(315). P. 560–566.

Beaves R.G., 1988. Net present value and rate of return: implicit and explicit reinvestment rate assumption. The Engineering Economist. Vol. 33(4). P. 275–302.

Bhandari S.B., 2009. Discounted payback period – some extensions. Journal of Business and Behavioral Sciences. Vol. 21(1). P. 28–38.

Bronshtein E.M., Akhmetova Yu.F., 2004. Set-valued characteristics of investments. Matematicheskoe modelirovanie. Vol. 16(2). P. 87–92. (In Russian).

Bronshtein E.M., Skotnikov D.A., 2007. The limit profitability of financial operations. Journal of Applied and Industrial Mathematics. Vol. 1. P. 165–174.

Brounen D., de Jong A., Koedijk K., 2004. Corporate finance in Europe: confronting theory with practice. Financial Management. Vol. 33(4). P. 71–101.

Cantor D.G., Lippman S.A., 1983. Investment selection with imperfect capital markets. Econometrica. Vol. 51(4). P. 1121–1144.

Chambers C.P., Echenique F., 2018. On multiple discount rates. Econometrica. Vol. 86(4). P. 1325–1346.

Chichilnisky G., 1996. An axiomatic approach to sustainable development. Social Choice and Welfare. Vol. 13. P. 231–257.

Drugeon J.-P., Ha-Huy T., Nguyen T.D.H., 2019. On maximin dynamic programming and the rate of discount. Economic Theory. Vol. 67. P. 703–729.

Ebert J.E.J., Prelec D., 2007. The fragility of time: time-insensitivity and valuation of the near and far future. Management Science. Vol. 53(9). P. 1423–1438.

Fishburn P.C., Rubinstein A., 1982. Time preference. International Economic Review. Vol. 23(3). P. 677–694.

Gordon M.J., 1955. The payoff period and the rate of profit. The Journal of Business. Vol. 28(4). P. 253–260.

Götze U., Northcott D., Schuster P., 2015. Investment appraisal: methods and models. Springer.

Gronchi S., 1986. On investment criteria based on the internal rate of return. Oxford Economic Papers. Vol. 38(1), 174–180.

Hajdasiński M.M., 1993. The payback period as a measure of profitability and liquidity. The Engineering Economist. Vol. 38(3). P. 177–191.

Hartman J.C., Schafrick I.C., 2004. The relevant internal rate of return. The Engineering Economist. Vol. 49(2). P. 139–158.

Harvey C.M., 1986. Value functions for infinite period planning. Management Science. Vol. 32(9). P. 1123–1139.

Hazen G.B., 2003. New perspective on multiple internal rates of return. The Engineering Economist. Vol. 48(1). P. 31–51.

Jameson G., 1970. Ordered linear spaces. Springer-Verlag.

Laibson D., 1997. Golden eggs and hyperbolic discounting. Quarterly Journal of Economics. Vol. 112(2). P. 443–478.

Lin A.Y.S., 1976. The modified internal rate of return and investment criterion. The Engineering Economist. Vol. 21(4). P. 237–247.

Loewenstein G., Prelec D., 1992. Anomalies in intertemporal choice: evidence and an interpretation. The Quarterly Journal of Economics. Vol. 107. P. 573–597.

Luan N.N., Yen N.D., 2020. A representation of generalized convex polyhedra and applications. Optimization. Vol. 69(3). P. 471–492.

Magni C.A., 2010. Average internal rate of return and investment decisions: A new perspective. The Engineering Economist. Vol. 55(2). P. 150−181.

Magni C.A., 2016. Capital depreciation and the underdetermination of rate of return: a unifying perspective. Journal of Mathematical Economics. Vol. 67. P. 54–79.

Messerschmidt M., 2015. Geometric duality theory of cones in dual pairs of vector spaces. Journal of Functional Analysis. Vol. 269(7). P. 2018–2044.

Monteiro G.A., Slavík A., Tvrdý M., 2018. Kurzweil-Stieltjes integral. Theory and applications. World Scientific.

Neyman A., 2023. Additive valuations of streams of payoffs that satisfy the time value of money principle: A characterization and robust optimization. Theoretical Economics. Vol. 18. P. 303–340.

Norberg R., 1990. Payment measures, interest, and discounting: An axiomatic approach with applications to insurance. Scandinavian Actuarial Journal. Vol. 1. P. 14–33.

Norstrøm C.J., 1972. A sufficient condition for a unique nonnegative internal rate of return. Journal of Financial and Quantitative Analysis. Vol. 7(3). P. 1835−1839.

Promislow S.D., 1994. Axioms for the valuation of payment streams: a topological vector space approach. Scandinavian Actuarial Journal. Vol. 2. P. 151−160.

Promislow S.D., 1997. Classification of usurious loans. In Sherris M. (ed.). Proceedings of the 7th International AFIR Colloquium. Sydney: Institute of Actuaries of Australia. P. 739–759.

Promislow S.D., 2015. Fundamentals of actuarial mathematics. Wiley.

Promislow S.D., Spring D., 1996. Postulates for the internal rate of return of an investment project. Journal of Mathematical Economics. Vol. 26. P. 325–361.

Quah J. K.-H., Strulovici B., 2013. Discounting, values, and decisions. Journal of Political Economy. Vol. 121(5). P. 896–939.

Rader T., 1963. The existence of a utility function to represent preferences. Review of Economic Studies. Vol. 30. P. 229–232.

Rohde K., 2009. Decreasing relative impatience. Journal of Economic Psychology. 2009. Vol. 30(6). P. 831–839.

Ryan P.A., Ryan G.P., 2002. Capital budgeting practices of the Fortune 1000: how have things changed? Journal of Business and Management. Vol. 8(4). P. 355–364.

Sarnat M., Levy H., 1969. The relationship of rules of thumb to the internal rate of return: a restatement and generalization. The Journal of Finance. Vol. 24(3). P. 479–490.

Shull D.M., 1992. Efficient capital project selection through a yield-based capital budgeting technique. The Engineering Economist. Vol. 38(1). P. 1–18.

Spring D., 2012. General balance functions in the theory of interest. arXiv:1208.1479.

Teichroew D., Robichek A.A., Montalbano M., 1965. An analysis of criteria for investment and financial decisions under certainty. Management Science. Vol. 12(3). P. 151−179.

Tossavainen T., 2006. On the zeros of finite sums of exponential functions. Gazette of the Australian Mathematical Society. Vol. 33(1). P. 47–50.

Vilensky P.L., Smolyak S.A., 1999. Internal rate of return of project and its modifications. Audit and Financial Analysis. Vol. 4. P. 203–225.

Weber T.A., 2014. On the (non-)equivalence of IRR and NPV. Journal of Mathematical Economics. Vol. 52. P. 25–39.