How to Measure the Average Rate of Change?

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How to measure the average rate of change?

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Abstract

This paper contributes to the theory of average rate of change (ARC) measurement. The contribution is twofold. First, it relates ARC measurement to intertemporal choice. We show that an ARC of a variable can be identified with a discount rate which makes an economic agent indifferent between the initial and final temporal states of the variable. Furthermore, there is a one-to-one correspondence between ARC measures and one-parameter families of time preferences indexed by a discount rate. Second, we employ an axiomatic approach to generalize the conventional ARC measures (such as the difference quotient and the continuously compounded growth rate) in several directions: to variables with arbitrary connected domains, to not necessarily time-shift invariant dependence on dates, to sets of time points other than an interval, to a benchmark-based evaluation. The generalized ARC measures turn out to correspond to the existing time preference models such as the discounted utility and the relative discounting model of Ok and Masatlioglu [Ok, E.A., Masatlioglu, Y., 2007. A theory of (relative) discounting. J. Econ. Theory 137(1), 214–245].

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1. Introduction

Values represented as rates of change are ubiquitous in human lives. Employees are concerned with the rates of changes of their wages and retirement accounts, managers and firm owners are concerned with the rates of changes of production and revenues, investors are concerned with the rates of changes of their portfolio values, and policy-makers are concerned with the rates of changes of macroeconomic indicators, such as GDP, prices, and employment. The representation of variables in terms of changes is a natural feature of human psychology (Kahneman and Tversky, 1979; Tversky and Kahneman, 1991; Ariely, 2008). In this paper, we revisit the axiomatic foundations of the formulas used to measure the average rate of change (ARC). We explore the limits of applicability of these well-known formulas by generalizing the existing axioms and asking how much these axioms can be relaxed to accommodate some non-standard settings. Central to our discussion is the duality between ARC measurement and time preferences established in our earlier work (Alexeev and Sokolov, 2014).

When asked to measure the ARC of a variable over a time interval, an individual will likely use the formula

$$\frac{s'-s}{t'-t}, \tag{1}$$

where s (resp. s') is the value of an outcome variable at time t (resp. $t'>t$). For example, if an individual's retirement account was valued at \$10,000 in year 1950 and \$100,000 in year 2000, the formula above implies that over the period 1950–2000 the account grew by \$1,800 every year, on average. This amount represents the average absolute change (measured in dollars) in the retirement account per year. If the individual was interested in measuring the average relative change (in percent), he/she would use the formula

$$(s'/s)^{\frac{1}{t'-t}}-1. \tag{2}$$

In our example, this formula implies that during the period of observation the individual's retirement account was growing at 4.7% annually, on average.

While the application of these formulas is straightforward in many standard situations, the question of how to apply these formulas, with appropriate modifications, in more general situations is far from trivial. These more general situations arise, for example, when the outcome variable takes a form other than a scalar. While some of the generalizations we consider may appear less practical than others, our ultimate goal here is to establish the theoretical limits of applicability of the ARC formulas. The following six questions determine the directions along which we attempt to generalize the ARC formulas (1) and (2).

1. What are the analogues of the ARC formulas for an outcome variable taking values in an abstract space?

To illustrate, imagine that the individual is forecasting the ARC of his/her retirement account over some future period. The value of the retirement account at a future date is assumed to be random, and thus s and s' are function-valued rather than scalar-valued. How ARC can be measured in these settings? There is a variety of non-equivalent ways. Imagine that the individual comes up with two solutions to this new problem. Solution A involves resolving risk *ex post* by first computing the ARC (2) for each possible realization of s and s' and then taking the expectation of the ARCs. Solution B involves resolving risk *ex ante* by replacing s and s' with their expected values and then computing the ARC. Employing an axiomatic approach, we show that solution B (as well as calculating the expected log-return) is legitimate, while solution A is not. Note that solution A, the expected rate of return, is how, e.g., the ARC of a stock's price is sometimes evaluated. In general, we show that the relevant modification to the ARC formulas in this case involves replacing s (s') with $u(s)$ ($u(s')$), where u is a real-valued function.

Other examples with non-scalar outcome variables are inflation measurement (the outcome variable – price vector – is vector-valued), stock market return measurement (the outcome variable – vector of stocks prices – is vector-valued), ARC of income inequality (the outcome variable – income distribution – is function-valued), fire spread rate (the outcome variable – shape of burned area – is set-valued). The function u in these examples may represent a price index, a stock market index, an index of income inequality, and a measure of fire spread distance (for a fire spread rate measure in one dimension) or the burned area in km^2 (for a fire spread rate measure in two dimensions), respectively. In general, u can be interpreted as a utility function representing preferences over the domain of the outcome variable. This highlights that information solely on the values of the outcome variable is not sufficient to measure ARC, the functional form of an ARC measure for the same outcome variable may vary with a target consumer group (as is illustrated, e.g., by the existence of a variety of inflation and stock market performance measures).

2. How does a non-stationary counterpart to the ARC formulas look like?

The ARC measures (1) and (2) are stationary in the sense that they are invariant with respect to a time shift. While dependence of an ARC on time via the difference $t'-t$ seems intuitive (since an ARC is usually interpreted as a measure of change of the outcome variable per time unit), the stationarity assumption that this difference induces can be too restrictive in some applications. Moreover, values t and t' may have an interpretation other than time (see the next question, for a discussion); in this case, the stationarity assumption, even if it is well defined, may have no economic content. To illustrate, suppose that the individual is interested in measuring the ARC of the real (inflation-adjusted) value of the retirement account. Since inflation fluctuated greatly during the period of 1950–2000, the ARC will not be invariant to a time shift, e.g., from 1951 to 2001, and hence will not be stationary. We show that a proper modification of the ARC formula (2) is defined implicitly as a solution d of the equation $sg(d,t)=s'g(d,t')$, where $g(d,\cdot)$ can be interpreted as the discount function that corresponds to the discount rate d. That is, the functional form of an ARC measure depends on the structure of intertemporal preferences of a target consumer group.

3. Is there an analogue of the ARC formulas if observations of the outcome variable are ordered by a variable (not necessarily time) whose domain is a linear order?

The ARC formulas assume that observations of the outcome variable are ordered by time. Depending on the application, time can be modeled either as continuous or discrete; this results in formally distinct models. Moreover, in some applications, observations can be ordered by a variable other than time. Examples include: distance, if one measures the ARC of a car value per mile driven; a physical property of an object, if one measures, e.g., the ARC of power consumption of a kettle per thickness of a limescale layer; a number of trials, if one measures the ARC of a reaction time per number of practice trials taken, as is common in learning curve estimations. In signal processing and statistics, the signals are analyzed in time domain as well as in the frequency domain, so both time and frequency can serve as an ordering variable. The domain of such an ordering variable may not be a subset of the reals with the usual order. Suppose that an individual is interested in measuring the ARC of a car value and uses the lexicographically ordered pair (vehicle age, miles driven) to order observations. All these particular cases can be unified by the assumption that the domain of an ordering variable constitutes a linear order and it would be desirable to generalize the ARC formulas along this direction. We show that in fact no such generalization exists unless the domain is order isomorphic to a subset of the reals with the usual order.

4. What is a benchmark-based counterpart to the ARC formulas?

Benchmarking is a universal practice in various fields, such as portfolio management or strategic planning. It involves measuring the outcome variable relative to some reference object. How to incorporate benchmarking in ARC measurement? Suppose the individual is interested in measuring ARC of the retirement account relative to the value of his/her partner's retirement account. The individual considers evaluating ARC of the difference (resp. quotient) of the values of his/her and the partner's accounts. We show that this intuitive approach is actually the only relevant benchmark-based counterpart of the ARC formula (1) (resp. (2)).

5. How to measure ARC over a set of time points other than an interval?

Though ARC is typically measured over a single period of time, a generalization is of interest. In particular, one may want to know how ARCs over disjoint intervals can be aggregated. To illustrate, suppose that the individual in our example holds most of the retirement portfolio in stocks. The individual is aware of the "January effect" in finance (Thaler, 1987), an abnormal calendar pattern in January stock returns relative to other months, and wishes to evaluate the strength of this effect on the portfolio. Hence, he/she needs to measure ARC of the retirement account over the January months only (as well as over the remaining months). The individual considers using formula (1) by computing the ARC over each January month and taking the arithmetic mean. We show that this is indeed a proper way of measuring the ARC in this case, and that in general one would use the time-weighted arithmetic mean (equivalently, the resulting ARC must be the quotient of a measure of change of the outcome variable to elapsed time).

6. What is a path-dependent counterpart to the ARC formulas?

The ARC measures (1) and (2) are path-independent in the sense that they depend on the outcome variable only through its values at the endpoints of the interval rather than on the entire path. Path-dependent ARC formulas are of interest in various fields, including measurement of aggregate price/quantity change (Balk, 2008, chapter 6), productivity (Richter, 1966; Jorgenson and Griliches, 1967) and welfare (Bruce, 1977; Cysne, 2003) measurement. E.g., most if not all statistical agencies calculate their consumer price index as a chained index, which in general produces a path-dependent measure of inflation. Though the conventional line integral approach, initiated by Divisia and Montgomery, suggests some important classes of path-dependent ARC measures, the general structure of path-dependent ARC remains unclear. We show that a proper path-dependent modification of the ARC formula (1) involves replacing the nominator with the integral over $[t,t']$ whose integrand value at $\tau \in [t,t']$ is completely determined by the germ of the path of the outcome variable at time τ. The obtained integral representation comprises, as special cases, most known path-dependent indices, including the Divisia index.

Answering these six questions allows us to substantially revise and complement our earlier work (Alexeev and Sokolov, 2014) (henceforth, AS), in which some results in the directions of the first two questions were derived. In our analysis, we rely on the difference measurement foundation (specifically, Theorem 5.3 in Wakker (1988)), additive representations (Krantz et al., 1971, chapter 6; Wakker, 1988, section 4), and conditional utility formalism (Krantz et al., 1971, chapter 8). As mentioned previously, the duality between ARC measurement and time preferences, established in AS, is central to our discussion. Section 2 revisits this duality in the light of a recent study of Doyle (2013) who advocates analyzing time preference models by isolating the rate parameter governing the degree of discounting in a discounting equation. We show that his approach is in fact a universal recipe for constructing an ARC measure. In particular, an ARC measure can be identified with a discount rate that makes an individual indifferent between the initial and final values of the outcome variable. This duality implies a one-to-one correspondence between ARC measures and one-parameter families of time preferences indexed by a discount rate. We show that the ARC formulas (1) and (2) correspond to exponential discounting. In section 2.1, an experimental procedure that elicits an ARC via a series of binary intertemporal choices is proposed. Since a large body of empirical literature reports evidence against exponential discounting, the "genuine" ARC obtained this way will likely differ from (1) and (2). In section 2.2, we derive the ARC measures that correspond to some common time preference models, including the discounted utility and the relative discounting model of Ok and Masatlioglu (2007). Having examined the duality between the ARC measurement and time preferences, in section 3 we apply an axiomatic approach to answer questions 1–3. The characterized ARC measures turn out to correspond to the common time preference models discussed in section 2.2. Due to the duality, the obtained characterizations of the ARC measures can be equivalently viewed as axiomatic foundations for the corresponding families of time preferences. Section 4 shows how benchmarking can be incorporated into ARC measurement and provides an answer to question 4. Section 5 addresses the remaining two questions.

Our work contributes to measurement theory. From an axiomatic viewpoint, several particular issues related to the highlighted questions are also analyzed in utility theory, index number theory, and investment appraisal. Question 1 under the special case when the time interval is fixed is related to the literature on difference (change, improvement, preference intensity) measurement (Krantz et al., 1971, chapter 4; Shapley, 1975; Wakker, 1988, section 5; Köbberling, 2006; Gerasimou, 2017). Frenzen (1994, section 2) and Aczél et al. (1996) address the problem of change measurement using a more demanding approach. Grodal and Vind (Vind, 2003, chapters 8, 11, 12) generalize some of these results in the directions of questions 5 and 6. Particular measures of change are studied in various fields, including measurement of aggregate price/quantity change, productivity measurement, welfare change measurement, and investment appraisal. In particular, in the special case when the outcome variable is a price or quantity vector, questions 1, 4, and 6 are studied in index number theory (Diewert and Nakamura, 1993; Balk, 2008). If the outcome variable is money, questions 2, 5, and 6 are related to the literature on investment appraisal that characterizes the internal rate of return (Promislow and Spring, 1996; Vilensky and Smolyak, 1999). In particular, Promislow and Spring (1996) propose a very general measure-theoretic construct that can be applied to generalize formula (2) in the directions of questions 2, 5, 6.

2. Average rate of change and time preference

We begin with basic definitions and notation. R_{++}, R_{+}, and R are the sets of positive, nonnegative, and all real numbers, respectively. A binary relation $\succeq$ on a set X is said to be trivial if $\succeq = \mathrm{X}\times\mathrm{X}$ and non-trivial otherwise. The inverse relation of $\succeq$, its symmetric and asymmetric (strict) parts are denoted by $\succeq^{-1} := \{(x,x')\in \mathrm{X}^2 : x' \succeq x\}$, $\sim := \succeq \cap \succeq^{-1}$, and $\succ := \succeq \setminus \succeq^{-1}$, respectively. The (strict) upper and (strict) lower contour sets of $\succeq$ are denoted by ($\mathrm{U}_{\succ}(x) := \{x'\in\mathrm{X} : x' \succ x\}$) $\mathrm{U}_{\succeq}(x) := \{x'\in\mathrm{X} : x' \succeq x\}$ and ($\mathrm{L}_{\succ}(x) := \{x'\in\mathrm{X} : x \succ x'\}$) $\mathrm{L}_{\succeq}(x) := \{x'\in\mathrm{X} : x \succeq x'\}$, respectively. If X is a topological space, then $\succeq$ is said to be continuous provided that it is closed in the product topology on $\mathrm{X}\times\mathrm{X}$. In the case of a total preorder (i.e., a complete and transitive binary relation) this definition of continuity is equivalent to another common one: for each $x\in\mathrm{X}$ the upper and lower contour sets, $\mathrm{U}_{\succeq}(x)$ and $\mathrm{L}_{\succeq}(x)$, are closed in X. Relation $\succeq$ is (continuously) representable if there exists a (continuous) function $I:\mathrm{X}\to\mathrm{R}$ such that $x \succeq x' \Leftrightarrow I(x)\ge I(x')$; in this case I is said to represent $\succeq$.

Let S be a connected topological space representing the domain of a variable of interest. An element of S is referred to as a state. Let $(\mathrm{T},\ge)$ be a linearly ordered topological space (i.e., a linearly ordered set endowed with the order topology) with at least 3 elements. The relations $>$, $\le$, $<$ and intervals $[t,t']$, $[t,t')$, etc. are defined as usual. If T is a subset of R, then we will always assume it to be equipped with the usual order. We interpret an element of T as time for simplicity, however, in some applications observations can be ordered by a variable other than time. Put $\mathrm{T}^2_{<} := \{(t,t')\in\mathrm{T}^2 : t<t'\}$. A function $f:\mathrm{T}\to\mathrm{R}$ is called strictly increasing (resp. decreasing) if it represents $\ge$ (resp. $\le$). Define $\mathrm{V} := \{(s,t;s',t')\in(\mathrm{S}\times\mathrm{T})^2 : t<t'\}$,

$W := V \cup V^{-1} = \{(s,t;s',t') \in (S\times T)^2 : t \neq t'\}$, and $\Delta := \{(s,t;s',t') \in (S\times T)^2 : t = t'\}$.[1] An element of V is interpreted as an ordered pair of dated observations of the outcome variable. We equip V with the relative topology induced from the product topology on $(S\times T)^2$.

A non-trivial total preorder $\succeq$ is assumed to be defined on V. This preorder is called an *ARC ordering* and ranks elements of V by the ARC: the statement $(s,t;s',t') \succeq (r,\tau;r',\tau')$ means that the ARC over the period from t to t' with the initial state s and the final state s' is no less than the ARC over the period from τ to τ' with r and r' as the initial and final states, respectively. A numerical representation of $\succeq$ (if any) will be referred to as an ARC measure, or simply an ARC. Given $(t,t') \in T^2_<$, by $\succeq_{t,t'}$ we denote the total preorder on S^2 defined by $(s,s') \succeq_{t,t'} (r,r') \Leftrightarrow (s,t;s',t') \succeq (r,\tau;r',\tau')$. The relations $\succ_{t,t'}$ and $\sim_{t,t'}$ are defined as usual. A numerical representation of $\succeq_{t,t'}$ is interpreted as a measure of change over the period from t to t'. This construction establishes a link between change and average rate of change measurements.

We now consider the relationship between ARC measurement and intertemporal choice. Namely, we show that an ARC ordering can be described by means of a family of time preferences (a TP-family, for short) over the space $S\times T$ of dated states. This relationship, first, provides an alternative interpretation of an ARC and, second, can be used to elicit an ARC ordering from an economic agent.

In a survey paper, Doyle (2013) analyzed a variety of time preference models isolating a discount rate parameter that equalizes two temporal rewards. The notion of a TP-family introduced below is an attempt to formalize Doyle's idea. It describes a one-parameter family of time preferences over dated states indexed by a discount rate with the property that for any two nonsimultaneous dated states there is a unique discount rate that makes an agent indifferent between them. A somewhat similar idea of representing time preference by means of a parametric family of preference relations is proposed in Vansnick (1987).

A collection $\{\trianglerighteq_d, d \in D\}$ of binary relations over $S\times T$ indexed by a linearly ordered set $(D, \succeq)$ (with $\succ$ defined as usual) is called a *TP-family* if the following three conditions hold:

1º. $\forall d \in D$ $\trianglerighteq_d$ is non-trivial and complete (total),

2º. $(s,t) \trianglerighteq_d (s',t')$, $t < t'$ $\Rightarrow$ $(s,t) \triangleright_{d'} (s',t')$ $\forall d' \succ d$,

3º. $\forall (s,t)$, (s',t'), $t \neq t'$ $\exists d \in D$: $(s,t) \sim_d (s',t')$,

where $\sim_d$ and $\triangleright_d$ are the symmetric and asymmetric parts of $\trianglerighteq_d$.

Condition 1º is standard and assumes that any two dated states are comparable. Condition 2º allows us to interpret an element of D as a subjective discount rate (a degree of impatience) and $\trianglerighteq_d$ as time preference governed by the degree of impatience d. Indeed, in the most general sense, one can define a degree of impatience as a characteristic of time preferences that, when increased, makes the earlier of any two dated outcomes more preferable. This is exactly what condition 2º asserts.[2] For any two dated states (s,t) and (s',t') with $t < t'$, a solution $d \in D$ of the equation

[1] We identify subsets of $(S\times T)^2$ (resp. $(S\times T)^4$) with binary relations on $S\times T$ (resp. $(S\times T)^2$) so that the operation $^{-1}$ is well defined.

[2] The following condition can be considered as a natural complement to 2º: $(s,t) \trianglerighteq_d (s',t)$ $\Rightarrow$ $(s,t) \trianglerighteq_{d'} (s',t)$ for any $d' \in D$, that is, preference between simultaneous outcomes is unaffected by the degree of impatience. However, since a TP-family is not of our interest in itself, but serves as a

$(s,t) \sim_d (s',t')$ is called the *internal rate of return* (*IRR*) and is denoted by $IRR(s,t;s',t')$. The IRR is a discount rate which makes an agent indifferent between the two dated states. Conditions 3º and 2º guarantee that the IRR exists and is unique so that the mapping $IRR : \mathrm{V} \to \mathrm{D}$ (called the *IRR mapping*) is well defined. Condition 3º holds regardless of the static preferences between s and s' so that a TP-family is rich enough to comprise positive, zero, and negative time preferences. In what follows, the subfamily $\{\trianglerighteq_d,\, d \in IRR(\mathrm{V})\}$ is called the *core* of $\{\trianglerighteq_d,\, d \in \mathrm{D}\}$. Note that the core itself is a TP-family.

A TP-family is said to be *continuous* if so are all its elements. Given binary relations $\trianglerighteq$ and $\trianglerighteq'$ on $\mathrm{S} \times \mathrm{T}$, we write $\trianglerighteq \approx \trianglerighteq'$ if $\trianglerighteq \cap \mathrm{W} = \trianglerighteq' \cap \mathrm{W}$. TP-families $\{\trianglerighteq_d,\, d \in \mathrm{D}\}$, $(\mathrm{D}, \succeq)$ and $\{\trianglerighteq'_{d'},\, d' \in \mathrm{D}'\}$, $(\mathrm{D}', \succeq')$ are said to be *isomorphic* if there exists an order isomorphism φ from $(\mathrm{D}, \succeq)$ to $(\mathrm{D}', \succeq')$ such that $\trianglerighteq_d \approx \trianglerighteq'_{\varphi(d)}\ \forall\, d \in \mathrm{D}$.

Each TP-family $\{\trianglerighteq_d,\, d \in \mathrm{D}\}$ induces an ARC ordering $\succeq$ by the rule:

$$v \succeq v' \iff IRR(v) \succeq IRR(v'). \tag{3}$$

An equivalent representation follows from the definition of a TP-family and is given by

$$(s,t;s',t') \succeq (r,\tau;r',\tau') \iff \{d \in \mathrm{D} : (s',t') \trianglerighteq_d (s,t)\} \supseteq \{d \in \mathrm{D} : (r',\tau') \trianglerighteq_d (r,\tau)\}. \tag{4}$$

In other words, $(s,t;s',t') \succeq (r,\tau;r',\tau')$ if and only if for any $d \in \mathrm{D}$, $(r',\tau') \trianglerighteq_d (r,\tau) \Rightarrow (s',t') \trianglerighteq_d (s,t)$.

The induced ARC ordering has two possible behavioral interpretations. The first interpretation is inspired by representation (3): given a four-tuple $(s,t;s',t') \in \mathrm{V}$, its ARC is a measure of perceived average rate of utility growth while passing from the dated state (s,t) to (s',t'). The second interpretation is inspired by (4): given $(s,t;s',t') \in \mathrm{V}$, its ARC is a measure of stability of preference for (s',t') over (s,t). Indeed, a TP-family can be considered as a tool to describe an agent's (a group of agents') time preference $\trianglerighteq_d$ governed by the subjective discount rate (degree of impatience) parameter d. The parameter reflects the state of nature and is affected by various exogenous factors such as age, health, wealth, the degree of uncertainty about the future, inflation, etc. Then one can interpret $(s,t;s',t') \succeq (r,\tau;r',\tau')$ as that (s',t') is preferred to (s,t) "more often" than (r',τ') to (r,τ), since the set of states of nature (discount rates $d \in \mathrm{D}$) such that $(s',t') \trianglerighteq_d (s,t)$ contains the set of states of nature such that $(r',\tau') \trianglerighteq_d (r,\tau)$.

The following lemma emphasizes the role of a TP-family in ARC measurement. It shows that representation (3) is general enough to produce any ARC ordering. Furthermore, discount rates can be identified with real numbers if and only if the induced ARC ordering is representable.

Lemma 1.

For a binary relation $\succeq$ on V, the following statements hold.

(a) $\succeq$ is an (continuous) ARC ordering if and only if it is induced by a (continuous) TP-family.

(b) $\succeq$ is a representable ARC ordering if and only if it is induced by a TP-family whose core is indexed by a set order isomorphic to a subset of R.

Two TP-families induce the same ARC ordering if and only if they have isomorphic cores.

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tool to induce an ARC ordering (see Lemma 1 below), we do not assume the condition to hold as it does not affect the induced ARC ordering.

The proof of Lemma 1 does not use the connectedness of S; so it generalizes Proposition 2 in AS, which relates continuously representable ARC orderings to continuous TP-families under the additional assumption of monotonicity. Lemma 1 shows that, in the context of ARC measurement, elements of a TP-family outside the core can be disregarded; moreover, they turn out to be degenerate from a behavioral point of view, at least in the case of a continuous TP-family.[3]

According to (3), if the index set $(\mathrm{D}, \underline{\succ})$ of a TP-family is a subset of R with the usual order, then the induced ARC ordering $\succeq$ is representable and the IRR mapping is precisely a numerical representation of $\succeq$. In view of this, Table 1 in Doyle (2013, p. 121) lists exactly numerical representations for the ARC orderings induced by the most common TP-families over money. Several particular representations are considered in details in the next section.

2.1 Examples and nonexamples

In this section we derive ARC orderings that correspond to some common time preference models. In all the examples below, $(\mathrm{D}, \underline{\succ})$ is a subset of R with the usual order.

We start with the exponential discounting model. A TP-family is said to be *exponential discounting* if $(\mathrm{D}, \underline{\succ})$ and $(\mathrm{T}, \geq)$ are R with the usual order and

$$(s,t) \trianglerighteq_d (s',t') \iff u(s)e^{-dt} \geq u(s')e^{-dt'},$$

where $u: \mathrm{S} \to \mathrm{R}_{++}$ is a continuous instantaneous utility function with unbounded $\ln u$. Its IRR mapping (i.e., the function that takes each $(s,t;s',t') \in \mathrm{V}$ to the solution d of the equation $u(s)e^{-dt} = u(s')e^{-dt'}$) is given by

$$(s,t;s',t') \mapsto \frac{\tilde{u}(s') - \tilde{u}(s)}{t'-t}, \tag{5}$$

where $\tilde{u} := \ln u$. That is, a numerical representation of the induced ARC ordering is simply the average rate of growth of an agent's utility.

A TP-family is said to be *multiplicative discounting* if $\mathrm{D} = \mathrm{R}$ and

$$(s,t) \trianglerighteq_d (s',t') \iff u(s)g(d,t) \geq u(s')g(d,t'),$$

where $u: \mathrm{S} \to \mathrm{R}_{++}$ is a continuous function with unbounded $\ln u$ and the function $g: \mathrm{R} \times \mathrm{T} \to \mathrm{R}_{++}$ satisfies the following properties: for each $d \in \mathrm{R}$ $g(d,\cdot)$ is continuous and for each $(t,t') \in \mathrm{T}^2_<$ $d \mapsto g(d,t)/g(d,t')$ is a strictly increasing homeomorphism of R onto R_{++}. Here $g(d,\cdot)$ is a discount function that corresponds to the discount rate d. For any $t_0 \in \mathrm{T}$, a reference point, g can be normalized such that $g(\cdot,t_0) \equiv 1$. Under this normalization, $g(\cdot,t)$ is continuous and strictly decreasing (resp. increasing) for every $t > t_0$ (resp. $t < t_0$) and the value $g(d,t)$ represents the

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[3] A continuous time preference $\trianglerighteq_d$ with $d \notin \mathit{IRR}(\mathrm{V})$ is state insensitive in the sense that for any $t \neq t'$, either $(s,t) \triangleright_d (s',t')$ $\forall s,s' \in \mathrm{S}$ or $(s',t') \triangleright_d (s,t)$ $\forall s,s' \in \mathrm{S}$. Indeed, for any $t \neq t'$, the sets $\mathrm{A} := \{(s,s') \in \mathrm{S}^2 : (s,t) \trianglerighteq_d (s',t')\}$ and $\mathrm{B} := \{(s,s') \in \mathrm{S}^2 : (s',t') \trianglerighteq_d (s,t)\}$ are disjoint (since $d \notin \mathit{IRR}(\mathrm{V})$), closed (by continuity of $\trianglerighteq_d$) and $\mathrm{A} \cup \mathrm{B} = \mathrm{S}^2$ (due to completeness of $\trianglerighteq_d$). Since S^2 is connected, either A or B is empty. A similar routine argument shows that, if T is connected, then there exist at most two elements of a continuous TP-family that are not in the core, namely, $(s,t) \trianglerighteq_d (s',t') \iff t \geq t'$ (in this case, d is the least elements of D) and $(s,t) \trianglerighteq_d (s',t') \iff t \leq t'$ (in this case, d is the greatest elements of D).

discount factor from t to t_0 that corresponds to the discount rate d. The condition that for any $(t,t') \in \mathrm{T}^2_<$, $d \mapsto g(d,t)/g(d,t')$ is strictly increasing (in other terms, $\ln g$ has strictly decreasing differences) signifies that, with increase of d, the earlier of any two dated outcomes becomes more preferable. In particular, if T is a real interval and g is differentiable in t, then the condition implies that the instantaneous discount rate, $-\partial \ln g(d,t)/\partial t$, is non-decreasing in d. The definition of a multiplicative discounting TP-family implicitly assumes T to be order isomorphic to a subset of R with the usual order. Indeed, for any $t < t'$ and $d < d'$, we have $g(d,t)/g(d,t') < g(d',t)/g(d',t')$. Rearranging the terms in the inequality, we get that for each $d < d'$ the function $t \mapsto g(d,t)/g(d',t)$ is strictly increasing. The family induces the ARC ordering with a numerical representation that sends each $(s,t;s',t') \in \mathrm{V}$ to the solution d of the equation

$$u(s)g(d,t) = u(s')g(d,t'). \tag{6}$$

A multiplicative discounting TP-family with the discount function of the form $g(d,t) = e^{-d\phi(t)}$, where $\phi : \mathrm{T} \to \mathrm{R}$ is strictly increasing, is called *uniform*. The identity function ϕ on $\mathrm{T} \subseteq \mathrm{R}$ (exponential discounting) and $\phi(t) = \alpha^{-1}\ln(1+\alpha t)$, $\alpha > 0$ (generalized hyperbolic discounting (Loewenstein and Prelec, 1992)) on $\mathrm{T} = \mathrm{R}_+$ are common examples. If T is a real interval, then the instantaneous discount rate, $-\partial \ln g(d,t)/\partial t = d\phi'(t)$, exists almost everywhere and a change of d results in a uniform change of the rates; this justifies the name of the TP-family. The family induces the ARC ordering with a representation

$$(s,t;s',t') \mapsto \frac{\tilde{u}(s') - \tilde{u}(s)}{\phi(t') - \phi(t)}, \tag{7}$$

where $\tilde{u} := \ln u$. From a behavioral point of view, one can think of ϕ in (7) as a subjective time perception function, which describes how fast time is perceived to pass in an agent's mind (Ahlbrecht and Weber, 1995). Like calendar time, perceived time is an interval scale (see Lemma 5(d) in the next section).

We proceed with the relative discounting model of Ok and Masatlioglu (2007) and Dubra (2009) (a similar model is studied independently by Scholten and Read (2006)) that covers a variety of transitive and intransitive time preference models, including multiplicative, subadditive (Read, 2001), and similarity-based (Rubinstein, 2003) models. A TP-family $\{\trianglerighteq_d,\, d \in \mathrm{D}\}$ is said to be *relative discounting* if D is a proper real interval symmetric with respect to zero and

$$(s,t) \trianglerighteq_d (s',t') \;\Leftrightarrow\; u(s)h(d,t,t') \geq u(s'), \tag{8}$$

where $u : \mathrm{S} \to \mathrm{R}_{++}$ is a continuous function such that $\{\ln u(s') - \ln u(s),\, s,s' \in \mathrm{S}\} = \mathrm{D}$; a function $h : \mathrm{D} \times \mathrm{T}^2 \to \mathrm{R}_{++}$ is such that $h(d,t',t) = 1/h(d,t,t')$ and for each $(t,t') \in \mathrm{T}^2_<$ $\ln h(\cdot,t,t')$ is strictly increasing and onto D. Here $h(d,\cdot,\cdot)$ is a relative discount function that corresponds to the discount rate d (the value $h(d,t,t')$ is interpreted as the discount factor that discounts the utility at time t to t' under the discount rate d). Representation (8) is a modification of that of Ok and Masatlioglu (2007) and Dubra (2009) to a connected state space and to a general ordering variable whose domain constitutes a linear order. Perhaps more importantly, in contrast to Ok and Masatlioglu (2007) and Dubra (2009), we do not impose any regularity assumptions on $h(d,\cdot,\cdot)$ such as continuity, surjectivity, or monotonicity in each argument. In particular, this allows the model to accommodate present/future bias, the tendency to over/under estimate immediate utility. In the context of the relative discounting model with $\mathrm{T} = \mathrm{R}_+$, these effects are manifested in a discontinuous at $(0,0)$ (the present) relative discount function and can be captured, for instance, by the quasi-hyperbolic discounting (Phelps and Pollak, 1968; Laibson, 1997). The TP-family induces the ARC ordering with a numerical representation that sends each $(s,t;s',t') \in \mathrm{V}$ to the solution d of the equation $u(s)h(d,t,t') = u(s')$.

A relative discounting TP-family is said to be *stationary* (Ok and Masatlioglu, 2007, section 3.3) if $\mathrm{T} = \mathrm{R}$ and the relative discount function can be expressed in the form $h(d,t,t') = g(d,t'-t)$ for some $g : \mathrm{D} \times \mathrm{R} \to \mathrm{R}_{++}$.

We close this section with an example of a time preference model that does not constitute a TP-family. A collection $\{\trianglerighteq_d,\ d \in \mathrm{D}\}$ of hyperbolic discounting time preferences,

$$(s,t) \trianglerighteq_d (s',t') \iff u(s)/(1+dt) \geq u(s')/(1+dt'),$$

with $\mathrm{T} = \mathrm{R}_+$ and a non-constant continuous $u : \mathrm{S} \to \mathrm{R}$, does not satisfy condition 3º for any index set $\mathrm{D} \subseteq \mathrm{R}$. Indeed, since S is connected and u is continuous and non-constant, there exist $s, s' \in \mathrm{S}$ such that $u(s')/u(s) > 1$. If we put $t = 1$ and $t' = u(s')/u(s)$, then there is no $d \in \mathrm{R}$ that solves $u(s)/(1+dt) = u(s')/(1+dt')$; a contradiction with condition 3º. Therefore, the collection does not constitute a TP-family and does not produce an ARC ordering.

2.2 Eliciting ARC ordering: an experimental design

Lemma 1 shows that an ARC ordering is completely determined by time preferences of an agent associated with the problem. In practice, nearly always formulas (1) and (2) (or, more generally, (5)) are used to calculate the ARC. As it is shown (section 2.1), these formulas correspond to exponential discounting. However, a large body of experimental research on choice over time documents evidence against exponential discounting (see Frederick et al. (2002) for a survey). That is, the usual ARC formulas are probably not suitable for some applications. In this section, we use representation (4) to suggest an experimental procedure that elicits a genuine ARC ordering $\succeq$ using a series of binary intertemporal choices.

Consider an experiment in which a researcher attempts to manipulate a subject's preferences over the space of dated outcomes $\mathrm{S} \times \mathrm{T}$ using exogenous variation in some treatment variable.[4] Assume that for each value of the treatment variable, the subject's preferences belong to the same TP-family. Given $(s,t;s',t'),(r,\tau;r',\tau') \in \mathrm{V}$, the subject is asked to state his/her preferences between (s,t) and (s',t'), as well as between (r,τ) and (r',τ') for different levels of the treatment variable (and hence, under different subjective discount rates d). Then either for each value of the treatment variable, the subject prefers (r,τ) to (r',τ'), whenever he/she prefers (s,t) to (s',t') (a necessary condition for $(s,t;s',t') \succeq (r,\tau;r',\tau')$) or vice versa (a necessary condition for $(r,\tau;r',\tau') \succeq (s,t;s',t')$). If neither of the two possibilities holds, then the subject's time preferences do not actually belong to the same TP-family for all values of the treatment variable. These implications hold independently of a particular time preference model. Moreover, if the domain of the treatment variable is order isomorphic to $(\mathrm{D}, \geq)$, then for any finite set $\mathrm{A} \subset \mathrm{V}$, there is a set I

[4] While we are aware of no robust method to manipulate a subject's time preference, the existing literature suggests several potential manipulations, such as presenting various levels of inflation (Ostaszewski et al., 1998; Kawashima, 2006) or changing the clock speed (Ghafurian and Reitter, 2018).

($\#\mathrm{I} \leq \#\mathrm{A}$) of trial values of the treatment variable such that the function that takes each $(s,t;s',t') \in \mathrm{A}$ to the number of trial values from I such that (s',t') is preferred to (s,t) is a numerical representation of the restriction of $\succeq$ to A.

A similar experiment can be conducted with a group of subjects in order to test whether their time preferences belong to the same TP-family (probably with distinct degrees of impatience d) and, if this is the case, elicit the induced ARC ordering $\succeq$. A testable implication is that for each $(s,t;s',t'),(r,\tau;r',\tau') \in \mathrm{V}$ either each subject that prefers (s,t) to (s',t') also prefers (r,τ) to (r',τ') (a necessary condition for $(s,t;s',t') \succeq (r,\tau;r',\tau')$), or vice versa (a necessary condition for $(r,\tau;r',\tau') \succeq (s,t;s',t')$), or neither (the subjects' time preferences do not belong to the same TP-family).

3. Axiomatizations

In this section, we use an axiomatic approach to suggest answers to questions 1–3 from the Introduction. Though any non-constant real-valued function on V is a numerical representation of an (representable) ARC ordering, one may feel that some ARC measures are more relevant than others. The conditions (axioms) on an ARC ordering introduced below seek to formalize this feeling. Various combinations of them turn out to provide axiomatic foundations for the examples of ARC orderings introduced in section 2.1.

(i) for every $(t,t') \in \mathrm{T}^2_<$, $\succeq_{t,t'}$ is continuous.

(ii) $(s,s') \succeq_{t,t'} (r,r')$ & $(s',s'') \succeq_{t',t''} (r',r'') \Rightarrow (s,s'') \succeq_{t,t''} (r,r'')$. Furthermore, if either antecedent inequality is strict, so is the conclusion.

(iii) $(s,s) \sim_{t,t'} (r,r)$.

(iv) for every $(t,t') \in \mathrm{T}^2_<$ and $v \in \mathrm{V}$, there exist $s,s' \in \mathrm{S}$ such that $(s,t;s',t') \sim v$.

(v) $(s,t;s',t') \sim (s',t';s'',t'') \Rightarrow (s,t;s',t') \sim (s,t;s'',t'')$.

Besides the usual continuity assumption on $\succeq$, we distinguish continuity w.r.t. state (i) since, in contrast to time, state (e.g., macroeconomic data) is often measured with error. Thereby, continuity w.r.t. state is a compelling property that provides assurance that small measurement errors result in a minor perturbation of the ordering.

In the context of difference measurement, condition (ii) is known as the weak monotonicity axiom (Krantz et al., 1971, Axiom 3, p. 151) or strong concatenation (Köbberling, 2006, p. 381; Wakker, 1988, Definition 5.2). To motivate the condition, consider two variables, A and B, over the same time period. According to (ii), if A has a higher ARC than B over each subperiod, then A must also dominate B over the consolidated period. Condition (ii) implies that there exists a collection $\{\succcurlyeq^t,\ t \in \mathrm{T}\}$ of total preorders over S such that an underlying TP-family can be chosen to satisfy

$$(s,t) \trianglerighteq_d (s',t') \ \&\ s' \succcurlyeq^{t'} s'' \Rightarrow (s,t) \trianglerighteq_d (s'',t'). \tag{9}$$

Furthermore, if either antecedent inequality in (9) is strict, so is the conclusion. In particular, this implies $(s,t) \trianglerighteq_d (s',t) \Leftrightarrow s \succcurlyeq^t s'$. That is, static preferences are not affected by the degree of impatience d and transitive, while the only possible source of intransitivity of a time preference $\trianglerighteq_d$ is the passage of time. The last assertion is consistent, e.g., with how intransitivity is modeled in the relative discounting time preference model of Ok and Masatlioglu (2007).

Condition (iii) is a weak form of the assumption that an ARC is equal to zero whenever there is no change. Put differently, a measure of change induced by an ARC ordering can be normalized to zero whenever there is no change. The condition holds if and only if all elements of an underlying TP-family are orthogonal in the sense of axiom A2 in Dubra (2009). Condition (iii) is known as the neutrality axiom in the context of difference measurement (Köbberling, 2006, p. 378) (see also Frenzen (1994, Property P_1) and Aczél et al. (1996, Property 4)) and can also be considered as an ordinal formulation of the identity test in index number theory (Balk, 2008, p. 58).

Condition (iv) requires that the state space is rich enough for an ARC to be set arbitrarily high/low by changing states. Put differently, delay can always be compensated by the change in a state. Axiom (iv) is also a necessary condition to determine equivalent average rate of change (by analogy with equivalent change in Frenzen (1994, section 2) and Aczél et al. (1996)). In terms of an underlying TP-family, (iv) is a weak form of the outcome sensitivity axiom of Ok and Masatlioglu (2007). Since S is connected, the combination of structural conditions (i) and (iv) can be concisely stated as follows: for every $(t,t') \in \mathrm{T}^2_<$ and $v \in \mathrm{V}$, the sets $\{(s,s') \in \mathrm{S}^2 : (s,t;s',t') \succeq v\}$ and $\{(s,s') \in \mathrm{S}^2 : v \succeq (s,t;s',t')\}$ are closed (i.e., $\succeq$ is continuous provided that T is equipped with the discrete topology) and non-empty.

According to condition (v), if ARCs over two subsequent periods are equal, the ARC over the consolidated period will be the same. In terms of an underlying TP-family, condition (v) states transitivity of the indifference relations $\sim_d$ restricted to temporally spaced outcomes: for any $d \in \mathrm{D}$ and pairwise distinct $t,t',t'' \in \mathrm{T}$, if $(s,t) \sim_d (s',t')$ and $(s',t') \sim_d (s'',t'')$, then $(s,t) \sim_d (s'',t'')$. A weaker form of (v), the standard sequence condition, is used to characterize difference representations (Köbberling, 2006, p. 379).

We proceed with the independence result.

Lemma 2.

(a) If there is a non-constant continuous real-valued function on S, then conditions (i)–(iv) are independent (i.e., any three of them do not imply the fourth). Otherwise, they are inconsistent.

(b) If S is non-pseudocompact (i.e., there is an unbounded continuous real-valued function on S) and T is order isomorphic to a subset of R, then conditions (i)–(v) are independent. Otherwise, they are inconsistent.

Most, or probably all, common ARC measures (such as (1), (2), and, more generally, (5)) are defined on $\mathrm{T} \subseteq \mathrm{R}$ and generate ARC orderings satisfying conditions (i)–(v). An interesting consequence of Lemma 2(b) is that there is no natural generalization of them to the case when T is not order isomorphic to a subset of R.

Our main objective here is to give a complete characterization of the class of ARC orderings that satisfy the introduced conditions (axioms). Conditions (i)–(v) do not impose any direct restrictions on the domains of the outcome and ordering variables and the form of dependence on dates; thus, the class of ARC orderings we are characterizing can be suggested as an answer to questions 1–3 from the Introduction. We begin with a preliminary result that describes the structure of an ARC ordering satisfying (i)–(iii). The following lemma is a direct consequence of Theorem 5.3 in Wakker (1988) and is a key tool for proving the subsequent results. It shows that the relation $\succeq_{t,t'}$ induced by an ARC ordering satisfying (i)–(iii) is time-independent and admits a difference representation.

Lemma 3.

For an ARC ordering $\succeq$, the following two statements are equivalent:

(a) (i)–(iii) hold;

(b) the relation $\succeq_{t,t'}$ is independent of the choice of $(t,t') \in \mathrm{T}^2_<$ and there exists a continuous function $\tilde{u} : \mathrm{S} \to \mathrm{R}$ such that $(s,s') \mapsto \tilde{u}(s') - \tilde{u}(s)$ represents $\succeq_{t,t'}$.

Lemma 3 agrees with the theoretical literature on difference measurement (Krantz et al., 1971, section 4.4; Shapley, 1975; Wakker, 1988, section 5; Frenzen, 1994, section 2; Aczél et al., 1996; Köbberling, 2006) showing that a measure of change is (up to an order-preserving transformation) a utility difference. The only difference is that we characterize a collection of measures of change with the same difference representation rather than an individual measure. In the context of price/quantity change measurement an equivalent quotient (rather than difference) representation also receives strong support from index number theory assuming the circularity (transitivity) test (Balk, 2008, p. 78) to hold. From the proof it follows that Lemma 3 remains valid if condition (i) is replaced by a weaker assumption of continuity of $\succeq_{t,t'}$ for some $(t,t') \in \mathrm{T}^2_<$.

An important special case that covers most of the practical applications arises when $(\mathrm{T}, \geq)$ is a subset of the reals with the usual order. This case turns out to be particularly convenient, since, as shown in the next lemma, then every continuous ARC ordering satisfying (ii) and (iii) is representable.

Lemma 4.

The following statements are equivalent:

(a) $(\mathrm{T}, \geq)$ is order isomorphic to a subset of R with the usual order;

(b) T is second countable;

(c) any continuous ARC ordering $\succeq$ that satisfies (ii) and (iii) is continuously representable, that is, there exist continuous functions $\tilde{u} : \mathrm{S} \to \mathrm{R}$ and $I : \{\tilde{u}(s') - \tilde{u}(s),\, s, s' \in \mathrm{S}\} \times \mathrm{T}^2_< \to \mathrm{R}$ such that I is strictly increasing in the first argument and $(s,t;s',t') \mapsto I(\tilde{u}(s') - \tilde{u}(s), t, t')$ represents $\succeq$.

The following is the main result of this section.

Proposition 1.

For an ARC ordering $\succeq$, the following statements hold.

(a) $\succeq$ satisfies (i)–(iv) if and only if it is induced by a relative discounting TP-family.

(b) $\succeq$ satisfies (i)–(v) if and only if it is induced by a multiplicative discounting TP-family.

Proposition 1 relates ARC orderings that satisfy natural axioms to basic time preference models. In the rest of the section, we introduce refinements of this result by imposing additional regularity conditions that make sense, both for an ARC ordering and an underlying TP-family. All these conditions impose restrictions on the form of dependence on dates (or, in terms of an underlying TP-family, on a discount function) rather than on values of the outcome variable.

Most conventional discount models do not mix positive, negative, and zero time preferences. Put differently, a generic discount function is assumed to be either strictly decreasing (for positive time preference), or strictly increasing (for negative time preference), or identically equal to 1 (for zero time preference). A typical example is provided by a uniform multiplicative discounting TP-family, where $\trianglerighteq_d$ with $d > 0$ (resp. $d < 0$, $d = 0$) exhibits positive (resp. negative, zero) time preference. The conditions introduced below ensure such a separation for an underlying TP-family. In order to formulate them, we introduce the following sets: $\mathrm{V}_1 := \{(s,t;s',t') \in \mathrm{V} : (s,s') \succ_{t,t'} (s,s)\}$, $\mathrm{V}_0 := \{(s,t;s',t') \in \mathrm{V} : (s,s') \sim_{t,t'} (s,s)\}$, $\mathrm{V}_{-1} := \{(s,t;s',t') \in \mathrm{V} : (s,s) \succ_{t,t'} (s,s')\}$. V_1, V_0, and V_{-1} decompose V into the growth, permanence, and decrease sets, respectively.

(vi) if $(s,t;s',t') \in \mathrm{V}_1$ (resp. $(s,t;s',t') \in \mathrm{V}_{-1}$), then $(s,\tau;s',\tau') \succ (s,t;s',t')$ (resp. $(s,t;s',t') \succ (s,\tau;s',\tau')$) provided that $[\tau,\tau'] \subset [t,t']$.

According to (vi), delay is undesirable (resp. desirable) in the case of growth (resp. decrease) of the variable. We also consider a related condition:

(vi)' $(s,t;s,t') \sim (s,\tau;s,\tau')$.

Condition (vi)' is in some sense dual to (iii). The combination of them (namely, $(s,t;s,t') \sim (r,\tau;r,\tau')$) implies that V_0 forms an equivalence class with respect to $\sim$. It is straightforward to show that an ARC ordering $\succeq$ satisfies (iii) and (vi)' if and only if it is induced by a TP-family $\{\trianglerighteq_d, d \in \mathrm{D}\}$, $(\mathrm{D}, \succeq)$ with the property that there exists $d_0 \in \mathrm{D}$ such that $\trianglerighteq_d$ is a positive (i.e., $(s,t) \triangleright_d (s,t')$ $\forall (t,t') \in \mathrm{T}^2_<$), negative (i.e., $(s,t') \triangleright_d (s,t)$ $\forall (t,t') \in \mathrm{T}^2_<$), or zero (i.e., $(s,t) \sim_d (s,t')$) time preference, if, respectively, $d \succ d_0$, $d_0 \succ d$, or $d = d_0$. One can also show that under (iii) and (iv), condition (vi)' holds if and only if $v_i \succ v_j$ for any $v_i \in \mathrm{V}_i$, $v_j \in \mathrm{V}_j$, $i > j$. That is, an element of the growth set has a strictly greater ARC, than an element of the permanence/decrease set and an element of the permanence set has a strictly greater ARC, than an element of the decrease set. In what follows, by "(vi)/(vi)'" we mean "(vi) or (vi)'".

Corollary 1.

For an ARC ordering $\succeq$, the following statements hold.

(a) $\succeq$ satisfies (i)–(iv), (vi) if and only if it is induced by a relative discounting TP-family whose index set D is open and whose relative discount function h can be normalized such that $h(0,\cdot,\cdot) \equiv 1$, $h(d,\cdot,\cdot)$ is strictly decreasing (resp. increasing) in the first argument and strictly increasing (resp. decreasing) in the second argument, whenever $d > 0$ (resp. $d < 0$).

(b) $\succeq$ satisfies (i)–(iv), (vi)' if and only if it is induced by a relative discounting TP-family whose relative discount function h can be normalized such that $h(0,\cdot,\cdot) \equiv 1$.

(c) $\succeq$ satisfies (i)–(v), (vi)/(vi)' if and only if it is induced by a multiplicative discounting TP-family whose discount function g can be normalized such that $g(d,\cdot)$ is strictly decreasing (resp. strictly increasing, identically equal to 1), whenever $d > 0$ (resp. $d < 0$, $d = 0$).

The class of discount functions described in Corollary 1(c) is characterized by Vilensky and Smolyak (1999, sections 3.3, 3.4) (under the additional assumption of continuous differentiability) and turns out to be important in investment appraisal in connection with the extension of the concept of internal rate of return to not necessarily exponentially discounted cash flows. The special case of an ARC ordering induced by a multiplicative discounting TP-family with a continuous discount function g satisfying the properties specified in Corollary 1(c) and the identity $g(-d,t) = 1/g(d,t)$ is also axiomatized in AS, Theorem 5.

Some time preference models assume that any gain can be compensated by a long enough passage of time; that is, the limit of a discount function as time approaches $+\infty$ is 0. The next condition, in the conjunction with (vi)', ensures this kind of behavior for an underlying TP-family.

(vii) for every $v \in \mathrm{V}_1$ (resp. $v \in \mathrm{V}_{-1}$) and $s, s' \in \mathrm{S}$, there exists $(t,t') \in \mathrm{T}^2_<$ such that $v \succeq (s,t;s',t')$ (resp. $(s,t;s',t') \succeq v$).

Being applied to the ARC measures (1) and (2), condition (vii) states that the ARC can be set arbitrary close to zero by increasing the delay. It is straightforward to show that for an ARC ordering $\succeq$ satisfying (iii) and (vi)', condition (vii) holds if and only if it is induced by a TP-family $\{\trianglerighteq_d, d \in \mathrm{D}\}$ with the property that for any $s, s' \in \mathrm{S}$ there exists $(t,t') \in \mathrm{T}^2_<$ (resp. $(t',t) \in \mathrm{T}^2_<$) such that $(s,t) \trianglerighteq_d (s',t')$, whenever $\trianglerighteq_d$ is a positive (resp. negative) time preference. In particular, an ARC ordering induced by a relative discounting TP-family satisfies (vi)' and (vii) if and only if the relative discount function h can be normalized such that $h(0,\cdot,\cdot) \equiv 1$ and for any $d \neq 0$ the subset $h(d,\mathrm{T}^2)$ of $(\mathrm{D},\geq)$ is unbounded. Similarly, an ARC ordering induced by a multiplicative discounting TP-family satisfies (vi)/(vi)' and (vii) if and only if the discount function g can be chosen to satisfy the properties described in Corollary 1(c) and for any $d \neq 0$ $\ln g(d,\cdot)$ is unbounded.

A discount function is often assumed to be continuous. The usual continuity assumption imposed on $\succeq$, in the combination with the next condition (which is in some sense dual to (vii)), characterizes this property for an underlying relative discounting TP-family.

(viii) for any $(s,t;s',t') \in \mathrm{V}_1$ (resp. $(s,t;s',t') \in \mathrm{V}_{-1}$), $v \in \mathrm{V}$, $t^* \in \mathrm{T}$, there exists a neighborhood $N \subset \mathrm{T}^2$ of the point (t^*,t^*) such that $(s,\tau;s',\tau') \succ v$ (resp. $v \succ (s,\tau;s',\tau')$), whenever $(\tau,\tau') \in \mathrm{T}^2_< \cap N$.

If T has no isolated points, the condition states that the ARC of an element from V_1 (resp. V_{-1}) can be set arbitrarily high (resp. low) by reducing the delay. If T is a discrete space, condition (viii) is redundant. In terms of an underlying TP-family, condition (viii) postulates continuity of preferences in time in a neighborhood of a simultaneous choice pair and implies the absence of present/future bias. In order to demonstrate behavioral implications of condition (viii) on an underlying TP-family, suppose that $\mathrm{T} = \mathrm{R}$ and interpret the outcome variable as money. Assuming that the time preferences in an underlying TP-family are monotone in money, we have $\mathrm{V}_1 = \{(s,t;s',t') \in \mathrm{V} : s < s'\}$, $\mathrm{V}_{-1} = \{(s,t;s',t') \in \mathrm{V} : s > s'\}$. Then condition (viii) states that for any $d \in \mathrm{D}$, t^*, and $s' > s$, there exists $\varepsilon > 0$ such that $(s',t') \triangleright_d (s,t)$ for all $t,t' \in (t^* - \varepsilon, t^* + \varepsilon)$. Being applied to the present $t = 0$, this condition postulates the absence of present and future bias.

The next result shows that, in terms of an underlying relative discounting TP-family, the usual continuity assumption imposed on $\succeq$ implies continuity of the discount function $h(d,\cdot,\cdot)$ out of the diagonal; while condition (viii) postulates continuity of the discount function on the diagonal.

Corollary 2.

For a continuous ARC ordering $\succeq$, the following statements hold.

(a) $\succeq$ satisfies (ii)–(iv) if and only if it is induced by a relative discounting TP-family whose relative discount function restricted to $\mathrm{D} \times \mathrm{T}^2_<$ is continuous.

(b) $\succeq$ satisfies (ii)–(iv), (viii) if and only if it is induced by a continuous relative discounting TP-family (equivalently, by a relative discounting TP-family with a continuous relative discount function h).
(c) $\succeq$ satisfies (ii)–(v) if and only if it is induced by a continuous multiplicative discounting TP-family (equivalently, by a multiplicative discounting TP-family whose discount function g can be chosen to be continuous).

Doyle (2013, section 7.1) notices that a rate parameter (in terms of the present model, an ARC measure) in most time preference models is a separable function of the effects of time and state changes. The following condition provides such a separation.
(ix) if $(s,t;s',t') \notin \mathrm{V}_0$, then $(s,t;s',t') \sim (s,\tau;s',\tau')$ implies $(r,t;r',t') \sim (r,\tau;r',\tau')$ for any $r,r' \in \mathrm{S}$.
Condition (ix) is the classical separability (or independence) condition, which is used to characterize decomposable representations (Krantz et al., 1971, Definition 11, p. 301; Fishburn and Rubinstein, 1982, Axiom B2). In terms of an underlying TP-family, the condition is closely related to the irrelevance axiom in Dubra (2009, Axiom A3).

Proposition 2.

For an ARC ordering $\succeq$, the following statements hold.
(a) $\succeq$ is representable and satisfies (i)–(iii), (ix) if and only if there exist functions $\tilde{u}:\mathrm{S}\to\mathrm{R}$, $f:\mathrm{T}^2_< \to \mathrm{R}$, and $G:\mathrm{D}\times f(\mathrm{T}^2_<)\to\mathrm{R}$ with $\mathrm{D} := \{\tilde{u}(s') - \tilde{u}(s),\ s,s' \in \mathrm{S}\}$ such that $\tilde{u}$ is continuous, G is strictly increasing in the first argument, for each $d \in \mathrm{D}\setminus\{0\}$ $G(d,\cdot)$ is injective, and $(s,t;s',t') \mapsto G(\tilde{u}(s') - \tilde{u}(s), f(t,t'))$ represents $\succeq$.
(b) (i)–(iv), (vi), (ix) hold if and only if $\succeq$ is induced by a relative discounting TP-family with the relative discount function whose restriction to $\mathrm{T}^2_<$ is represented in the form $h(d,t,t') = c(d, f(t,t'))$ for some functions $f:\mathrm{T}^2_<\to\mathrm{R}$ and $c:\mathrm{D}\times f(\mathrm{T}^2_<)\to\exp(\mathrm{D})$ such that f is strictly decreasing (resp. increasing) in the first (resp. second) argument, $c(d,\cdot)$ is strictly increasing (resp. strictly decreasing, identically equal to 1), whenever $d>0$ (resp. $d<0$, $d=0$).
(c) If T is connected (i.e., a linear continuum), then $\succeq$ satisfies (ii)–(v), (ix) and is continuous if and only if it is induced by a uniform multiplicative discounting TP-family with a continuous time perception function ϕ.

Proposition 2(c) provides an axiomatization of an ARC ordering induced by a continuous uniform multiplicative discounting TP-family over a connected set of time points. The connectedness assumption cannot be omitted. A special case of an ARC ordering induced by a continuous uniform multiplicative discounting TP-family with T being an open real interval and ϕ being onto R is also characterized in AS (Theorems 6 and 7), however those characterizations are based on the axioms (in their notation, (vi) and (vii)) that are problematic to test.

Finally, we consider conditions on an ARC ordering that produce representation (5). In the next two conditions, $\mathrm{T}=\mathrm{R}$.
(x) $(s,t;s',t') \sim (s,t+\tau;s',t'+\tau)\ \forall\, \tau\in\mathrm{R}$.

Condition (x) states that an ARC ordering depends on initial and final dates of observations through their difference. The condition requires an underlying TP-family to be stationary (Fishburn and Rubinstein, 1982, Axiom A5). We also consider a related axiom:

(x)' $\quad (s,t;s',t') \succeq (r,\tau;r',\tau') \;\Rightarrow\; (s,at+b;s',at'+b) \succeq (r,a\tau+b;r',a\tau'+b) \;\forall\, a>0, b$.

According to (x)', $\succeq$ is invariant with respect to a change of time scale.

Proposition 3.

Let $\mathrm{T}=\mathrm{R}$ and $\succeq$ be an ARC ordering.

(a) $\succeq$ satisfies (ii), (iii), (x) and is continuous if and only if there exist continuous functions $\tilde{u}:\mathrm{S}\to\mathrm{R}$ and $G:\{\tilde{u}(s')-\tilde{u}(s),\ s,s'\in\mathrm{S}\}\times\mathrm{R}_{++}\to\mathrm{R}$ such that G is strictly increasing in the first argument and $(s,t;s',t')\mapsto G(\tilde{u}(s')-\tilde{u}(s),t'-t)$ represents $\succeq$.

(b) $\succeq$ satisfies (i)–(iv), (x) if and only if it is induced by a stationary relative discounting TP-family.

(c) $\succeq$ satisfies (ii), (iii)/(vii), (v), (x) and is continuous if and only if there exists a non-constant continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ such that (5) represents $\succeq$.

(d) $\succeq$ satisfies (ii)–(v), (x)/(x)' and is continuous if and only if it is induced by an exponential discounting TP-family.

(e) $\succeq$ satisfies (i)–(v), (x) if and only if there exist a continuous unbounded function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ and an additive function $A:\mathrm{R}_{++}\to\mathrm{R}$ (i.e., for any $\tau,\tau'\in\mathrm{R}_{++}$, $A(\tau)+A(\tau')=A(\tau+\tau')$) such that

$$(s,t;s',t')\mapsto\frac{\tilde{u}(s')-\tilde{u}(s)}{t'-t}+\frac{A(t'-t)}{t'-t}$$

represents $\succeq$.

(f) $\succeq$ satisfies (i)–(v), (vi)/(vi)'/(viii)/(ix), (x) if and only if it is induced by an exponential discounting TP-family.

(g) If $\mathrm{V}_0\neq\mathrm{V}$, then $\succeq$ satisfies (ii), (iii), (viii), (x)' and is continuous if and only if there exist a non-constant continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ and a strictly increasing continuous function $h:\{\tilde{u}(s')-\tilde{u}(s),\ s,s'\in\mathrm{S}\}\to\mathrm{R}$ such that $h(0)=0$ and

$$(s,t;s',t')\mapsto\frac{h(\tilde{u}(s')-\tilde{u}(s))}{t'-t} \tag{10}$$

represents $\succeq$.

An analog of part (b) of Proposition 3 remains valid if T is a linearly ordered group. With obvious changes, (a), (c), (e), and (f) hold if T is a linearly ordered group that is connected under the order topology. Finally, analogues of (d) and (g) are valid if T is a linearly ordered vector space whose order topology makes it a topological vector space. The last two assertions follow from a result of Iseki (1951), who proved that if a non-trivial linearly ordered group is connected in the order topology, then it is isomorphic to the naturally ordered additive group of the reals. Axiomatizations closely related to (c) and (d) are obtained in AS, Theorems 6 and 7. However, these results do not cover each other.

Part (e) shows that a discontinuous with respect to time analogue of the ARC ordering with a representation (5) is too irregular to be useful. Indeed, since the image of any open interval under a discontinuous additive function is everywhere dense in R (Aczél, 1987, Theorem 4, p. 12), this ordering is inconsistent, e.g., with conditions (vi), (vi)', (viii), and (ix). On the other hand, in most practical applications, time is discrete and the irregularity is immaterial (since an additive function is linear over the rationals).

The ARC ordering with a representation (10) is induced by a TP-family that includes (in the notation of Doyle (2013, Table 1)) exponential, hyperbolic, arithmetic, and hyperboloid (G&M) discounting as special cases. The condition $\mathrm{V}_0 \neq \mathrm{V}$ in part (g) cannot be omitted. One can show that if $\mathrm{T} = \mathrm{R}$, then a continuous ARC ordering with $\mathrm{V}_0 = \mathrm{V}$ satisfies (iii) and (x)' if and only if it is represented by $(s,t;s',t') \mapsto \alpha t + \beta t'$ for some constants $\alpha^2 + \beta^2 \neq 0$. Since the case $\mathrm{V}_0 = \mathrm{V}$ is degenerate from an economic viewpoint, we do not elaborate on this.

Though the considered additional conditions (vi), (vi)', (vii)–(x), (x)' as well as continuity of $\succeq$ seem to be appealing, they turn out to be rather restrictive. In particular, each of them is in general inconsistent with the ARC ordering (induced by a multiplicative discounting TP-family) with a numerical representation in the form of the real (inflation-adjusted) return

$$(s,t;s',t') \mapsto \left(\frac{s'/p(t')}{s/p(t)} \right)^{\frac{1}{t'-t}} - 1. \tag{11}$$

Here elements of $\mathrm{S} = \mathrm{R}_{++}$ are interpreted as money, $\mathrm{T} = \mathrm{R}$, and p is a positive function representing a price index.

One can think of (i)–(x) as conditions on the elements of a TP-family rather than on the ARC ordering it induces. These conditions are closely related to well-known axioms (transitivity, stationarity, separability, etc.) used to characterize time preference. In this sense Propositions 1–3 can be treated as axiomatizations of time preferences described by exponential, multiplicative, relative, and stationary relative discounting models. Though our results parallel closely the known characterizations of those models by Fishburn and Rubinstein (1982), Ok and Masatlioglu (2007), and Dubra (2009), the difference is that we axiomatize the family of time preferences governed by a subjective discount rate parameter rather than a particular time preference relation. Such a family is a more realistic model of individual time preference since the subjective discount rate of an individual can be affected by various exogenous factors (such as wealth, the degree of uncertainty about the future, inflation, etc.). Our axiomatizations, however, are in general incomplete since they do not impose any restriction on the static preferences (i.e., on the restriction of a time preference over $\mathrm{S} \times \mathrm{T}$ to the set $\mathrm{S} \times \{t\}$), except completeness.

We close this section with the uniqueness result. The next lemma complements the characterizations of the ARC orderings provided in this section and parallels closely the well-known uniqueness results on the exponential, multiplicative, and relative discounting models (Fishburn and Rubinstein, 1982, Theorems 2 and 3; Ok and Masatlioglu, 2007, Proposition 1).

Lemma 5.

(a) Two exponential discounting TP-families with utility functions u and u' induce the same ARC ordering if and only if there exist positive constants a, b such that $u' = bu^a$.

(b) Two (stationary) relative discounting TP-families $\{\trianglerighteq_d, d \in \mathrm{D}\}$, $\{\trianglerighteq'_{d'}, d' \in \mathrm{D}'\}$ with utility functions u, u' and (stationary) relative discount functions h, h' (g, g') induce the same ARC ordering if and only if there exist positive constants a, b and an order isomorphism φ from $(\mathrm{D}', \geq)$ to $(\mathrm{D}, \geq)$ such that $u' = bu^a$, $h'(d',t,t') = h(\varphi(d'),t,t')^a$ ($g'(d',\tau) = g(\varphi(d'),\tau)^a$).

(c) Two multiplicative discounting TP-families with utility functions u, u' and discount functions g, g' induce the same ARC ordering if and only if there exist positive constants a, b, a function $c : \mathrm{R} \to \mathrm{R}_{++}$, and an order automorphism φ of $(\mathrm{R}, \geq)$ such that $u' = bu^a$, $g'(d,t) = c(d) g(\varphi(d), t)^a$.

(d) Two uniform multiplicative discounting TP-families with utility functions u, u' and time perception functions ϕ, ϕ' induce the same ARC ordering if and only if there exist constants $a,b,c \in \mathrm{R}_{++}$ and $q \in \mathrm{R}$ such that $u' = bu^a$, $\phi' = q + c\phi$.

4. Average rate of change relative to a benchmark

Benchmarking is a universal practice in various fields including decision making, finance, economics, and business. It is used to measure performance relative to some reference object or process. In particular, an ARC is often required to be evaluated relative to a benchmark. For instance, portfolio return is usually measured relative to a market index; economic growth can be treated as the ARC of the nominal GDP per capita relative to a price index; in comparative experiments, the ARC of a variable in a treatment group is measured relative to that variable in a control group (e.g., the learning rate of a patient is measured relative to a healthy control group (Weickert et al., 2009)).

This section aims to incorporate benchmarking into ARC measurement. Our main assumption here is that a benchmark has the same nature as the outcome variable. Given this assumption, an ARC relative to a benchmark can be described by means of a non-trivial total preorder $\succeq$ on the set $\{(s,\tilde{s},t;s',\tilde{s}',t') \in (\mathrm{S}^2 \times \mathrm{T})^2 : t < t'\}$. The first two elements in a triple $(s,\tilde{s},t) \in \mathrm{S}^2 \times \mathrm{T}$ are interpreted as dated observations of the outcome variable and the benchmark, respectively. In what follows, the total preorder $\succeq$ will be referred to as a *relative* ARC ordering.

In a way similar to section 2, a relative ARC ordering is tied to time preference. In particular, an analogue of Lemma 1 holds for relative ARC orderings. Analogous to the non-benchmark case, given a relative ARC ordering $\succeq$ and $(t,t') \in \mathrm{T}^2_<$, by $\succeq_{t,t'}$ we denote the total preorder on $\mathrm{S}^2 \times \mathrm{S}^2$ given by $(s,\tilde{s};s',\tilde{s}') \succeq_{t,t'} (r,\tilde{r};r',\tilde{r}')$ $\Leftrightarrow$ $(s,\tilde{s},t;s',\tilde{s}',t') \succeq (r,\tilde{r},t;r',\tilde{r}',t')$. A numerical representation of $\succeq_{t,t'}$ (if any) is interpreted as a measure of change relative to a benchmark. With obvious changes (s replaced by $(s,\tilde{s})$, etc.), conditions (i)–(x) make sense for a relative ARC ordering. An ARC ordering can be interpreted as a relative ARC ordering with a fixed benchmark. Specifically, given a function $p : \mathrm{T} \to \mathrm{S}$ representing dynamics of a benchmark value over T, an ARC ordering can be identified with the restriction of a relative ARC ordering to the set $\{(s, p(t), t; s', p(t'), t') : (s,t;s',t') \in \mathrm{V}\}$. Moreover, provided that p is identically constant, if a relative ARC ordering satisfies condition (i) (resp. (ii), (iii), (v)–(x)) then so does the ARC ordering it generates.

Consider the following two conditions on $\succeq_{t,t'}$.

B1. $(s,\tilde{s};s',\tilde{s}) \sim_{t,t'} (s,\tilde{r};s',\tilde{r})$.

B2. $(s,s;s',s') \sim_{t,t'} (r,r;r',r')$.

According to B1, an ARC relative to a benchmark is irrelevant of the benchmark value provided that the latter remains constant. For instance, the real (inflation-adjusted) ARC of an economic statistic is not affected by the price level provided that there is no inflation. Condition B2 is an ordinal formulation of the assumption that a measure of change relative to a benchmark is zero whenever the outcome variable mimics the benchmark.

The following result is a benchmark-based analogue of Lemma 3.

Lemma 6.

For a relative ARC ordering $\succeq$, the following two statements are equivalent:

(a) (i)–(iii), B1, B2 hold;

(b) the relation $\succeq_{t,t'}$ is independent of the choice of $(t,t') \in \mathrm{T}^2_<$ and there exists a continuous function $\tilde{u} : \mathrm{S} \to \mathrm{R}$ such that $(s,\tilde{s};s',\tilde{s}') \mapsto (\tilde{u}(s') - \tilde{u}(\tilde{s}')) - (\tilde{u}(s) - \tilde{u}(\tilde{s}))$ represents $\succeq_{t,t'}$.

The utility difference representation $\tilde{u}(s) - \tilde{u}(\tilde{s})$ for the worth of s relative to the benchmark $\tilde{s}$ obtained in Lemma 6 is a usual building block in the literature on decision making with a reference point, in particular, reference-dependent utility (Tversky and Kahneman, 1991; Sugden, 2003; Koszegi and Rabin, 2006; Bleichrodt, 2009) and regret theory (Diecidue and Somasundaram, 2017). Assuming conditions B1, B2 to hold and using Lemma 6, one can obtain in a straightforward manner (replacing $\tilde{u}(s') - \tilde{u}(s)$ with $(\tilde{u}(s') - \tilde{u}(\tilde{s}')) - (\tilde{u}(s) - \tilde{u}(\tilde{s}))$) benchmark-based counterparts to all the characterization results of section 3. In particular,

$$(s,\tilde{s},t;s',\tilde{s}',t') \mapsto \frac{(s' - \tilde{s}') - (s - \tilde{s})}{t' - t}, \; (s,\tilde{s},t;s',\tilde{s}',t') \mapsto \left(\frac{s'/\tilde{s}'}{s/\tilde{s}} \right)^{\frac{1}{t'-t}} - 1 \tag{12}$$

are the only proper benchmark-based analogues of the ARC measures (1) and (2), respectively. From the benchmark-based counterpart of Proposition 1(b) it follows that if a relative ARC ordering $\succeq$ satisfies B1, B2, (i)–(v), then for any benchmark dynamics $p : \mathrm{T} \to \mathrm{S}$ the restriction of $\succeq$ to the set $\{(s, p(t), t; s', p(t'), t') : (s,t;s',t') \in \mathrm{V}\}$ produces an ARC ordering induced by a multiplicative discounting TP-family. That is, the case of a fixed benchmark is covered by the non-benchmark case, as in example (11).

5. An extension[5]

In this section, a more general model of ARC measurement is outlined. It parallels closely the conditional utility formalism (Krantz et al., 1971, chapter 8) and generalizes the approach employed above in two directions. First, it permits calculation of an ARC over a set of time points other than an interval. The following example illustrates this inquiry. Suppose a researcher is interested in comparing the performance of the U.S. economy under Democratic vs. Republican presidents. In order to measure economic performance under, say, Democratic presidents, the researcher considers calculating the economic growth (productivity growth, stock market returns or any other measure of interest) during each Democratic presidential term since World War II using formula (5) and taking the time-weighted arithmetic mean. In this section, we show that this is the only proper extension of a measure of economic growth over an interval to a finite union of bounded disjoint intervals. Similar extensions for other ARC measures axiomatized in section 3 are also presented. Second, the model allows an ARC over a specified time interval to be dependent on the entire path of the

[5] The ideas in this section are inspired by an anonymous reviewer of our previous paper AS, to whom we are grateful.

outcome variable rather than its values at the endpoints. Path-dependent ARC measures and measures of change are of interest, e.g., in measurement of aggregate price/quantity change (Balk, 2008, chapter 6), productivity (Richter, 1966; Jorgenson and Griliches, 1967) and welfare (Bruce, 1977; Malaney, 1996, chapter 2; Cysne, 2003) measurement. Several characterization results on the structure of path-dependent measures of change are available in the literature. E.g., Richter (1966) provided an axiomatic foundation for the Divisia index (see also Malaney, 1996, chapter 1). Grodal and Vind (Vind, 2003, chapters 8, 11, 12) proposed a general construct, with the help of which the structure of path-dependent measures of change can be described. However, the general structure of path-dependent ARC measures remains unclear. In this section, a result in this direction is presented: we derive path-dependent counterparts for ARC measures induced by a relative and multiplicative discounting TP-family.

The model has two new primitives: a non-empty collection $\mathcal{P}$ of functions from T to S interpreted as possible paths of the variable and a non-empty collection $\mathcal{B}$ of subsets of T. Given $p \in \mathcal{P}$ and $B \in \mathcal{B}$, the restriction of p to B is denoted by p_B. Throughout this section, by an ARC ordering we mean a non-trivial total preorder $\succeq$ on $\mathcal{H}(\mathcal{P},\mathcal{B}) := \{p_B : p \in \mathcal{P}, B \in \mathcal{B}\}$.[6] The statement $p_A \succeq q_B$ means that the ARC of p over the set A is no less than the ARC of q over B. For any $B \in \mathcal{B}$, the restriction of $\succeq$ to $\mathcal{P}_B := \{p_B, p \in \mathcal{P}\}$, the ordering measuring change over B, is denoted by $\succeq_B$. To simplify the notation, we will write $p \succeq_B q$ instead of $p_B \succeq_B q_B$. Given a set $\mathcal{E}$, the collection of all finite unions of elements of $\mathcal{E}$ is denoted by $\mathcal{R}(\mathcal{E})$. Put $\mathcal{I} := \{(t,t'] : (t,t') \in \mathrm{T}^2_<\}$. For a function $f : \mathrm{T} \to \mathrm{R}$, by $\mu_f : \mathcal{R}(\mathcal{I}) \to \mathrm{R}$ we mean the finitely additive set function whose restriction to $\mathcal{I}$ is given by $\mu_f((t,t']) := f(t') - f(t)$.

ASSUMPTION A. T is order-dense (i.e., for all $(t,t'') \in \mathrm{T}^2_<$, there is $t' \in \mathrm{T}$ such that $t < t' < t''$); S is Hausdorff; $\mathcal{P}$ is the set of all right-continuous[7] functions from T to S.

Under assumption A the model reduces to the one considered in the previous sections provided that $\mathcal{B} = \mathcal{I}$ and the following path-independence condition holds:

(PI) $\quad p \sim_{(t,t']} q$, whenever $(t,t'] \in \mathcal{B}$, $p(t) = q(t)$, and $p(t') = q(t')$.

Note that under assumption A, condition (PI) is well defined (i.e., the values $p(t)$ and $q(t)$ are completely determined by $p_{(t,t']}$ and $q_{(t,t']}$). In what follows, under assumption A and condition (PI), we identify an ARC ordering $\succeq$ on $\mathcal{H}(\mathcal{P},\mathcal{I})$ with the ARC ordering $\succeq'$ on V given by $(p(t),t;p(t'),t') \succeq' (q(\tau),\tau;q(\tau'),\tau') \Leftrightarrow p_{(t,t']} \succeq q_{(\tau,\tau']}$.

The next five conditions are straightforward generalizations of (i)–(v), respectively.

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[6] Benchmark-based approach outlined in section 4 can be incorporated in the present model by means of a total preorder on the set $\{(p_B, \widetilde{p}_B) : p, \widetilde{p} \in \mathcal{P}, B \in \mathcal{B}\}$; we omit the details.

[7] A function from T to S is said to be right-continuous if it is continuous with respect to the right half-open interval topology (the topology on T generated by the subbasis consisting of the sets $\{\tau \in \mathrm{T} : \tau \geq t\}$, $\{\tau \in \mathrm{T} : \tau < t\}$, $t \in \mathrm{T}$). The right-continuity condition is essential provided that $\mathcal{I} \subseteq \mathcal{B}$. To motivate it assume that $\mathrm{S} = \mathrm{T} = \mathrm{R}$, $\mathcal{I} \subseteq \mathcal{B}$ and consider the path $p(t) = 1_{(1,+\infty)}(t)$ (where $1_{(1,+\infty)}$ is the indicator function of the set $(1,+\infty)$), which is discontinuous from the right. Then we expect the ARCs over the intervals $(0,1]$ and $(1,2]$ to be zero (since $p_{(0,1]}$ and $p_{(1,2]}$ are constant functions), while the ARC over the consolidated period $(0,2]$ is expected to be positive (since p grows over $(0,2]$). The right-continuity condition excludes this possibility.

(I) for any $B \in \mathcal{B}$, $\succeq_B$ is continuous in the topology of pointwise convergence (the topology on $\{p_B, p \in \mathcal{P}\}$ inherited from the product topology on S^B).

(II) $p \succeq_A q$ & $p \succeq_B q \Rightarrow p \succeq_{A\cup B} q$ provided that $A \cup B \in \mathcal{B}$ and $A \cap B = \varnothing$. Furthermore, if either antecedent inequality is strict, so is the conclusion.

(III) $p \sim_B q$, whenever p_B and q_B are constant functions.

(IV) for any $A \in \mathcal{B}$ and $q_B \in \mathcal{H}(\mathcal{P},\mathcal{B})$, there exists $p \in \mathcal{P}$ such that $p_A \sim q_B$.

(V) $p_A \sim p_B$ implies $p_A \sim p_{A\cup B}$ provided that $A \cup B \in \mathcal{B}$ and $A \cap B = \varnothing$.

We also consider the following strengthening of condition (IV):

(IV)' for every natural n, pairwise disjoint sets $A_1,\ldots,A_n \in \mathcal{B}$, and $q^{(1)}_{B_1},\ldots,q^{(n)}_{B_n} \in \mathcal{H}(\mathcal{P},\mathcal{B})$, there exists $p \in \mathcal{P}$ such that $p_{A_i} \sim q^{(i)}_{B_i}$, $i = 1,\ldots,n$.

Condition (IV)' requires the set $\mathcal{P}$ to be sufficiently rich. If $\mathcal{P} = \mathrm{S}^{\mathrm{T}}$, then conditions (IV) and (IV)' are equivalent. Since $\succeq$ is non-trivial, (IV) implies that $\varnothing \notin \mathcal{B}$, while (III) and (IV) imply that $\mathcal{B}$ does not contain singletons. On the other hand, the combination of conditions (I)–(IV) does not allow both the set $\mathcal{P}$ and the collection $\mathcal{B}$ to be sufficiently rich. For instance, one can show that, if $\mathcal{P} = \mathrm{S}^{\mathrm{T}}$ and $\mathcal{I} \subseteq \mathcal{B}$, then (I)–(IV) are inconsistent.[8]

Conditions similar to (II) and (V) are used in Krantz et al. (1971, chapter 8) to axiomatize conditional utility: (II) is a strong form of their independence axiom (ibid., Axiom 4, p. 380), while (V) is closely related to the union indifference axiom (ibid., Axiom 3, p. 380). A slightly stronger condition than (V), the averaging axiom, is used in axiomatizations of subjective expected utility and conditional expected utility (Bolker, 1966, 1967; Fishburn, 1972; Jeffrey, 1978; Ahn, 2008; Gravel et al., 2018). In terms of the present model, the averaging axiom asserts that an ARC over a consolidated period must fall between ARCs over its subperiods. Provided that $\mathcal{P}$ is a singleton and $\mathcal{B}$ is a σ-algebra, the structure of ARC orderings satisfying the averaging axiom and an independence condition (called impartiality) is characterized in the Bolker-Jeffrey version of expected utility theory.

The combination of conditions (II) and (V) has a straightforward cardinal interpretation. Namely, if (IV)' holds and I represents $\succeq$, then (II) and (V) hold if and only if for any disjoint sets $A, B \in \mathcal{B}$ with $A \cup B \in \mathcal{B}$, there exists a mean $M_{A,B}(\cdot,\cdot)$ (i.e., $\min\{a,b\} \le M_{A,B}(a,b) \le \max\{a,b\}$) such that $M_{A,B}$ is strictly increasing in both arguments and

$$I(p_{A\cup B}) = M_{A,B}(I(p_A), I(p_B)) \quad \forall p \in \mathcal{P}. \tag{13}$$

For instance, for the ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$ with a numerical representation $p_E \mapsto \mu_{\tilde{u}\circ p}(E)/\mu_\phi(E)$ (characterized in Corollary 3 below), where $\tilde{u}: \mathrm{S} \to \mathrm{R}$ and $\phi: \mathrm{T} \to \mathrm{R}$ is strictly increasing, $M_{A,B}$ is the time-weighted arithmetic mean,

$$M_{A,B}(a,b) = \frac{\mu_\phi(A)}{\mu_\phi(A\cup B)}a + \frac{\mu_\phi(B)}{\mu_\phi(A\cup B)}b.$$

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[8] Pick $E \in \mathcal{I}$ and a constant function q_E. Let p_E be an S-valued step function subordinated to a partition $E_1,\ldots,E_n \in \mathcal{I}$ of E (that is, p_{E_i}, $i = 1,\ldots,n$ are constant functions). From (II) and (III) it follows that $p_E \sim q_E$. Since the set of all step functions is dense in S^E (in the topology of pointwise convergence) and, by condition (I), the set $\{p_E \in \mathrm{S}^E : p_E \sim q_E\}$ is closed, $\succeq_E$ is trivial.

Our main goal here is to obtain a natural extension for ARC measures described in the previous sections to a more general class of sets of time points than intervals. We begin with the following simple result that provides conditions under which an ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{B}))$ (resp. the family $\{\succeq_B, B\in\mathcal{R}(\mathcal{B})\}$ of orderings measuring change) is completely determined by its restriction to $\mathcal{H}(\mathcal{P},\mathcal{B})$ (resp. by the subfamily $\{\succeq_B, B\in\mathcal{B}\}$).

Lemma 7.

Let $\mathcal{E}$ be a non-empty collection of subsets of T such that for any $E\in\mathcal{R}(\mathcal{E})$ there are $m\geq 1$, $n\geq 0$, and pairwise disjoint sets $A_1,\dots,A_{m+n}\in\mathcal{E}$ such that $\bigcup_{i=1}^{m} A_i = E$ and $\bigcup_{i=1}^{m+n} A_i\in\mathcal{E}$. Let $\succeq$ and $\succeq'$ be ARC orderings on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{E}))$.

(a) If $\succeq$ and $\succeq'$ satisfy (II), (IV)', and $\succeq_C=\succeq'_C$ for any $C\in\mathcal{E}$, then the equality remains valid for all $C\in\mathcal{R}(\mathcal{E})$.

(b) If $\succeq$ and $\succeq'$ satisfy (II), (IV)', (V), and their restrictions to $\mathcal{H}(\mathcal{P},\mathcal{E})$ coincide, then $\succeq=\succeq'$.

The proof of Lemma 7 does not use the order structure of the space T. Since the collection of intervals $\mathcal{I}$ satisfies the conditions imposed on $\mathcal{E}$ in Lemma 7, under the conditions of Lemma 7, there exists at most one extension of an ARC ordering (and the induced collection of orderings measuring change) described in the previous sections to $\mathcal{R}(\mathcal{I})$. The following proposition generalizes Lemma 3 and Proposition 1. It provides extensions of a measure of change in the form of a utility difference and an ARC measure induced by a relative or multiplicative discounting TP-family to sets of time points being finite unions of bounded intervals.

Proposition 4.

Let assumption A hold and $\succeq$ be an ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$.

(a) (PI) and (I)–(III) hold if and only if there exists a continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ such that for any $E\in\mathcal{R}(\mathcal{I})$ the map on $\mathcal{P}_E$ given by $p_E\mapsto\mu_{\tilde{u}\circ p}(E)$ represents $\succeq_E$.

(b) (PI) and (I)–(IV) hold if and only if there exist a non-constant continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ and a collection $\{\phi_E: n(E)\mathrm{D}\xrightarrow{\text{onto}}\mathrm{D},\ E\in\mathcal{R}(\mathcal{I})\}$ of strictly increasing continuous functions such that $p_E\mapsto\phi_E(\mu_{\tilde{u}\circ p}(E))$ represents $\succeq$. Here $\mathrm{D}:=\{\tilde{u}(s')-\tilde{u}(s),\ s,s'\in\mathrm{S}\}$ and $n(E):=\min\left\{n:\exists E_1,\dots,E_n\in\mathcal{I},\ \bigcup_{i=1}^{n}E_i=E\right\}$.

(c) (PI) and (I)–(V) hold if and only if there exist an unbounded continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ and a function $\tilde{g}:\mathrm{R}\times\mathrm{T}\to\mathrm{R}$ such that for each $C\in\mathcal{I}$ $d\mapsto-\mu_{\tilde{g}(d,\cdot)}(C)$ is a strictly increasing self-homeomorphism of R and the map that sends each $p_E\in\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$ to a unique solution d of the equation $\mu_{\tilde{u}\circ p}(E)+\mu_{\tilde{g}(d,\cdot)}(E)=0$ represents $\succeq$.

It follows from Lemma 3 and Proposition 4(a) that under assumption A and (PI), a collection $\{\succeq_E, E\in\mathcal{I}\}$ of total preorders ($\succeq_E$ is assumed to be defined on $\mathcal{P}_E$) satisfying (I)–(III) can be complemented in a unique way to a collection $\{\succeq_E, E\in\mathcal{R}(\mathcal{I})\}$ with the same properties. By Propositions 1(b) and 4(c), under assumption A and (PI), an ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{I})$ satisfying

(I)–(V) has a unique extension to $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$ with the same properties. From Proposition 4(b) it follows that an ARC over $E \in \mathcal{R}(\mathcal{I})$ is a function $\mu_{\tilde{u}\circ p}(E)$ and E. This representation supports the intuition of Doyle (2013, section 7.1) that an ARC over an interval E is the composition of the effects of reward over E and time perception of E for a more general set of time points than an interval.

The next two corollaries are direct consequences of Propositions 2(c), 3(d), 3(f), 4(c), and Lemma 7(b); their proofs are omitted. They provide extensions of ARC orderings induced by uniform multiplicative and exponential discounting TP-families to $\mathcal{R}(\mathcal{I})$.

Corollary 3.

Let T be connected, assumption A hold, and $\succeq$ be an ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$. The following two statements are equivalent:

(a) (PI), (II)–(V) hold and the restriction of $\succeq$ to $\mathcal{H}(\mathcal{P},\mathcal{I})$ satisfies (ix) and is continuous;

(b) there exist continuous functions $\tilde{u}:\mathrm{S}\to\mathrm{R}$ and $\phi:\mathrm{T}\to\mathrm{R}$ such that $\tilde{u}$ is unbounded, ϕ is strictly increasing, and $p_E \mapsto \mu_{\tilde{u}\circ p}(E)/\mu_{\phi}(E)$ represents $\succeq$.

Corollary 4.

Let $\mathrm{T}=\mathrm{R}$, assumption A hold, and $\succeq$ be an ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$. The following statements are equivalent:

(a) (PI), (II)–(V) hold and the restriction of $\succeq$ to $\mathcal{H}(\mathcal{P},\mathcal{I})$ satisfies (x)/(x)' and is continuous;

(b) (PI), (I)–(V) hold and the restriction of $\succeq$ to $\mathcal{H}(\mathcal{P},\mathcal{I})$ satisfies (vi)/(vi)'/(viii)/(ix) and (x);

(c) there exists an unbounded continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ such that $p_E \mapsto \mu_{\tilde{u}\circ p}(E)/\mu_{id}(E)$, where id is the identity function on R, represents $\succeq$.

The ARC obtained in Corollary 3 (resp. Corollary 4) is simply the average rate of utility growth per perceived (resp. operational) time over a considered set of time points. The representations of this type are closely related to those used in mono-set subjective expected utility theories (Jeffrey, 1965, 1978; Bolker, 1966, 1967; Domotor, 1978).

It is not hard to show that if $\mathcal{I}\subseteq\mathcal{B}\subseteq\mathcal{R}(\mathcal{I})$ and assumption A holds, then the combination of conditions (PI) and (III) is equivalent to the following one:

(III)' $p \sim_{(t,t']} q$, whenever $p(t)=p(t')$, and $q(t)=q(t')$.

Thus, Proposition 4 and Corollaries 3, 4 remain valid if conditions (PI) and (III) are replaced by (III)'.

Our final result does not assume neither condition (PI) nor (III) to hold (at the price of strengthening condition (IV)) and, therefore, provides intuition for the structure of path-dependent ARC measures.

Proposition 5.

Let assumption A hold and $\succeq$ be an ARC ordering on $\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$.

(a) If (I), (II), (IV)' hold, then there exist a proper real interval D, a map $U:\mathcal{P}\to\mathrm{R}^{\mathrm{T}}$, and a collection $\{\phi_E:\mu_{U(\mathcal{P})}(E)\xrightarrow{\text{onto}}\mathrm{D},\ E\in\mathcal{R}(\mathcal{I})\}$ of strictly increasing continuous functions such that $p_E \mapsto \phi_E\circ\mu_{U(p)}(E)$ represents $\succeq$.

(b) If (I), (II), (IV)', (V) hold, then there exist a proper real interval D, a map $U:\mathcal{P}\to\mathrm{R}^{\mathrm{T}}$, and a function $\widetilde{g}:\mathrm{D}\times\mathrm{T}\to\mathrm{R}$ such that for any $E\in\mathcal{I}$, $d\mapsto-\mu_{\widetilde{g}(d,\cdot)}(E)$ is strictly increasing and onto $\mu_{U(\mathcal{P})}(E)$ and the function that sends each $p_E\in\mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{I}))$ to a unique solution d of the equation $\mu_{U(p)}(E)+\mu_{\widetilde{g}(d,\cdot)}(E)=0$ represents $\succeq$.

Statements (a) and (b) become "if and only if" if for each $t\in\mathrm{T}$, $p\mapsto U(p)(t)$ is continuous; for any $E\in\mathcal{I}$, $\mu_{U(p)}(E)=\mu_{U(q)}(E)$ if $p_E=q_E$; for any pairwise disjoint $E_1,\ldots,E_n\in\mathcal{R}(\mathcal{I})$, $\{(\mu_{U(p)}(E_1),\ldots,\mu_{U(p)}(E_n)),p\in\mathcal{P}\}=\prod_{i=1}^{n}\mu_{U(\mathcal{P})}(E_i)$.

If, in addition to the conditions of part (a)/(b), condition (III) holds, then U can be chosen to satisfy $U(p)\equiv 0$, whenever p is a constant function. If condition (PI) holds, then there is a function $\widetilde{u}:\mathrm{S}\times\mathrm{T}\to\mathrm{R}$ such that $U(p)(t)=\widetilde{u}(p(t),t)$; for each $t\in\mathrm{T}$, $\widetilde{u}(\cdot,t)$ is continuous and for any $E\in\mathcal{I}$, $\mu_{U(\mathcal{P})}(E)=\mathrm{R}$.

Proposition 5(b) solves the aggregation problem for ARC measurement. In order to motivate it assume that we are given a finite number of ARCs over disjoint sets, what is the ARC over the union of the sets? As we know, under conditions (II), (IV)', and (V), the answer must be an average of the ARCs (Eq. (13)), but which one (arithmetic, geometric, harmonic)? Shall it be weighted or unweighted? For instance, from Corollary 4 it follows that if the ARC is measured by formula (5), then, regardless of $\widetilde{u}$, the aggregation function is the time-weighted arithmetic mean. Propositions 4(c) and 5(b) show that, in general, the aggregation function is a generalized quasi-arithmetic mean (for properties of a generalized quasi-arithmetic mean, see Matkowski (2010)) regardless of whether the ARC is path-independent or path-dependent. Indeed, let I be the representation of $\succeq$ obtained in Proposition 4(c) or 5(b). Given $p\in\mathcal{P}$, pairwise disjoint sets $E_1,\ldots,E_n\in\mathcal{R}(\mathcal{I})$, and the ARCs $I(p_{E_1}),\ldots,I(p_{E_n})$, we have

$$I(p_{E_1\cup\ldots\cup E_n})=\left(\sum_{i=1}^{n}\varphi_{E_i}\right)^{-1}\left(\sum_{i=1}^{n}\varphi_{E_i}\circ I(p_{E_i})\right),\tag{14}$$

where the functions $\varphi_{E_i}(d):=-\mu_{\widetilde{g}(d,\cdot)}(E_i)$, $i=1,\ldots,n$ are strictly increasing and continuous. Using the inclusion-exclusion principle, formula (14) can be generalized to overlapping sets.

From the proof it follows that the "if" parts of Proposition 5(a), (b) remain valid if condition (IV)' holds for $n=1,2,3$. It is not hard to show that the "if" part of Proposition 5(a) remains valid if (IV)' is replaced by (III) and (IV), provided that T has no least or greatest elements (we omit the details). In this case, Proposition 5(a) is a direct generalization of Proposition 4(b) to path-independence ARC. From Proposition 4(c) it follows that under assumption A, (PI), (I), (II), (III), (V), conditions (IV) and (IV)' are equivalent; so that Proposition 5(b) generalizes Proposition 4(c).

From Proposition 5 it follows that under (PI), (I), (II), (IV)', (V), the restriction of $\succeq$ to $\mathcal{H}(\mathcal{P},\mathcal{I})$ is induced by a TP-family $\{\trianglerighteq_d,\ d\in\mathrm{R}\}$ with discounted time-dependent utility, given by

$$(s,t)\trianglerighteq_d(s',t')\ \Leftrightarrow\ u(s,t)g(d,t)\geq u(s',t')g(d,t'),$$

where $u:=e^{\widetilde{u}}$, $g:=e^{\widetilde{g}}$.

A measure of change $\mu_{U(p)}$ obtained in Proposition 5(a), (b) is a path-dependent counterpart for a measure of change in the form of utility difference. We close this section with a discussion of the structure of $\mu_{U(p)}$ in the special case when $T = R$ and $U(p)$ is differentiable at all but countably many points and continuous. These assumptions are consistent with how change is usually measured in continuous time using Divisia approach. In this case, since $\mu_{U(p)}(E) = \mu_{U(q)}(E)$ if $p_E = q_E$, the derivative of $U(p)$ at t, if it exists, is completely determined by the germ $[p]_t$ of p at t (the equivalence class of functions that agree with p at a neighborhood of t). Therefore, there is a function $f : \{[p]_\tau : \tau \in R\} \to R$ such that $\tau \mapsto f([p]_\tau)$ is Henstock-Kurzweil integrable on any bounded interval and for any $E \in \mathcal{R}(\mathcal{I})$,

$$\mu_{U(p)}(E) = \int_E f([p]_\tau) d\tau \,, \tag{15}$$

where the integral is the Henstock-Kurzweil integral. The integral representation (15) comprises, as a special case, the Divisia price/quantity index (indeed, the time derivative of the logarithm of the Divisia index is completely determined by the germ of the vector-valued function describing the underlying price and quantity dynamics).

6. Conclusion

This paper contributes to the problem of average rate of change measurement. We revisit the axiomatic foundations of the ARC formulas and explore the limits of their applicability by asking how much these axioms can be relaxed to accommodate some non-standard settings. Much of our discussion relies on the duality between ARC measurement and time preferences that we established in our earlier work (AS). In particular, we show (Lemma 1) that isolating a discount rate parameter in a discount equation, as proposed by Doyle (2013), is a general way to construct an ARC measure.[9] An ARC derived this way is a tool to measure a perceived average rate of growth of an agent's utility. We provide both heuristic and axiomatic characterizations of a variety of ARC measures. We show that ARC measures can be derived from some basic time preference models, as well as from some natural axiomatic foundations. We also describe the structure of path-dependent ARCs (Proposition 5) and solve the aggregation problem for ARC measurement. Below we summarize the answers to our six questions about the generalizability of the ARC formulas (1) and (2) that we stated at the beginning of the paper.

1. What are the analogues of the ARC formulas for an outcome variable taking values in an abstract space?

Up to an order-preserving transformation, (5) is the only stationary (time-shift invariant) generalization of the ARC measures (1) and (2) for an outcome variable with a connected domain (Proposition 3(c), (d), (f)). Put differently, the ARC should be evaluated using a two-stage procedure which first assigns a numerical value to the outcome variable and then applies formula (1).

2. How does a non-stationary counterpart to the ARC formulas look like?

Every non-stationary ARC measure that satisfies some natural axioms is defined implicitly as a solution d of Eq. (6) (Proposition 1(b)), where u and $g(d,\cdot)$ can be interpreted, respectively, as an instantaneous utility function and a discount function that corresponds to the discount rate d. In other words, under some natural conditions discounted utility is the most general time preference model consistent with ARC measurement.

[9] Moreover, Table 1 in Doyle (2013, p. 121) comprises precisely ARC measures induced by most currently popular time preference models over money.

3. Is there an analogue of the ARC formulas if observations of the outcome variable are ordered by a variable (not necessarily time) whose domain is a linear order?

There is no natural counterpart of the ARC formulas (1) and (2) for a general ordering variable unless its domain T is order isomorphic to a subset of R with the usual order (Lemma 2(b)). The case $\mathrm{T} \subseteq \mathrm{R}$ is particularly convenient due to its representability under mild conditions (Lemma 4).

4. What is a benchmark-based counterpart to the ARC formulas?

The benchmark-based counterparts to the ARC measures (1) and (2) are given by (12). The case of a fixed benchmark is covered by a non-stationary non-benchmark ARC measure, as in (11).

5. How to measure ARC over a set of time points other than an interval?

A natural extension of the common ARC measure (5) to sets of time points being finite unions of bounded proper intervals is still the quotient of a measure of change to elapsed time (Corollary 4).

6. What is a path-dependent counterpart to the ARC formulas?

A path-dependent analogue of (1) involves replacing the measure of change (i.e., the nominator in (1)) with its path-dependent counterpart given by (15) with $E = (t, t']$.

We close with a discussion of the topological assumptions used in the paper, the continuity assumptions imposed on $\succeq$ (continuity of $\succeq$, conditions (i) and (viii)) and connectedness of the state space S. Topological assumptions are made to preserve the analytical tractability of the problem; however, they often suffer from lack of economic interpretation and inability to be falsified by a finite number of observations. Though this is the case for the continuity assumptions used in the paper, they may have an economic interpretation in the presence of supplementary conditions.[10] Indeed, from Proposition 3(d), (e), (f) it follows that under (i)–(v) and (x), the following five conditions are equivalent: continuity of $\succeq$, (vi), (vi)', (viii), and (ix). Since conditions (vi), (vi)', and (ix) have an economic interpretation and are falsifiable, so are the two remaining conditions. In the TP-family domain, condition (viii) itself has a clear behavioral content, the absence of present/future bias. While in this paper we deal with a connected state space, our key result, Lemma 1, which relates ARC measurement to time preference, does not use the connectedness and continuity assumptions. At least for the common ARC measures (1) and (2), connectedness is a necessary condition for the notion of equivalent average rate of change (by analogy with equivalent change in Frenzen (1994, section 2) and Aczél et al. (1996)) – condition (iv) – to be well defined. Some results on representations of ARC orderings are also available in AS, where we replace the connectedness assumption with separability, which is a non-restrictive assumption in an empirically relevant special case of countable state space.

7. Appendix. Proofs

Proof of Lemma 1.

(a). Let $\succeq$ be an ARC ordering. Since $\succeq$ is a total preorder, the quotient set $\mathrm{D} := \mathrm{V}/\sim$ is linearly ordered in a natural way. Let $\rangle$ be the corresponding linear order on D. Define the family $\{\trianglerighteq_d,\ d \in \mathrm{D}\}$, $(\mathrm{D}, \rangle)$ of binary relations on $\mathrm{S} \times \mathrm{T}$ by $\trianglerighteq_{[v]} := \mathrm{L}_{\succeq}(v) \cup \mathrm{U}_{\succeq}(v)^{-1} \cup \Delta$, where $[v] \in \mathrm{D}$ is the equivalence class of $v \in \mathrm{V}$. By construction, $\trianglerighteq_{[v]}$ is well defined (i.e., is independent of a representative of $[v]$), non-trivial, and complete:

[10] We borrow this idea from Wakker (1988).

$$\trianglerighteq_{[v]} \cup \trianglerighteq_{[v]}^{-1} = \mathrm{L}_{\succeq}(v) \cup \mathrm{U}_{\succeq}(v)^{-1} \cup \Delta \cup \mathrm{L}_{\succeq}(v)^{-1} \cup \mathrm{U}_{\succeq}(v) \cup \Delta^{-1} = (\mathrm{S}\times\mathrm{T})^2 .$$

It is straightforward to show that 2º and 3º hold and (3) induces $\succeq$.

Let us show that if $\succeq$ is continuous, then so is $\trianglerighteq_{[v]}$. Since T is equipped with the order topology, the sets Δ, $\mathrm{V} \cup \Delta$, and $\mathrm{V}^{-1} \cup \Delta$ are closed in $(\mathrm{S}\times\mathrm{T})^2$. Let cl and cl_{V} be the topological closure operators in $(\mathrm{S}\times\mathrm{T})^2$ and V, respectively. We have $\mathrm{cl}(\mathrm{U}_{\succeq}(v)^{-1}) \cap \mathrm{V} \subseteq \mathrm{cl}(\mathrm{V}^{-1} \cup \Delta) \cap \mathrm{V} = (\mathrm{V}^{-1} \cup \Delta) \cap \mathrm{V} = \varnothing$, $\mathrm{cl}(\Delta) \cap \mathrm{V} = \varnothing$, and, therefore,

$$\mathrm{cl}(\trianglerighteq_{[v]}) \cap \mathrm{V} = \left(\mathrm{cl}(\mathrm{L}_{\succeq}(v)) \cap \mathrm{V}\right) \cup \left(\mathrm{cl}(\mathrm{U}_{\succeq}(v)^{-1}) \cap \mathrm{V}\right) \cup \left(\mathrm{cl}(\Delta) \cap \mathrm{V}\right) = \mathrm{cl}_{\mathrm{V}}(\mathrm{L}_{\succeq}(v)) = \mathrm{L}_{\succeq}(v) .$$

Since $(s,t;s',t') \mapsto (s',t';s,t)$ is a self-homeomorphism of $(\mathrm{S}\times\mathrm{T})^2$, a similar argument shows that $\mathrm{cl}(\trianglerighteq_{[v]}) \cap \mathrm{V}^{-1} = \mathrm{U}_{\succeq}(v)^{-1}$. Thus,

$$\mathrm{cl}(\trianglerighteq_{[v]}) = \left(\mathrm{cl}(\trianglerighteq_{[v]}) \cap \mathrm{V}\right) \cup \left(\mathrm{cl}(\trianglerighteq_{[v]}) \cap \mathrm{V}^{-1}\right) \cup \left(\mathrm{cl}(\trianglerighteq_{[v]}) \cap \Delta\right) = \mathrm{L}_{\succeq}(v) \cup \mathrm{U}_{\succeq}(v)^{-1} \cup \Delta = \trianglerighteq_{[v]} .$$

Conversely, given a TP-family $\{\trianglerighteq_d, d \in \mathrm{D}\}$, conditions 1º–3º imply that the binary relation $\succeq$ on V defined by (3)(4) is a non-trivial total preorder. If the TP-family is continuous, then so is $\succeq$. Indeed, for every $v \in \mathrm{V}$, $\mathrm{L}_{\succeq}(v)$ (resp. $\mathrm{U}_{\succeq}(v)$) is closed in V as the intersection of the closed set $\trianglerighteq_{IRR(v)}$ (resp. $\trianglerighteq_{IRR(v)}^{-1}$) and V.

(b). Trivial.

Assume that $\succeq$ is induced by TP-families $\{\trianglerighteq_d, d \in \mathrm{D}\}$, $(\mathrm{D}, \underline{\succ})$ and $\{\trianglerighteq'_{d'}, d' \in \mathrm{D}'\}$, $(\mathrm{D}', \underline{\succ}')$. Since the induced relation $\succeq$ does not depend on elements of the TP-families outside the cores, in what follows we assume that the TP-families coincide with their cores. That is,

$$\forall\, d \in \mathrm{D} \text{ (resp. } d' \in \mathrm{D}') \ \exists (s,t),\ (s',t'),\ t \neq t' :\ (s,t) \sim_d (s',t') \text{ (resp. } (s,t) \sim'_{d'} (s',t')). \qquad (16)$$

Let $IRR : \mathrm{V} \to \mathrm{D}$ and $IRR' : \mathrm{V} \to \mathrm{D}'$ be the corresponding IRR mappings and let $\varphi : \mathrm{D} \to \mathrm{D}'$ be the mapping that takes each $d \in \mathrm{D}$ to $IRR'(v)$, where $v \in \mathrm{V}$ is a solution of the equation $IRR(v) = d$. Conditions 2º, 3º, and (16) imply that φ is well defined. It is straightforward to verify that φ is an order isomorphism from $(\mathrm{D}, \underline{\succ})$ to $(\mathrm{D}', \underline{\succ}')$. By (3)(4), for any $v \in \mathrm{V}$

$\trianglerighteq_{IRR(v)} \cap \mathrm{V} = \mathrm{L}_{\succeq}(v) = \trianglerighteq'_{IRR'(v)} \cap \mathrm{V} = \trianglerighteq'_{\varphi(IRR(v))} \cap \mathrm{V}$, $\trianglerighteq_{IRR(v)}^{-1} \cap \mathrm{V} = \mathrm{U}_{\succeq}(v) = \trianglerighteq'^{-1}_{IRR'(v)} \cap \mathrm{V} = \trianglerighteq'^{-1}_{\varphi(IRR(v))} \cap \mathrm{V}$,

and, therefore, $\trianglerighteq_d \approx \trianglerighteq'_{\varphi(d)}$ for all $d \in \mathrm{D}$.

The converse is straightforward. ■

Proof of Lemma 2.

The inconsistency result for part (a) (resp. (b)) follows from Proposition 1(a) (resp. Proposition 1(b)) below, where it is proved that an ARC ordering satisfies (i)–(iv) (resp. (i)–(v)) if and only if it is induced by a relative (resp. multiplicative) discounting TP-family and, therefore, there is a non-constant continuous real-valued function on S (resp. S is non-pseudocompact and T is order isomorphic to a subset of R).

We shall prove the independence result only for part (b). The remaining statement can be established in a similar manner. To show independence we provide five examples of ARC orderings that satisfy four of the conditions (i)–(v) while violating the fifth. In all the examples $\mathrm{T} \subseteq \mathrm{R}$. Note that since S is connected and non-pseudocompact, there exists a continuous real-valued function on S whose image is an open interval.

The ARC ordering represented by (5) with discontinuous $\tilde{u}:\mathrm{S}\xrightarrow{\text{onto}}\mathrm{R}$ satisfies all the conditions except continuity.

The ARC ordering represented by

$$(s,t;s',t')\mapsto\begin{cases}\dfrac{\tilde{u}(s')-\tilde{u}(s)}{t'-t} & \text{if } \tilde{u}(s')\geq\tilde{u}(s)\\ 0 & \text{otherwise}\end{cases},$$

where $\tilde{u}:\mathrm{S}\xrightarrow{\text{onto}}\mathrm{R}$ is a continuous function, meets all the conditions except (ii).

Let $\{u(\cdot,t):\mathrm{S}\xrightarrow{\text{onto}}\mathrm{R}_{++},\, t\in\mathrm{T}\}$ be a collection of continuous functions and let $g:\mathrm{R}\times\mathrm{T}\to\mathrm{R}_{++}$ satisfy the conditions imposed on a discount function in a multiplicative discounting TP-family. If there exist τ and τ' such that the function $s\mapsto u(s,\tau)/u(s,\tau')$ is non-constant, then the ARC ordering induced by the TP-family $\{\trianglerighteq_d,\, d\in\mathrm{R}\}$,

$$(s,t)\trianglerighteq_d(s',t') \iff u(s,t)g(d,t)\geq u(s',t')g(d,t'),$$

satisfies all the conditions except (iii).

The ARC ordering represented by (5), where $\tilde{u}:\mathrm{S}\to\mathrm{R}$ is a non-constant bounded continuous function, meets all the conditions except (iv).

Finally, the ARC ordering represented by $(s,t;s',t')\mapsto\tilde{u}(s')-\tilde{u}(s)$, where $\tilde{u}:\mathrm{S}\xrightarrow{\text{onto}}\mathrm{R}$ is a continuous function, satisfies all the conditions except (v). ■

Proof of Lemma 3.

(a)$\Rightarrow$(b). Applying (ii) with $s=s'$ and $r=r'$ (resp. $s'=s''$ and $r'=r''$) and using (iii), we get

$$(s,s')\succeq_{t,t''}(r,r') \iff (s,s')\succeq_{t',t''}(r,r') \text{ (resp. } (s,s')\succeq_{t,t'}(r,r')\text{)}.$$

Therefore, the relation $\succeq_{t,t'}$ is independent of the choice of t and t'. Put $\succcurlyeq:=\succeq_{t,t'}$. By (ii), if $(s,s')\succcurlyeq(r,r')$ and $(s',s'')\succcurlyeq(r',r'')$, then $(s,s'')\succcurlyeq(r,r'')$ and if either antecedent inequality is strict, so is the conclusion. Applying this implication with $s=s''$, $r=r''$ and using (iii), we get: $(s,s')\succcurlyeq(r,r') \iff (r',r)\succcurlyeq(s',s)$. Thus, $\succcurlyeq$ satisfies the conditions of Theorem 5.3 in Wakker (1988) and there exists a continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ such that $(s,s')\mapsto\tilde{u}(s')-\tilde{u}(s)$ represents $\succcurlyeq$.

(b)$\Rightarrow$(a). Straightforward. ■

Proof of Lemma 4.

(a)$\Rightarrow$(b). Cater (1999, Theorem II).

(b)$\Rightarrow$(c). Let (b) hold and let $\succeq$ be a continuous ARC ordering that satisfies (ii) and (iii). Put $\mathrm{D}:=\{\tilde{u}(s')-\tilde{u}(s),\, s,s'\in\mathrm{S}\}$, where $\tilde{u}$ is defined in Lemma 3. By construction, D is a real interval symmetric with respect to zero. $\succeq$ induces a total preorder $\succeq'$ on $\mathrm{V}':=\mathrm{D}\times\mathrm{T}^2_<$ by

$$(\tilde{u}(s')-\tilde{u}(s),t,t')\succeq'(\tilde{u}(r')-\tilde{u}(r),\tau,\tau') \iff (s,t;s',t')\succeq(r,\tau;r',\tau').$$

The relations $\succ'$ and $\sim'$ are defined as usual.

We claim that $\succeq'$ is continuous, provided that D is endowed with the usual topology as a subset of R. If $\mathrm{D}=\{0\}$, the claim is trivial, so we assume that D is a proper interval in what follows. In order to simplify the exposition, we define $U(s,s'):=\tilde{u}(s')-\tilde{u}(s)$, $\mathrm{X}:=\mathrm{S}^2$, $\mathrm{Y}:=\mathrm{T}^2_<$, and identify V with $\mathrm{X}\times\mathrm{Y}$ in the rest of the proof. First, notice that since $\succeq$ is continuous, for any $x\in\mathrm{X}$, $y\in\mathrm{Y}$, $v\in\mathrm{V}$, the sets $\{x'\in\mathrm{X}:(x',y)\succ v\}$ and $\{y'\in\mathrm{Y}:(x,y')\succ v\}$ are open. Therefore, so is $\{y\in\mathrm{Y}:(d,y)\succ' v'\}$ for any $d\in\mathrm{D}$ and $v'\in\mathrm{V}'$. If $y\in\mathrm{Y}$ and $v\in\mathrm{V}$ are such that $\{x\in\mathrm{X}:(x,y)\succ v\}\neq\varnothing,\mathrm{X}$, then, by connectedness of X, completeness and continuity of $\succeq$, there exists $x'\in\mathrm{X}$ such that $(x',y)\sim v$. In this case $\{x\in\mathrm{X}:(x,y)\succ v\}=\{x\in\mathrm{X}:U(x)>U(x')\}$ by the definition of U. Therefore, for any $y\in\mathrm{Y}$ and $v'\in\mathrm{V}'$, the set $\{d\in\mathrm{D}:(d,y)\succ' v'\}$ is of the form $\varnothing$, D, or $(d',+\infty)\cap\mathrm{D}$ for some $d'\in\mathrm{R}$. Now pick $v'\in\mathrm{V}'$ and $v_0'=(d_0,y_0)\in\mathrm{U}_{\succ'}(v')$. If d_0 is the least element of D, then $\mathrm{D}\times\{y:(d_0,y)\succ' v'\}\subset\mathrm{U}_{\succ'}(v')$ is an open neighborhood of v_0'. Otherwise, there exists $d^*\in\mathrm{D}$ such that $d^*<d_0$ and $(d^*,y_0)\succ v'$; then $\left((d^*,+\infty)\cap\mathrm{D}\right)\times\{y:(d^*,y)\succ' v'\}\subset\mathrm{U}_{\succ'}(v')$ is an open neighborhood of v_0'. Thus, $\mathrm{U}_{\succ'}(v')$ is open in V'. A similar argument shows that $\mathrm{L}_{\succ'}(v')$ is open in V'. Since R and T are second countable, so is V' and, by a theorem of Debreu (1964, Proposition 3), $\succeq'$ is continuously representable.

(c)$\Rightarrow$(a). Let (c) hold. The ARC ordering $\succeq$ defined by $(s,t;s',t')\succeq(r,\tau;r',\tau')\ \Leftrightarrow\ t'\geq\tau'$ is continuous, satisfies (ii) and (iii) with trivial $\succeq_{t,t'}$, and, therefore, is representable. If $(\mathrm{T},\geq)$ has no least element, then (a) holds due to representability of $\succeq$. Otherwise, there exists a function $f:\mathrm{T}\setminus\{t_0\}\to\mathrm{R}_{++}$ that represents the restriction of $\geq$ to $\mathrm{T}\setminus\{t_0\}$, where t_0 is the least element of T. Extend f to T by setting $f(t_0):=0$. The constructed extension represents $\geq$. ■

Proof of Proposition 1.

The "if" parts are straightforward, so we need only to prove the "only if" parts.

(a). In view of representation (3), we have to show that there exists a relative discounting TP-family whose IRR mapping represents $\succeq$. Let $\tilde{u}$, D, V', Y, and $\succeq'$ be as in the proof of Lemma 4. Since (iv) holds and $\succeq$ is non-trivial, the interval D is proper. Pick $y_0\in\mathrm{Y}$. It follows from Lemma 3 and condition (iv) that the function $I:\mathrm{V}'\to\mathrm{D}$ that sends each $v'\in\mathrm{V}'$ to a solution $d\in\mathrm{D}$ of the equation $v'\sim'(d,y_0)$ is well defined and represents $\succeq'$. Let J be the inverse of I with respect to the first argument, that is, $I(c,y)=d\ \Leftrightarrow\ J(d,y)=c$. J is well defined and since for every $y\in\mathrm{Y}$, $I(\cdot,y)$ is a strictly increasing self-homeomorphism of D, so is $J(\cdot,y)$. From the definitions of $\succeq'$ and J it follows that the function that takes each $(s,t;s',t')\in\mathrm{V}$ to the solution $d\in\mathrm{D}$ of the equation $J(d,t,t')=\tilde{u}(s')-\tilde{u}(s)$ represents $\succeq$. This function is exactly the IRR mapping of the relative discounting TP-family with the utility function $u=e^{\tilde{u}}$ and the relative discount function h whose restriction to V' is e^J. This completes the proof.

(b). According to (v),

$$J(d,t,t')+J(d,t',t'')=J(d,t,t'') \tag{17}$$

provided that $J(d,t,t')+J(d,t',t'')\in\mathrm{D}$. We claim that $\mathrm{D}=\mathrm{R}$. Indeed, if, contrary to the claim, D is bounded from above, then, given $t<t'<t''$, there exists $d_0<\sup\mathrm{D}$ that solves $J(d_0,t,t')+J(d_0,t',t'')=\sup\mathrm{D}$. Then from (17) it follows that $\lim_{d\uparrow d_0}J(d,t,t'')=\sup\mathrm{D}$. Since $d\mapsto J(d,t,t'')$ is strictly increasing and onto D, we arrive to a contradiction.

Extend the domain of J to $\mathrm{R}\times\mathrm{T}^2$ by setting $J(d,t,t):=0$ and $J(d,t',t):=-J(d,t,t')$ for $(t,t')\in\mathrm{T}^2_<$. Then the Sincov functional equation (17) holds for all $(d,t,t',t'')\in\mathrm{R}\times\mathrm{T}^3$. Its general solution is given by $J(d,t,t')=\tilde{g}(d,t)-\tilde{g}(d,t')$ for some function $\tilde{g}$ (Aczél, 1966, section 8.1.3). Thus, the function that takes each $(s,t;s',t')\in\mathrm{V}$ to the solution $d\in\mathrm{R}$ of the equation $\tilde{g}(d,t)-\tilde{g}(d,t')=\tilde{u}(s')-\tilde{u}(s)$ represents $\succeq$. Since this function is the IRR mapping of the multiplicative discounting TP-family with the utility function $u=e^{\tilde{u}}$ and the discount function $g=e^{\tilde{g}}$, this completes the proof. ■

Proof of Corollary 1.

We shall prove only part (c). The remaining statements can be proved in a similar fashion.

The "if" part is clear. For the "only if" part, it is straightforward to show that for an ARC ordering induced by a multiplicative discounting TP-family condition (vi) implies (vi)'; so we assume (vi)' to hold in what follows. Given $(\tau,\tau')\in\mathrm{T}^2_<$, the discount function g can be normalized such that $g(0,\tau)=g(0,\tau')=1$. Then condition (vi)' implies $g(0,\cdot)\equiv 1$. Pick $(t,t')\in\mathrm{T}^2_<$, since $d\mapsto g(d,t)/g(d,t')$ is strictly increasing, $g(d,t)/g(d,t')>1$ (resp. $g(d,t)/g(d,t')<1$), whenever $d>0$ (resp. $d<0$). ■

Proof of Corollary 2.

The "if" parts are straightforward, so we need only to prove the "only if" parts. Let D, Y, y_0, I, J, $\tilde{g}$, and $\succeq'$ be as in the proof of Proposition 1.

(a). Since $\succeq'$ is continuous (the proof of continuity of $\succeq'$ in Lemma 4 does not use second countability of T), for any $d\in\mathrm{D}$ the strict lower level set of $J(d,\cdot)$ is open. Indeed, $\{y\in\mathrm{Y}: c>J(d,y)\}=\{y\in\mathrm{Y}: I(c,y)>d\}=\{y\in\mathrm{Y}:(c,y)\succ'(d,y_0)\}$ if $c\in\mathrm{D}$, and equals $\varnothing$ or Y otherwise. A similar argument shows that the strict upper level set of $J(d,\cdot)$ is open so that $J(d,\cdot)$ is continuous. Since J is separately continuous and $J(\cdot,y)$ is monotone for every $y\in\mathrm{Y}$, J is continuous (Grushka, 2019).

(b). Extend the domain of J to $\mathrm{D}\times\mathrm{T}^2$ by setting $J(d,t,t):=0$ and $J(d,t',t):=-J(d,t,t')$ for $(t,t')\in\mathrm{T}^2_<$. Since the extension is continuous and monotone in the first argument, it is sufficient to show that for any $d\in\mathrm{D}$ and $t\in\mathrm{T}$, $J(d,\cdot,\cdot)$ is continuous at the point (t,t). Choose $\varepsilon>0$; without loss of generality, we may assume that $\varepsilon\in\mathrm{D}$. Pick $s,s'\in\mathrm{S}$ such that $\tilde{u}(s')-\tilde{u}(s)=\varepsilon$ (resp. $\tilde{u}(s')-\tilde{u}(s)=-\varepsilon$). By (viii), there exist neighborhoods $A,A'\subset\mathrm{T}$ (resp. $B,B'\subset\mathrm{T}$) of t such that $J(d,\tau,\tau')<\varepsilon$ (resp. $J(d,\tau,\tau')>-\varepsilon$), whenever $(\tau,\tau')\in\mathrm{T}^2_<\cap(A\times A')$ (resp. $(\tau,\tau')\in\mathrm{T}^2_<\cap(B\times B')$). Then the set $C:=A\cap A'\cap B\cap B'$ is a neighborhood of t and $|J(d,\tau,\tau')|<\varepsilon$, whenever $(\tau,\tau')\in C^2$.

(c). For any date t_0, $\tilde{g}$ can be normalized such that $\tilde{g}(\cdot,t_0)\equiv 0$. Under this normalization, $\tilde{g}$ is continuous and monotone in the first argument. From part (a) and the pasting lemma it follows that $\tilde{g}$ is also continuous in the second argument. Therefore, g must be jointly continuous. ■

Proof of Proposition 2.

(a). The "if" part is straightforward. For the "only if" part, it follows from representability of $\succeq$ and Lemma 3 that there exist functions $\widetilde{u}:\mathrm{S}\to\mathrm{R}$ and $I:\{\widetilde{u}(s')-\widetilde{u}(s),\ s,s'\in\mathrm{S}\}\times\mathrm{T}^2_<\to\mathrm{R}$ such that $\widetilde{u}$ is continuous, I is strictly increasing in the first argument, and $(s,t;s',t')\mapsto I(\widetilde{u}(s')-\widetilde{u}(s),t,t')$ represents $\succeq$. Put $\mathrm{D}:=\{\widetilde{u}(s')-\widetilde{u}(s),\ s,s'\in\mathrm{S}\}$. If $\mathrm{D}=\{0\}$, the statement holds with $G(0,z)=z$ and $f(t,t')=I(0,t,t')$. Otherwise, fix $d_0\in\mathrm{D}\setminus\{0\}$ and put $f(t,t'):=I(d_0,t,t')$. By (ix), if $f(t,t')=f(\tau,\tau')$, then $I(d,t,t')=I(d,\tau,\tau')$ for any $d\in\mathrm{D}$. Thus, the function $G:\mathrm{D}\times f(\mathrm{T}^2_<)\to\mathrm{R}$ given by $G(d,f(t,t')):=I(d,t,t')$ is well defined. G is strictly increasing in the first argument (since so is I) and, by (ix), for each $d\in\mathrm{D}\setminus\{0\}$ $G(d,\cdot)$ is injective.

(b). Follows from Corollary 1(a) and (ix) in a similar fashion as in the proof of part (a).

(c). The "only if" part needs proofs. First, we show that continuity of $\succeq$ and conditions (ii)–(v), (ix) imply (vi)'. In view of (v), it is sufficient to show that $(s,t;s,t')\sim(s,t';s,t'')$ for any $t<t'<t''$. By Proposition 1(b), $\succeq$ is induced by a multiplicative discounting TP-family. In particular, it is representable, so that $\succeq$ satisfies the conditions of part (a). Let G and f be as in part (a). Pick $t<t'<t''$ and assume by way of contradiction that $G(0,f(t,t'))\neq G(0,f(t',t''))$. By (iv), there exist non-zero constants c, c' such that $G(c,f(t,t'))=G(0,f(t',t''))$, $G(0,f(t,t'))=G(c',f(t',t''))$. Applying (v) to these equalities, we obtain $G(c,f(t,t'))=G(c,f(t,t''))$, $G(c',f(t',t''))=G(c',f(t,t''))$. Applying (ix) to the last two equalities, we get $G(0,f(t,t'))=G(0,f(t,t''))$, $G(0,f(t',t''))=G(0,f(t,t''))$, a contradiction with $G(0,f(t,t'))\neq G(0,f(t',t''))$.

Let $\widetilde{g}$ be the logarithm of the discount function of the underlying multiplicative discounting TP-family. By Corollaries 1(c) and 2(c), $\widetilde{g}$ can be normalized such that $\widetilde{g}$ is continuous and $\widetilde{g}(d,\cdot)$ is strictly decreasing (resp. strictly increasing, identically equal to 0), whenever $d>0$ (resp. $d<0$, $d=0$). Put $\phi:=\widetilde{g}(-1,\cdot)$. By definition, ϕ is strictly increasing and continuous. By (ix),

$$\phi(t)-\phi(t')=\phi(\tau)-\phi(\tau')\ \Rightarrow\ \widetilde{g}(d,t)-\widetilde{g}(d,t')=\widetilde{g}(d,\tau)-\widetilde{g}(d,\tau'). \tag{18}$$

The implication (18) means that $\widetilde{g}(d,t)-\widetilde{g}(d,t')=F(d,\phi(t)-\phi(t'))$ for some function F. As shown in (Aczél, 1987, p. 23–25), then there exists a function a such that $F(d,z)=a(d)z$. The conditions imposed on $\widetilde{g}$ imply that $a(0)=0$, a is strictly decreasing and onto R. ■

In what follows, we need the following lemma.

Lemma 8.

Let $\mathrm{T}=\mathrm{R}$. For a continuous ARC ordering, each of the following implies condition (x):

(a) (vi)', (viii), (x)';

(b) (ii), (iii), (viii), (x)', and $\mathrm{V}_0\neq\mathrm{V}$.

Proof.

(a). Pick $v=(s,t;s',t')\in\mathrm{V}$. If $v\in\mathrm{V}_0$, (x) follows from (vi)' and (x)'. Now suppose that $v\in\mathrm{V}_1$ (the case of $v\in\mathrm{V}_{-1}$ proceeds in a similar manner) and assume by way of contradiction that $(s,t;s',t')\succ(s,t+\tau;s',t'+\tau)$ for some τ. Since $\succeq$ is continuous, there exists $\tau'\in(\tau-(t'-t),\tau)$

such that $(s,t;s',t') \succ (s,t+\tau;s',t'+\tau')$. Set $t_0 := t$, $t'_0 := t'$, $t_1 := t+\tau$, $t'_1 := t'+\tau'$. Let $a \in (0,1)$ and $b \in \mathrm{R}$ solve the system of linear equations $at_0 + b = t_1$, $at'_0 + b = t'_1$. Put $t_{k+1} := at_k + b$, $t'_{k+1} := at'_k + b$, $k = 1,2,\ldots$ The sequences $\{t_k\}_{k=0}^{\infty}$, $\{t'_k\}_{k=0}^{\infty}$ converge to $b/(1-a)$ and satisfy $t_k < t'_k$, $k = 0,1,\ldots$ Put $v_k := (s,t_k;s',t'_k)$, $k = 1,2,\ldots$ By (x)', $v \succ v_k$, $k = 1,2,\ldots$, a contradiction with (viii).

(b). Let I and D be as in Lemma 4 and its proof. Since $\mathrm{V}_0 \neq \mathrm{V}$, we have $\mathrm{D} \neq \{0\}$. From the proof of part (a) it follows that $I(c,t,t') = I(c,t+\tau,t'+\tau)$ for any $c \neq 0$. Since I is continuous, the equality remains valid for $c = 0$. ■

Proof of Proposition 3.

(a). Follows from Lemma 4 and (x).

(b). Follows from Proposition 1(a) and (x).

(c). The "only if" part needs proof. Let $\succeq$ be a continuous ARC ordering. First, assume that (ii), (iii), (v), (x) hold. Let G and $\widetilde{u}$ be as in part (a) and let $\mathrm{D} := \{\widetilde{u}(s') - \widetilde{u}(s),\ s,s' \in \mathrm{S}\}$. If $\mathrm{D} \neq \{0\}$, without loss of generality, we may assume that $1 \in \mathrm{D}$. Given $c \in \mathrm{D}$ and $\tau \in \mathrm{R}_{++}$, pick $s,s',s'' \in \mathrm{S}$ and $t,t',t'' \in \mathrm{R}$ such that $\widetilde{u}(s') - \widetilde{u}(s) = \widetilde{u}(s'') - \widetilde{u}(s') = c/2$ and $t'-t = t''-t' = \tau/2$. Condition (v) with those s, s', s'' and t, t', t'' yields $G(c,\tau) = G(c/2,\tau/2)$. Iterating the last identity, we get $G(c,\tau) = G(rc,r\tau)$ provided that r is a positive rational number such that $rc \in \mathrm{D}$. It follows from continuity of G that $G(0,\cdot)$ is identically constant and $G(c,\tau) = G(\operatorname{sgn} c, \tau/|c|)$, $c \in \mathrm{D} \setminus \{0\}$. Since G is strictly increasing in the first argument, (5) represents $\succeq$. Since $\succeq$ is non-trivial, $\mathrm{D} \neq \{0\}$.

Now assume that (ii), (vii), (v), (x) hold. First, note that continuity of $\succeq$ and conditions (v), (x) imply (vi)'. By (x), the relation $\succeq_{t,t'}$ depends on t and t' through the difference $t'-t$, so we set $\succeq_\tau := \succeq_{t,t+\tau}$ (with $\succ_\tau$ and $\sim_\tau$ defined as usual). Define the total preorder $\trianglerighteq$ on S by $s \trianglerighteq s'$ $\Leftrightarrow$ $(r,s) \succeq_\tau (r,s')$ (with $\triangleright$ defined as usual). It follows from (ii) that $\trianglerighteq$ is independent of the choice of r and τ in the definition and $s \trianglerighteq s'$ $\Leftrightarrow$ $(s',r) \succeq_\tau (s,r)$. It is sufficient to establish that (iii) holds. Assume by way of contradiction that $(s,s) \succ_\tau (r,r)$ for some r, s, and τ. Then either $s \triangleright r$ or $r \triangleright s$. Assume that $s \triangleright r$ (the case of $r \triangleright s$ proceeds in a similar manner). Pick $\tau < \tau'$. Since $(r,s) \succ_{\tau'-\tau} (s,s) \succ_{\tau'-\tau} (r,r)$, the connectedness of S and the continuity of $\succeq_{\tau'-\tau}$ imply that there exists $s \triangleright r' \triangleright r$ such that $(r,r') \sim_{\tau'-\tau} (s,s)$. Since for any $t < t'$, $(r,t;s,t'),(r,\tau;r',\tau') \in \mathrm{V}_1$ and $(r,t;s,t') \succ (s,t;s,t') \sim (s,\tau;s,\tau') \sim (r,\tau;r',\tau')$, we arrive to a contradiction with (vii).

(d). The "if" part is obvious. If (x) holds in the "if" part, the "only if" part follows from part (c) and (iv). If (x)' holds in the "if" part, then, since a continuous ARC ordering induced by a multiplicative discounting TP-family satisfies (viii), the "only if" part follows from Lemma 8(b) and part (c).

(e). The "if" part is straightforward. For the "only if" part, by Proposition 1(b), the ARC ordering is induced by a multiplicative discounting TP-family. Put $\varphi(d,t'-t) := \widetilde{g}(d,t) - \widetilde{g}(d,t')$, where $\widetilde{g}$ is as in the proof of Proposition 1(b). By (x), $\varphi : \mathrm{R} \times \mathrm{R}_{++} \to \mathrm{R}$ is well defined and additive with respect to the second argument. For any $d < d'$ the function $\tau \mapsto \varphi(d',\tau) - \varphi(d,\tau)$ is positive, additive, and, hence, linear (Aczél, 1987, Corollary 5, p. 13). Therefore, there exist a function $\phi:\mathrm{R}\to\mathrm{R}$ and an additive function $A:\mathrm{R}_{++}\to\mathrm{R}$ such that $\varphi(d,\tau)=\phi(d)\tau-A(\tau)$. Since $\varphi(\cdot,\tau)$ is an increasing self-homeomorphism of R, so is ϕ.

(f). Follows from part (e) and the fact that an additive function bounded on an interval of positive length is linear (Aczél, 1987, Corollary 5, p. 13).

(g). The "if" part is straightforward. For the "only if" part, it follows from Lemma 8(b) that the ARC ordering satisfies the conditions imposed on $\succeq$ in part (a). Let $\tilde{u}$ and G be as in part (a). Since $\mathrm{V}_0\neq\mathrm{V}$, the interval $\mathrm{D}:=\{\tilde{u}(s')-\tilde{u}(s),\ s,s'\in\mathrm{S}\}$ is proper. By (x)', $\forall a>0$

$$G(d,\tau)\geq G(d',\tau') \iff G(d,a\tau)\geq G(d',a\tau'). \tag{19}$$

From (19) with $d=d'$ and continuity of G, it follows that for any $d\in\mathrm{D}$, $G(d,\cdot)$ is either identically constant or strictly monotone. Condition (viii) and (19) imply that for any $d>0$ (resp. $d<0$), $G(d,\cdot)$ is strictly decreasing (resp. increasing) and maps R_{++} onto $G(\mathrm{D}\cap\mathrm{R}_{++},\mathrm{R}_{++})$ (resp. $G(-\mathrm{D}\cap\mathrm{R}_{++},\mathrm{R}_{++})$). Since G is continuous, $G(0,\cdot)$ is identically constant. Put

$$f(z):=\begin{cases}G(1,1/z) & \text{if } z>0\\ G(0,1) & \text{if } z=0\\ G(-1,-1/z) & \text{if } z<0\end{cases}.$$

Thus defined function is strictly increasing and maps R onto $G(\mathrm{D},\mathrm{R}_{++})$. Therefore, the function $H:=f^{-1}\circ G$ is well defined and $(s,t;s',t')\mapsto H(\tilde{u}(s')-\tilde{u}(s),t'-t)$ represents $\succeq$. Applying (19), we get $H(d,a\tau)=H(d,\tau)/a\ \ \forall a>0$. Thus, (10) with $h(d):=H(d,1)$ represents $\succeq$. ■

Proof of Lemma 5.

We shall prove only part (c). The remaining statements can be proved in a similar fashion.

Let two multiplicative discounting TP-families with utility functions u, u' and discount functions g, g' induce the same ARC ordering $\succeq$. Put $\tilde{u}:=\ln u$, $\tilde{u}':=\ln u'$, $\tilde{g}:=\ln g$, $\tilde{g}':=\ln g'$. Then for any $(t,t')\in\mathrm{T}^2_<$, the total preorder $\succeq_{t,t'}$ is represented by both $(s,s')\mapsto\tilde{u}(s')-\tilde{u}(s)$ and $(s,s')\mapsto\tilde{u}'(s')-\tilde{u}'(s)$. From the uniqueness result on difference measurement (Krantz et al., 1971, Theorem 2, p. 151) it follows that there exist constants $a>0$ and $\tilde{b}$ such that $\tilde{u}'=a\tilde{u}+\tilde{b}$. Since the TP-families coincide with their cores and, therefore, by Lemma 1 are isomorphic, there exists an order automorphism φ of R such that for any $d\in\mathrm{R}$ and $(s,t;s',t')\in\mathrm{W}$

$$\tilde{u}'(s)+\tilde{g}'(d,t)\geq\tilde{u}'(s')+\tilde{g}'(d,t') \iff \tilde{u}(s)+\tilde{g}(\varphi(d),t)\geq\tilde{u}(s')+\tilde{g}(\varphi(d),t'). \tag{20}$$

Since $\tilde{u}$ and $\tilde{u}'$ are unbounded, for any $t\neq t'$ and d there exist $s,s'\in\mathrm{S}$ such that (20) holds with equalities. Multiplying the second equality by a and subtracting from the first, we obtain

$$\tilde{g}'(d,t)-a\tilde{g}(\varphi(d),t)=\tilde{g}'(d,t')-a\tilde{g}(\varphi(d),t'). \tag{21}$$

Eq. (21) holds for any $t\neq t'$ and d if and only if there exists a function $\tilde{c}:\mathrm{R}\to\mathrm{R}$ such that $\tilde{g}'(d,t)=a\tilde{g}(\varphi(d),t)+\tilde{c}(d)$. Now the required result holds with $b:=e^{\tilde{b}}$ and $c:=e^{\tilde{c}}$.

The converse is straightforward. ■

Proof of Lemma 6.

(a)$\Rightarrow$(b). Since S^2 is connected, we can apply Lemma 3 to show that the relation $\succeq_{t,t'}$ is independent of the choice of $(t,t')\in\mathrm{T}^2_<$ and there exists a continuous function $u:\mathrm{S}^2\to\mathrm{R}$ such that

$(s,\tilde{s};s',\tilde{s}') \mapsto u(s',\tilde{s}') - u(s,\tilde{s})$ represents $\succeq_{t,t'}$. By B1, there exist continuous $\tilde{u},\tilde{u}' : \mathrm{S}^2 \to \mathrm{R}$ such that $u(s,\tilde{s}) = \tilde{u}(s) - \tilde{u}'(\tilde{s})$. It follows from B2 that $\tilde{u}$ and $\tilde{u}'$ differ by a constant.

(b)$\Rightarrow$(a). Straightforward. ■

Proof of Lemma 7.

(a). Pick $E \in \mathcal{R}(\mathcal{E}) \setminus \mathcal{E}$. There are $m \geq 2$, $n \geq 1$, and pairwise disjoint sets $A_1,\dots,A_{m+n} \in \mathcal{E}$ such that $\bigcup_{i=1}^{m} A_i = E$ and $B := \bigcup_{i=1}^{m+n} A_i \in \mathcal{E}$. Given $p,q \in \mathcal{P}$, by (IV)', there exists $p' \in \mathcal{P}$ such that

$$\begin{aligned} p'_{A_i} &\sim p_{A_i},\ i = 1,\dots,m, \\ p'_{A_i} &\sim q_{A_i},\ i = m+1,\dots,m+n. \end{aligned} \tag{22}$$

Applying (II) to (22), we obtain

$$p_E \succeq q_E \;\Leftrightarrow\; p'_E \succeq q_E \;\Leftrightarrow\; p'_B \succeq q_B. \tag{23}$$

Since $A_1,\dots,A_{m+n}, B \in \mathcal{E}$ and $\succeq_C = \succeq'_C$ for any $C \in \mathcal{E}$, (22) and (23) hold with $\succeq$ replaced by $\succeq'$.

(b). Pick $p_E \in \mathcal{H}(\mathcal{P},\mathcal{R}(\mathcal{E}))$. It is sufficient to show that there is $q_{A_1} \in \mathcal{H}(\mathcal{P},\mathcal{E})$ such that $q_{A_1} \sim p_E \;\Leftrightarrow\; q_{A_1} \sim' p_E$. There are pairwise disjoint sets $A_1,\dots,A_m \in \mathcal{E}$ such that $\bigcup_{i=1}^{m} A_i = E$. By (IV)', there exists $q \in \mathcal{P}$ such that

$$q_{A_1} \sim p_E,\ i = 1,\dots,m. \tag{24}$$

From (V) it follows that (24) is equivalent to the system

$$\begin{aligned} q_{A_{i+1}} &\sim q_{A_i},\ i = 1,\dots,m-1, \\ q_E &\sim p_E. \end{aligned} \tag{25}$$

Since $A_1,\dots,A_m \in \mathcal{E}$ and the restrictions of $\succeq$ and $\succeq'$ to $\mathcal{H}(\mathcal{P},\mathcal{E})$ coincide, the first $m-1$ equations in (25) hold with $\sim$ replaced by $\sim'$. From part (a) it follows that the last equation in (25) (and, therefore, (24)) remains valid with $\sim$ replaced by $\sim'$. ■

In order to proof Proposition 4, we need the following lemma.

Lemma 9.

Let U be a real interval and let $\mathrm{D} := \mathrm{U} - \mathrm{U}$. For a sequence $a_1,\dots,a_n$ of real numbers, the following statements are equivalent:

(a) there exist $u_0,\dots,u_n \in \mathrm{U}$ such that $a_i = u_i - u_{i-1}$, $i = 1,\dots,n$;

(b) $\sum_{i=j}^{k} a_i \in \mathrm{D}$ for any $1 \leq j \leq k \leq n$;

(c) $\max_{k\in\{0,\dots,n\}} \sum_{i=1}^{k} a_i - \min_{k\in\{0,\dots,n\}} \sum_{i=1}^{k} a_i \in \mathrm{D}$ (with the convention that an empty sum is zero).

Proof.

(a)$\Rightarrow$(b). $\sum_{i=j}^{k} a_i = u_k - u_{j-1} \in \mathrm{D}$.

(b)$\Rightarrow$(c). We have

$$\max_{k\in\{0,\dots,n\}}\sum_{i=1}^{k}a_i-\min_{k\in\{0,\dots,n\}}\sum_{i=1}^{k}a_i=\sum_{i=1}^{k^*}a_i-\sum_{i=1}^{k_*}a_i=\begin{cases}\sum_{i=k_*+1}^{k^*}a_i & \text{if } k^*\ge k_*\\ -\sum_{i=k^*+1}^{k_*}a_i & \text{if } k^*<k_*\end{cases}\in \mathrm{D},$$

where k^* (resp. k_*) is a value of k at which the maximum (resp. minimum) is attained.

(c)$\Rightarrow$(a). Put $u'_k:=\sum_{i=1}^{k}a_i$, $k=0,\dots,n$. Since $\max_k u'_k-\min_k u'_k\in \mathrm{D}\cap \mathrm{R}_+$, there exists a constant c such that $u'_0+c,\dots,u'_n+c\in \mathrm{U}$. The sequence $u_k:=u'_k+c$, $k=0,\dots,n$ satisfies the desired property. ■

Proof of Proposition 4.

The "if" parts are straightforward, so we need only to prove the "only if" parts.

(a). The restriction of $\succeq$ to $\mathcal{H}(\mathcal{P},\mathcal{I})$ satisfies (i)–(iii). By Lemma 3, there exists a continuous function $\tilde{u}:\mathrm{S}\to\mathrm{R}$ such that for any $E\in\mathcal{I}$ $p_E\mapsto\mu_{\tilde{u}\circ p}(E)$ represents $\succeq_E$. Put $\mathrm{U}:=\tilde{u}(\mathrm{S})$, $\mathrm{D}:=\mathrm{U}-\mathrm{U}$.

We introduce the following definition. Given $t_0<\dots<t_n$, a sequence $a_1,\dots,a_n$ of real numbers is said to be *admissible* if there exists $p\in\mathcal{P}$ such that $\mu_{\tilde{u}\circ p}((t_{i-1},t_i])=a_i$, $i=1,\dots,n$. It follows from assumption A that a sequence $a_1,\dots,a_n$ is admissible if and only if it satisfies one of the equivalent conditions of Lemma 9.

Pick $E\in\mathcal{R}(\mathcal{I})\setminus\mathcal{I}$ and let $A_1,\dots,A_m\in\mathcal{I}$ be pairwise disjoint intervals such that $\bigcup_{i=1}^{m}A_i=E$. Without loss of generality, we may assume that the intervals $A_i=(t_i,t'_i]$, $i=1,\dots,m$ are chosen such that $t'_i<t_{i+1}$, $i=1,\dots,m-1$ and put $B_i:=(t'_i,t_{i+1}]$, $i=1,\dots,m-1$, $C:=E\cup\bigcup_{i=1}^{m-1}B_i=(t_1,t'_n]$. Note that for each $a_1,\dots,a_m\in\mathrm{D}$ there exists $q\in\mathcal{P}$ such that $\mu_{\tilde{u}\circ q}(A_i)=a_i$, since the sequence $a_1,-(a_1+a_2)/2,a_2,-(a_2+a_3)/2,\dots,a_m$ is admissible. Now pick $p\in\mathcal{P}$ and set $a_i:=\mu_{\tilde{u}\circ p}(A_i)$, $i=1,\dots,m$, $b_i:=\mu_{\tilde{u}\circ p}(B_i)$, $i=1,\dots,m-1$, $\bar{a}:=\mu_{\tilde{u}\circ p}(E)/m=\frac{1}{m}\sum_{i=1}^{m}a_i$. It is sufficient to show that

$$\exists p^*\in\mathcal{P}:\ p^*\sim_E p \text{ and } \mu_{\tilde{u}\circ p^*}(A_i)=\bar{a},\ i=1,\dots,m. \tag{26}$$

Indeed, if (26) holds for every $p\in\mathcal{P}$, then for any $p,q\in\mathcal{P}$

$$p\succeq_E q\ \Leftrightarrow\ p^*\succeq_E q^*\ \Leftrightarrow\ \mu_{\tilde{u}\circ p^*}(A_1)\ge\mu_{\tilde{u}\circ q^*}(A_1)\ \Leftrightarrow\ \mu_{\tilde{u}\circ p}(E)\ge\mu_{\tilde{u}\circ q}(E)$$

(here the second equivalence follows from (II)) as desired. Note that if (26) holds, then, by (II), $q\sim_E p$ for any $q\in\mathcal{P}$ satisfying $\mu_{\tilde{u}\circ q}(A_i)=\bar{a}$, $i=1,\dots,m$.

We claim that (26) holds for $m=2$. Indeed, since the sequence a_1,b_1,a_2 is admissible, so is $(a_1+a_2)/2,b_1,(a_1+a_2)/2$ and, therefore, there exists $p^*\in\mathcal{P}$ such that $\mu_{\tilde{u}\circ p^*}(A_1)=(a_1+a_2)/2$, $\mu_{\tilde{u}\circ p^*}(B_1)=b_1$, $\mu_{\tilde{u}\circ p^*}(A_2)=(a_1+a_2)/2$. Now $p^*\sim_E p$ follows from $p^*\sim_C p$, $p^*\sim_{B_1} p$, and (II).

Now assume that $m\ge 3$. Set $a^{(0)}:=(a_1,\dots,a_m)$ and define

$$a^{(k)} := \left((a_1^{(k-1)} + a_2^{(k-1)})/2, (a_1^{(k-1)} + a_2^{(k-1)})/2, a_3^{(k-1)}, \ldots, a_m^{(k-1)}\right) \text{ if } k \bmod m = 1,$$

$$a^{(k)} := \left(a_1^{(k-1)}, (a_2^{(k-1)} + a_3^{(k-1)})/2, (a_2^{(k-1)} + a_3^{(k-1)})/2, a_4^{(k-1)}, \ldots, a_m^{(k-1)}\right) \text{ if } k \bmod m = 2,$$

$$\ldots$$

$$a^{(k)} := \left(a_1^{(k-1)}, \ldots, a_{m-2}^{(k-1)}, (a_{m-1}^{(k-1)} + a_m^{(k-1)})/2, (a_{m-1}^{(k-1)} + a_m^{(k-1)})/2\right) \text{ if } k \bmod m = m-1,$$

$$a^{(k)} := \left((a_1^{(k-1)} + a_m^{(k-1)})/2, a_2^{(k-1)}, \ldots, a_{m-1}^{(k-1)}, (a_1^{(k-1)} + a_m^{(k-1)})/2\right) \text{ if } k \bmod m = 0,$$

for $k = 1, 2, \ldots$ Let $p^{(k)} \in \mathcal{P}$ be such that $\mu_{\tilde{u} \circ p^{(k)}}(A_i) = a_i^{(k)}$, $i = 1, \ldots, m$. Since (26) holds for $m = 2$, we have $p^{(k)} \sim_E p^{(k-1)}$ and, therefore, $p^{(k)} \sim_E p$.

The sequence $u_k := a^{(k)}_{1+((k-1) \bmod m)}$, $k = 1, 2, \ldots$ satisfies the recurrence relation $u_k = (u_{k-1} + u_{k+1-m})/2$, $k = 2, 3, \ldots$ and the initial conditions $u_k := a_{m+k}$, $k = 3-m, \ldots, 0$, $u_1 := (a_1 + a_2)/2$. Since $\{u_k\}_{k=1}^{\infty}$ converges to $\bar{a}$ (Tay et al., 2004, Theorems 2.1, 2.2), there exists n such that the sequence $a_1^{(n)}, -\bar{a}, a_2^{(n)}, -\bar{a}, \ldots, a_m^{(n)}$ is admissible. Therefore, $p^{(n)}$ can be chosen such that $\mu_{\tilde{u} \circ p^{(n)}}(B_i) = -\bar{a}$, $i = 1, \ldots, m-1$. Let $p^* \in \mathcal{P}$ be such that $\mu_{\tilde{u} \circ p^*}(A_i) = \bar{a}$, $i = 1, \ldots, m$ and $\mu_{\tilde{u} \circ p^*}(B_i) = -\bar{a}$, $i = 1, \ldots, m-1$. As $\sum_{i=1}^{m} a_i^{(k)} = m\bar{a}$ for any k,

$$\mu_{\tilde{u} \circ p^{(n)}}(C) = \bar{a} = \mu_{\tilde{u} \circ p^*}(C) \;\Rightarrow\; p^{(n)} \sim_C p^* \;\Rightarrow\; p^{(n)} \sim_E p^*,$$

where the last implication follows from (II). Since $p^{(n)} \sim_E p$, we are done.

(b). Let $\tilde{u} : \mathrm{S} \to \mathrm{R}$ and D be as in the proof of part (a). Since $\succeq$ is non-trivial and (IV) holds, D is a proper interval. As the function $p_A \mapsto \mu_{\tilde{u} \circ p}(A)$, $A \in \mathcal{I}$ represents $\succeq_A$ and is onto D, from (IV) it follows that there is a numerical representation $I : \mathcal{H}(\mathcal{P}, \mathcal{R}(\mathcal{I})) \to \mathrm{D}$ of $\succeq$ such that for each $E \in \mathcal{R}(\mathcal{I})$ the restriction of I to $\mathcal{P}_E$ is onto D. Since $p_E \mapsto \mu_{\tilde{u} \circ p}(E)$ is a numerical representation of $\succeq_E$, there is a strictly increasing function $\phi_E : n(E)\mathrm{D} \xrightarrow{\text{onto}} \mathrm{D}$ such that the restriction of I to $\mathcal{P}_E$ is given by $p_E \mapsto \phi_E(\mu_{\tilde{u} \circ p}(E))$.

(c). Let $\succeq'$ be the restriction of $\succeq$ to $\mathcal{H}(\mathcal{P}, \mathcal{I})$. The total preorder $\succeq'$ is non-trivial (if, contrary to the claim, it were trivial, then, by (V), so would be $\succeq$) and satisfies (i)–(v). By Proposition 1(b), it is induced by a multiplicative discounting TP-family and it is straightforward to show that $\succeq'$ satisfies (IV)'.

Let us show that (IV)' remains valid for $\succeq$. Pick pairwise disjoint $A_1, \ldots, A_n \in \mathcal{R}(\mathcal{I})$ and $q_{B_1}^{(1)}, \ldots, q_{B_n}^{(n)} \in \mathcal{H}(\mathcal{P}, \mathcal{R}(\mathcal{I}))$. Since $\succeq$ satisfies (IV), there exist $q'^{(1)}_{B'_1}, \ldots, q'^{(n)}_{B'_n} \in \mathcal{H}(\mathcal{P}, \mathcal{I})$ such that $q'^{(i)}_{B'_i} \sim q^{(i)}_{B_i}$, $i = 1, \ldots, n$. Since $\succeq'$ satisfies (IV)', there exists $p \in \mathcal{P}$ such that $p_{A_i^{(j)}} \sim' q'^{(i)}_{B'_i}$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$, where $\{A_i^{(j)}\}_{j=1}^{m_i}$ is a partition of A_i into intervals from $\mathcal{I}$. By (V), $p_{A_i} \sim q'^{(i)}_{B'_i} \sim q^{(i)}_{B_i}$, $i = 1, \ldots, n$.

Let $\tilde{u}$ and $\tilde{g}$ be as in the proof of Proposition 1(b). The conditions imposed on $\tilde{g}$ imply that the map that sends each $p_E \in \mathcal{H}(\mathcal{P}, \mathcal{R}(\mathcal{I}))$ to a solution d of the equation $\mu_{\tilde{u} \circ p}(E) + \mu_{\tilde{g}(d,\cdot)}(E) = 0$ is well defined and represents an ARC ordering $\succcurlyeq$ on $\mathcal{H}(\mathcal{P}, \mathcal{R}(\mathcal{I}))$. Since $\succcurlyeq$ and $\succeq$ satisfy (II), (IV)', (V) and $\succeq'$ is the restriction of $\succcurlyeq$ to $\mathcal{H}(\mathcal{P},\mathcal{I})$, from Lemma 7(b) it follows that $\succcurlyeq$ and $\succeq$ coincide. ■

Proof of Proposition 5.

The "only if" parts are straightforward, so we need only to prove the "if" parts.

(a). First, we show that for any $E \in \mathcal{R}(\mathcal{I})$, $\mathcal{P}_E$ is connected in the topology of pointwise convergence. Since $\mathcal{P}_E$ is a product of spaces of the form $\mathcal{P}_{(t,t']}$, it is sufficient to consider the case $E \in \mathcal{I}$. Given a partition $\pi_n = \{(t,t_1),[t_1,t_2),...,[t_{n-1},t_n),[t_n,t']\}$ of $E=(t,t']$, let $\mathcal{S}(\pi_n) \subset \mathcal{P}_E$ be the set of S-valued step functions subordinated to π_n. Put $\mathcal{S} := \bigcup \mathcal{S}(\pi_n)$, where the union is taken over all (finite) partitions π_n of E. $\mathcal{S}(\pi_n)$ is homeomorphic to S^{n+1} and, therefore, is connected. Since $\bigcap \mathcal{S}(\pi_n)$, the set of constant functions on E, is non-empty, $\mathcal{S}$ is connected. Finally, since $\mathcal{S}$ is dense in $\mathcal{P}_E$, $\mathcal{P}_E$ is connected.

Given $n \geq 2$ and $E_i \in \mathcal{R}(\mathcal{I})$, $i=1,...,n$ with pairwise disjoint closures, put $E := \bigcup_{i=1}^{n} E_i$. The relation $\succeq_E$ is said to have an additive representation if there exists a collection $\{u_{E_i} : \mathcal{P}_{E_i} \to \mathrm{R},\ i=1,...,n\}$ of continuous functions such that $p_E \mapsto \sum_{i=1}^{n} u_{E_i}(p_{E_i})$ represents $\succeq_E$. Fix $q \in \mathcal{P}$, an additive representation is called normalized, if $u_{E_i}(q_{E_i}) = 0$. The uniqueness result on an additive representation (Krantz et al., 1971, Theorem 2, p. 257 and Theorem 14, p. 302) implies that a normalized additive representation is defined up to a positive multiplier (if $\{u'_{E_1},...,u'_{E_n}\}$ is another normalized additive representation of the same relation, then there is $\alpha > 0$ such that $u'_{E_i} = \alpha u_{E_i}$, $i=1,...,n$), provided that at least two of $u_{E_1},...,u_{E_n}$ are non-constant. From (II) it follows that if $\{u_{E_i},\ i=1,...,n\}$ is a normalized additive representation for $\succeq_E$, then so is $\{u_{E_i},\ i \in I\}$ for $\succeq_{E(I)}$, where $I \subset \{1,...,n\}$, $|I| \geq 2$, $E(I) := \bigcup_{i \in I} E_i$. Note that since (IV)' holds and $\succeq$ is non-trivial, so are $\succeq_{E_i}$, $i=1,...,n$. Hence, conditions (I) and (II) ensure (Krantz et al., 1971, Theorem 14, p. 302; Wakker, 1988, Theorem 4.1) the existence of a normalized additive representation of $\succeq_E$, provided that $n \geq 3$.

Pick intervals $C_i \in \mathcal{I}$, $i=1,2,3$ with pairwise disjoint closures and put $C := C_1 \cup C_2$. Let $\{u_{C_1},u_{C_2},u_{C_3}\}$ be a normalized additive representation of $\succeq_{C \cup C_3}$. To each $\succeq_E$, $E \in \mathcal{R}(\mathcal{I}) \setminus \{C_1,C_2,C_3\}$ we assign a continuous representation $u_E : \mathcal{P}_E \to \mathrm{R}$ as follows. We consider four cases. In what follows, cl is the topological closure operator in T.

Case 1. $E \in \mathcal{J}_1 := \{A \in \mathcal{R}(\mathcal{I}) : \mathrm{cl}(C) \cap \mathrm{cl}(A) = \varnothing\}$. Let $\{u'_{C_1},u'_{C_2},u'_E\}$ be a normalized additive representation of $\succeq_{C \cup E}$. Since $\{u'_{C_1},u'_{C_2}\}$ is a normalized additive representation of $\succeq_C$, there is $\alpha > 0$ such that $u_{C_i} = \alpha u'_{C_i}$, $i=1,2$. Put $u_E := \alpha u'_E$. Note that u_E is well defined, i.e., is independent of a particular normalized additive representation $\{u'_{C_1},u'_{C_2},u'_E\}$ of $\succeq_{C \cup E}$.

Case 2. $E \in \mathcal{J}_2 := \{A \in \mathcal{R}(\mathcal{I}) \setminus \mathcal{J}_1 : \mathrm{cl}(C \cup A) \neq \mathrm{T}\}$. There are $D_1, D_2 \in \mathcal{J}_1 \cap \mathcal{I}$ such that $C \cup E, D_1, D_2$ have pairwise disjoint closures. Let u_{D_1}, u_{D_2} be the functions defined in Case 1. There is a unique function u_E such that $\{u_{D_1}, u_{D_2}, u_E\}$ is a normalized additive representation of $\succeq_{D_1 \cup D_2 \cup E}$. We have to show that u_E is well defined, i.e., is independent of the choice of D_1, D_2. Let $D_1', D_2' \in \mathcal{J}_1 \cap \mathcal{I}$ be another intervals such that $C \cup E, D_1', D_2'$ have pairwise disjoint closures and let u_E' be the function such that $\{u_{D_1'}, u_{D_2'}, u_E'\}$ is a normalized additive representation of $\succeq_{D_1' \cup D_2' \cup E}$. Since the closures of D_1, D_2, D_1', D_2' are intervals, there are $D_1'' \in \{D_1, D_2\}$, $D_2'' \in \{D_1', D_2'\}$ with disjoint closures. Without loss of generality, we may assume that $D_1'' = D_1$ and $D_2'' = D_2'$. Let u_E'' be the function such that $\{u_{D_1}, u_{D_2'}, u_E''\}$ is a normalized additive representation of $\succeq_{D_1 \cup D_2' \cup E}$. Since $\{u_{D_1}, u_E\}$ and $\{u_{D_1}, u_E''\}$ (resp. $\{u_{D_2'}, u_E'\}$ and $\{u_{D_2'}, u_E''\}$) are normalized additive representations of $\succeq_{D_1 \cup E}$ (resp. $\succeq_{D_2' \cup E}$), we conclude that $u_E = u_E'' = u_E'$.

Case 3. $\mathrm{cl}(C \cup E) = \mathrm{T}$ and $\mathrm{cl}(E) \neq \mathrm{T}$. There are $D_1, D_2 \in \mathcal{J}_2 \cap \mathcal{I}$ such that E, D_1, D_2 have pairwise disjoint closures. Let u_{D_1}, u_{D_2} be the functions defined in Case 2. There is a unique function u_E such that $\{u_{D_1}, u_{D_2}, u_E\}$ is a normalized additive representation of $\succeq_{D_1 \cup D_2 \cup E}$. In a way similar to Case 2, we can show that u_E is well defined, i.e., is independent of the choice of D_1, D_2.

Before considering the remaining case $\mathrm{cl}(E) = \mathrm{T}$, we show that

$$u_{E_1 \cup E_2}(p_{E_1 \cup E_2}) = u_{E_1}(p_{E_1}) + u_{E_2}(p_{E_2}) \tag{27}$$

for any disjoint $E_1, E_2 \in \mathcal{R}(\mathcal{I})$ such that $\mathrm{cl}(E_1 \cup E_2) \neq \mathrm{T}$. The proof is straightforward if E_1, E_2 have disjoint closures. Hence assume that the closures intersect. Since $\mathrm{cl}(E_1 \cup E_2) \neq \mathrm{T}$, there is $E_3 \in \mathcal{I}$ such that $E_1 \cup E_2, E_3$ have disjoint closures. Put $\mathrm{U}_i := u_{E_i}(\mathcal{P}_{E_i})$, $i = 1,2,3$. Since $\succeq_{E_i}$ is non-trivial, $\mathcal{P}_{E_i}$ is connected, and u_{E_i} is continuous, U_i is a proper interval. By condition (II), there are strictly increasing in each argument functions $g : (\mathrm{U}_1 + \mathrm{U}_3) \times \mathrm{U}_2 \to \mathrm{R}$, $h : \mathrm{U}_1 \times (\mathrm{U}_3 + \mathrm{U}_2) \to \mathrm{R}$ such that

$$\begin{aligned} g(u_{E_1}(p_{E_1}) + u_{E_3}(p_{E_3}), u_{E_2}(p_{E_2})) &= g(u_{E_1 \cup E_3}(p_{E_1 \cup E_3}), u_{E_2}(p_{E_2})) = u_{E_1 \cup E_3 \cup E_2}(p_{E_1 \cup E_3 \cup E_2}) \\ &= h(u_{E_1}(p_{E_1}), u_{E_3 \cup E_2}(p_{E_3 \cup E_2})) = h(u_{E_1}(p_{E_1}), u_{E_3}(p_{E_3}) + u_{E_2}(p_{E_2})), \end{aligned} \tag{28}$$

where the first (resp. the forth) equality follows from the fact that E_1, E_3 (resp. E_3, E_2) have disjoint closures and (27). Denoting $x_i = u_{E_i}(p_{E_i})$, $i = 1,2,3$ and using condition (IV)', we conclude that

$$g(x_1 + x_3, x_2) = h(x_1, x_3 + x_2) \tag{29}$$

holds for all $(x_1, x_2, x_3) \in \mathrm{U}_1 \times \mathrm{U}_2 \times \mathrm{U}_3$. From a result of Maksa (2000, Lemma 1) it follows that there is a strictly increasing function $\varphi : \mathrm{U}_1 + \mathrm{U}_2 + \mathrm{U}_3 \to \mathrm{R}$ such that $u_{E_1 \cup E_2 \cup E_3}(p_{E_1 \cup E_2 \cup E_3}) = \varphi(u_{E_1}(p_{E_1}) + u_{E_2}(p_{E_2}) + u_{E_3}(p_{E_3}))$. Thus, $\{u_{E_1} + u_{E_2}, u_{E_3}\}$ (where by $u_{E_1} + u_{E_2}$ we mean the function $p_{E_1 \cup E_2} \mapsto u_{E_1}(p_{E_1}) + u_{E_2}(p_{E_2})$) is a normalized additive representation of $\succeq_{(E_1 \cup E_2) \cup E_3}$ and (27) follows.

Case 4. $\mathrm{cl}(E) = \mathrm{T}$. Put $u_E(p_E) := u_{E_1}(p_{E_1}) + u_{E_2}(p_{E_2}) + u_{E_3}(p_{E_3})$, where $E_1, E_2, E_3 \in \mathcal{I}$ is a partition of E. From (27) it follows that u_E is well defined, i.e., is independent of the choice of

E_1, E_2, E_3. We have to show that u_E represents $\succeq_E$. Put $\mathrm{U}_i := u_{E_i}(\mathcal{P}_{E_i})$, $i = 1,2,3$. Reproducing the argument used to obtain Eqs. (28) and (29), we conclude that there is a strictly increasing function $\varphi : \mathrm{U}_1 + \mathrm{U}_2 + \mathrm{U}_3 \to \mathrm{R}$ such that $p_E \mapsto \varphi(u_{E_1}(p_{E_1}) + u_{E_2}(p_{E_2}) + u_{E_3}(p_{E_3}))$ (and, therefore, u_E) represents $\succeq_E$.

From the definition of u_E in Case 4 it follows that (27) remains valid if $\mathrm{cl}(E_1 \cup E_2) = \mathrm{T}$. Therefore, for any $p \in \mathcal{P}$, $E \mapsto u_E(p_E)$ is a finitely additive set function on $\mathcal{R}(\mathcal{I})$. For a fixed $t_0 \in \mathrm{T}$, by $U : \mathcal{P} \to \mathrm{R}^{\mathrm{T}}$ denote the map given by

$$U(p)(t) = \begin{cases} -u_{(t,t_0]}(p_{(t,t_0]}) & \text{if } t < t_0 \\ 0 & \text{if } t = t_0 \\ u_{(t_0,t]}(p_{(t_0,t]}) & \text{if } t > t_0 \end{cases}.$$

By construction, for each $t \in \mathrm{T}$, $p \mapsto U(p)(t)$ is continuous; for any $E \in \mathcal{I}$, $\mu_{U(p)}(E) = \mu_{U(q)}(E)$ if $p_E = q_E$. It is straightforward to show that for any $E \in \mathcal{R}(\mathcal{I})$, $p_E \mapsto \mu_{U(p)}(E)$ represents $\succeq_E$.

From (IV)' it follows that there is a numerical representation $I : \mathcal{H}(\mathcal{P}, \mathcal{R}(\mathcal{I})) \xrightarrow{\text{onto}} \mathrm{D} := u_C(\mathcal{P}_C)$ of $\succeq$ such that for each $E \in \mathcal{R}(\mathcal{I})$, the restriction of I to $\mathcal{P}_E$ is onto D. Since $p_E \mapsto \mu_{U(p)}(E)$ is a numerical representation of $\succeq_E$, there is a strictly increasing function $\phi_E : \mu_{U(\mathcal{P})}(E) \xrightarrow{\text{onto}} \mathrm{D}$ such that the restriction of I to $\mathcal{P}_E$ is given by $p_E \mapsto \phi_E \circ \mu_{U(p)}(E)$. Since $\mu_{U(\mathcal{P})}(E)$ and D are intervals and ϕ_E is monotone, it is continuous. By (IV)', for any pairwise disjoint $E_1, \ldots, E_n \in \mathcal{R}(\mathcal{I})$, $\{(\mu_{U(p)}(E_1), \ldots, \mu_{U(p)}(E_n)), p \in \mathcal{P}\} = \prod_{i=1}^{n} \mu_{U(\mathcal{P})}(E_i)$.

(b). Let D, U, and $\{\phi_E, E \in \mathcal{R}(\mathcal{I})\}$ be as in part (a). By (IV)', for any disjoint $A, B \in \mathcal{R}(\mathcal{I})$ and $d \in \mathrm{D}$, there is $p \in \mathcal{P}$ such that $\mu_{U(p)}(A) = \phi_A^{-1}(d)$, $\mu_{U(p)}(B) = \phi_B^{-1}(d)$. Then condition (V) implies $\mu_{U(p)}(A \cup B) = \phi_{A \cup B}^{-1}(d)$, so that for any $d \in \mathrm{D}$, $E \mapsto \phi_E^{-1}(d)$ is a finitely additive set function on $\mathcal{R}(\mathcal{I})$: $\phi_A^{-1}(d) + \phi_B^{-1}(d) = \mu_{U(p)}(A) + \mu_{U(p)}(B) = \mu_{U(p)}(A \cup B) = \phi_{A \cup B}^{-1}(d)$. Therefore, there is a function $\tilde{g} : \mathrm{D} \times \mathrm{T} \to \mathrm{R}$ such that $\phi_E^{-1}(d) = -\mu_{\tilde{g}(d,\cdot)}(E)$, $E \in \mathcal{R}(\mathcal{I})$. The function $d \mapsto -\mu_{\tilde{g}(d,\cdot)}(E)$ is strictly increasing and onto $\mu_{U(\mathcal{P})}(E)$ (since so is ϕ_E^{-1}).

It is straightforward to show that if, in addition to the conditions of part (a)/(b), condition (III) holds, then U can be chosen to satisfy $U(p) \equiv 0$, whenever p is a constant function.

Now assume that, in addition to the conditions of part (a)/(b), condition (PI) holds. By (PI), $\mu_{U(p)}((t,t']) = h(p(t),t; p(t'),t')$ for some function $h : \mathrm{V} \to \mathrm{R}$. We have

$$h(s,t;s',t') + h(s',t';s'',t'') = h(s,t;s'',t''), \tag{30}$$

provided that $t < t' < t''$. Extend the domain of h to $(\mathrm{S} \times \mathrm{T})^2$ by setting $h(s',t';s,t) := -h(s,t;s',t')$, $(t,t') \in \mathrm{T}^2_<$ and $h(s,t;s',t) := h(s_0,t_0;s',t) - h(s_0,t_0;s,t)$, $s_0 \in \mathrm{S}$, $t_0 \neq t$. It is straightforward to show that the extension is well defined, i.e., is independent of the choice of s_0 and t_0. Then the Sincov functional equation (30) holds for all $(s,t;s',t';s'',t'') \in (\mathrm{S} \times \mathrm{T})^3$. Its general solution is given by $h(s,t;s',t') = \tilde{u}(s',t') - \tilde{u}(s,t)$ for some function $\tilde{u} : \mathrm{S} \times \mathrm{T} \to \mathrm{R}$. Since for any $E \in \mathcal{I}$,

$p \mapsto \mu_{U(p)}(E)$ is continuous, so is $\tilde{u}(\cdot,t)$ for each $t \in \mathrm{T}$. Condition (IV)' implies that for any $t < t' < t''$,

$$\tilde{u}(\mathrm{S},t') - \tilde{u}(\mathrm{S},t) + \tilde{u}(\mathrm{S},t'') - \tilde{u}(\mathrm{S},t') = \tilde{u}(\mathrm{S},t'') - \tilde{u}(\mathrm{S},t), \tag{31}$$

where $+$ (resp. $-$) is the Minkowski sum (resp. difference) operation. Since $\tilde{u}(\mathrm{S},t)$, $\tilde{u}(\mathrm{S},t')$, $\tilde{u}(\mathrm{S},t'')$ are proper intervals, Eq. (31) implies $\tilde{u}(\mathrm{S},t'') - \tilde{u}(\mathrm{S},t) = \mu_{U(\mathcal{P})}((t,t'']) = \mathrm{R}$. ■

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