EUROPEAN
UNIVERSITY AT
SAINT-PETERSBURG

Kirill Borissov

On equilibrium dynamics
with many agents
and wages paid ex ante

Working paper Ec-05/11

Department of Economics

St. Petersburg
2011

УДК 330.35
ББК 65.012.2
B78

Европейский университет в Санкт-Петербурге

Кирилл Борисов
Равновесная динамика с многими агентами и зарплатой, выплачиваемой в начале периода
на английском языке

Серия препринтов; Ec-05/11; Факультет экономики
Санкт-Петербург, 2011

Borissov K.
B78 On equilibrium dynamics with many agents and wages paid ex ante / Kirill Borissov: Working paper Ec-05/11; Department of Economics. — St. Petersburg, 2011. — 12 p.

A model of economic growth with many agents and borrowing constraints is considered under the assumption that wages are paid ex ante. It is shown that, in contrast to the traditional case where wages are paid ex post, the convergence of equilibrium paths to a steady-state equilibrium occurs regardless of specifications of technology.

Kirill Borissov, European University at St. Petersburg, 3 Gagarinskaya Str., Saint-Petersburg 191187, Russia, and Saint-Petersburg Institute for Economics and Mathematics (RAS), 1 Tchaikovskogo Str., Saint-Petersburg 191187, Russia. E-mail: kirill@eu.spb.ru.

Kirill Borissov thanks the Russian Foundation for Basic Research (grant No. 11-06-00183-a).

On equilibrium dynamics with many agents and wages paid *ex ante*

Kirill Borissov

European University in St. Petersburg, 3 Gagarinskaya Str., St. Petersburg, 191187 RUSSIA,

and St. Petersburg Institute for Economics and Mathematics (Russian Academy of Sciences),

1 Tchaikovsky Str., St. Petersburg, 191187 RUSSIA

(e-mail: kirill@eu.spb.ru)

Abstract

A model of economic growth with many agents and borrowing constraints is considered under the assumption that wages are paid *ex ante*. It is shown that, in contrast to the traditional case where wages are paid *ex post*, the convergence of equilibrium paths to a steady-state equilibrium occurs regardless of specifications of technology.

Keywords: Economic growth; Heterogeneous agents; Stability

JEL classification: D91; O41; C61

1. Introduction

The problem of timing of wage payments is an old one. The Physiocrats, some Classical economists and the founders of Austrian theory of capital assumed wages to be paid *ex ante*, that is, at the beginning of the (uniform) production period, and thus as belonging to the capital advanced in each industry. Nowadays the absolute majority of economists reckon wages as a part of the net product; that is, they took them to be paid *ex post*.

The problem of weather wages are paid *ex post* or *ex ante* is ignored by modern economic literature[1]. The assumption that wages are paid *ex post* is made through habit. However, no of the two assumptions seems more realistic than the other one because the period of wage payments is conceptually distinct from the production period, the two periods need not coincide, and the very assumption of a single period of production is a convenient simplification.

It may erroneously appear that the distinction between paying wages *ex post* and *ex ante* can be made only in discrete time models. However, this distinction does not disappear in continuous time models though the question of whether a discrete or continuous time framework should be chosen when

---

[1] To the best of my knowledge, the last paper discussing this problem was published many years ago (Harris, 1981)

modeling dynamic economic processes is related to the problem of timing of wage payments. This question is sometimes discussed in the modern literature (see, e.g., Bosi and Ragot (2010)).

In this note, I consider a model of economic growth with many agents and borrowing constraints under the assumption that wages are paid *ex ante* and show that such a model exhibits different stability properties than the standard model with *ex post* wage payments. It is well-known that in models with consumers differing in time-preference rates the existence of a non-negativity constraint on wealth can be the cause of cyclical and complicated dynamics (see, e.g., Becker and Foias (1987), Becker and Foias (1994), Sorger (1994), Bosi and Seegmuller (2010)). “Ultimately, whether the given Ramsey economy exhibits a stable solution ..., or one of the cyclic, or even chaotic equilibria emerges, turns on the economy’s elasticity of substitution in production” (Becker, 2006, p. 436). I show that if wages are paid *ex ante*, then complicated dynamics cannot arise and the convergence of the economy to the long-run steady state from an arbitrary initial state occurs regardless of specifications of technology.

In Section 2, I shortly explain what is the difference between paying wages *ex post* and *ex ante*. The model with many agents and borrowing constraints under the assumption that wages are paid *ex ante* is introduced in Section 3. The stability theorem is presented in Section 4.

2. What is the difference between paying wages *ex post* and *ex ante*

Let us first explain what is the difference between paying wages *ex post* and *ex ante*. Suppose that output Y is determined by a neoclassical production function

$$Y = F(K, L),$$

where K is the stock of fully depreciating real capital and L is the amount of labor used in production.

If wages are paid *ex post*, equilibrium on the capital and labor markets implies that the interest and wage rates, r and w, are given by

$$1 + r = F_K(K, L), \; w = F_L(K, L)$$

or, equivalently, by

$$1 + r = f'(k), \; w = f(k) - f'(k)k,$$

where $f(k) = F(k,1)$, $k = K/L$. This follows from the form of the profit maximization problem for the production sector:

$$\max\{F(K,L) - [(1+r)K + wL]\}.$$

If wages are paid *ex ante*, the profit maximization problem is as follows:

$$\max\{F(K,L) - (1+r)[K + wL]\}.$$

In this case the equilibrium interest and wage rates, r and w, are determined by

$$1+r = f'(k),\ w = \frac{f(k) - f'(k)k}{f'(k)}.$$

In what follows I assume that $f'(0) > 1$ and that labor supply is constant over time and normalized to 1.

To clarify the role of timing of wage payments, first consider two very simple discrete-time models in which all wages are spent on consumption and all capital income is invested. If wages are paid *ex post*, then the dynamics of is given by the following relationships:

$$s_{t+1} = (1+r_t)s_t,\ k_t = s_t,\ w_t = f(k_t) - f'(k_t)k_t,$$

$$1+r_t = f'(k_t),\ c_t = w_t.$$

where c_t and s_t are the time *t* consumption and savings. Hence

$$k_{t+1} = f'(k_t)k_t,\ t = 0,1,\dots. \tag{1}$$

The only non-zero steady state of the dynamic system given by (1) is the golden rule capital stock k^* defined as the solution to the equation $f'(k) = 1$. Stability properties of the model depend on whether total capital income $f'(k)k$ is increasing in the capital stock k or not. If $f'(k)k$ is always increasing in the capital stock k, then k^* is globally asymptotically stable: any sequence k_t given by (1) and starting from $k_0 > 0$ converges to k^*. At the same time, if $(f'(k^*)k^*)' < -1$, then the steady state k^* is unstable.

Now suppose that wages are paid *ex ante*. In this case the relationships defining the model are as follows:

$$s_{t+1} = (1+r_t)s_t\,,\ k_t + w_t = s_t\,,\ 1+r_t = f'(k_t)\,,$$

$$w_t = \frac{f(k_t) - f'(k_t)k_t}{f'(k_t)}\,,\ c_t = w_t\,.$$

Hence

$$f(k_{t+1})\,/\,f'(k_{t+1}) = f(k_t)\,. \qquad (2)$$

The dynamic system given by (2) has the same non-zero steady state k^* as above. Since both $f(k)/f'(k)$ and $f(k)$ are increasing in k, the convergence $k_t \to k^*$ occurs *regardless* of the assumptions on $f'(k)k$.

To elucidate the difference in stability properties of the two models, let us make a distinction between real capital and capital understood in the broad sense as the part of wealth which is used for production. In the model where wages are paid *ex post*, the part of wealth which is used for production is equal to the stock of real capital. If wages are paid *ex ante*, the stock of capital in the broad sense at time t, κ_t, is given by $\kappa_t = k_t + w_t$, where $w_t = \frac{f(k_t) - f'(k_t)k_t}{f'(k_t)}$. Therefore, $\kappa_t = g(k_t)$, where $g(k) = f(k)/f'(k)$. Thus, (2) can be rewritten as

$$\kappa_{t+1} = \varphi(\kappa_t)\,,$$

where $\varphi(\kappa) = f[g^{-1}(\kappa)]$. The function $\varphi(\kappa)$ shows the dependence of total capital income on the stock of capital in the broad sense. It is clear that this function is increasing regardless of the elasticity of substitution in production.

In the next section I introduce a model with heterogeneous agents and borrowing constraints, but first the following point on the Ramsey model with a representative agent should be made. One of the main properties of this model is that under the traditional assumption that wages are paid *ex post* optimal and equilibrium paths are the same thing. It is not difficult to show that the same is true under the assumption that wages are paid *ex ante*.

3. The model

There is a finite set $J = \{1,\ldots,|J|\}$ of consumers. Let $u : \Re_+ \to \Re$ be a felicity function satisfying

$$u'(0) = +\infty\,;\; u'(c) > 0\,\forall c > 0\,; u''(c) < 0\,\forall c > 0\,;\; u'(\infty) = 0\,.$$

This felicity function is the same for all consumers. As for the discount factors of consumers, $\beta_j,\; j \in J,$ they may differ. Each consumer $j \in J$ is endowed with $1/|J|$ units of labor force at each time. Savings must be non-negative so that future income cannot be discounted to the present. Since credit markets are imperfect, per-period budget constraints cannot be aggregated in a unique intertemporal budget constraint.

Given $(1+r_{-1})s_{-1}^j \geq 0$, $\{r_t\}_{t=0,1,\ldots}$ and $\{w_t\}_{t=0,1,\ldots}$ consumer $j \in J$ solves the following problem:

$$\max\{\sum_{t=0}^{\infty} \beta_j^t u(c_t^j) \mid s_t^j + c_t^j \leq (1+r_{t-1})s_{t-1}^j + w_t / |J|,$$
$$s_t^j \geq 0,\; t = 0,1,\ldots\}.$$

Here s_t^j and c_t^j are the savings and consumption of consumer *j* at time *t*.

Suppose that we are given a non-degenerate initial state of the economy represented by a tuple $\{k_{-1}^*,\, (1+r_{-1}^*)s_{-1}^{j*},\;\; j \in J\}$ such that $k_{-1}^*>0$, $(1+r_{-1}^*)s_{-1}^{j*} \geq 0,\;\; j \in J,$ and $\sum_{j\in J}(1+r_{-1}^*)s_{-1}^{j*} = f(k_{-1}^*)$. An *equilibrium path* starting from this initial state is defined as a sequence $\{r_t^*, w_t^*, k_t^*, \{s_t^{j\,*}\}_{j\in J}, \{c_t^{j\,*}\}_{j\in J}\}_{t=0,1,\ldots}$ satisfying the following conditions:

- $1 + r_t^* = f'(k_t^*),\;\; t = 0,1,\ldots;$
- $w_t^* = \dfrac{f(k_t^*) - f'(k_t^*)k_t^*}{f'(k_t^*)},\;\; t = 0,1,\ldots;$
- for each $j \in J$, $\{s_t^{j\,*}, c_t^{j\,*}\}_{t=0,1,\ldots}$ is a solution to

$$\max\{\sum_{t=0}^{\infty}\beta^t u(c_t)\,|\, s_0 + c_0 \le (1+r_{-1}^{j\,*})s_{-1}^{j\,*} + w_0^*,$$
$$s_t + c_t \le (1+r_{t-1}^*)s_{t-1} + w_t^*,\ s_{t-1}\ge 0,\ t=1,2,...\}; \tag{3}$$

- $$k_t^* + w_t^* = \sum_{j\in J} s_t^{j\,*},\ t=0,1,\dots .$$

It is not difficult to check that on any equilibrium path $\{r_t^*, w_t^*, k_t^*, \{s_t^{j\,*}\}_{j\in J}, \{c_t^{j\,*}\}_{j\in J}\}_{t=0,1,...}$ the natural balance holds:

$$k_t^* + \sum_{j\in J} c_t^{j\,*} = f(k_{t-1}^*),\ t=0,1,\dots,$$

and that

$$(1+r_t^*)\sum_{j\in J} s_t^{j\,*} = f(k_t^*),\ t=0,1,\dots .$$

By using the argument developed in Becker, Boyd III and Foias (1991), it is possible to show that *for any non-degenerate initial state there is an infinite equilibrium path starting from this initial state.*

A tuple $\{r^*, w^*, k^*, \{s^{j*}\}_{j\in J}, \{c^{j*}\}_{j\in J}\}$ is called a *steady-state equilibrium* if the sequence $\{r_t^*, w_t^*, k_t^*, \{s_t^{j\,*}\}_{j\in J}, \{c_t^{j\,*}\}_{j\in J}\}_{t=0,1,...}$ given for all $t=0,1,...$ by

$$r_t^* = r^*,\ w_t^* = w^*,\ k_t^* = k^*,\ s_t^{j\,*} = s^{j*},\ j\in J,\ c_t^{j\,*} = c^{j*},\ j\in J,$$

is an equilibrium path starting from the initial state $\{k^*,\ (1+r^*)s^{j*},\ j\in J\}$

Assume for simplicity that $\beta_1 > \beta_j,\ j\ne 1$.

It is easy to show that, as in the case where wages are paid *ex post* (see Becker (1980)), in a steady-state equilibrium all capital in the broad sense belongs to the most patient consumer, whereas all other consumers spend all their wages on consumption. Namely, *there is a unique steady-state equilibrium* $\{r^*, w^*, k^*, \{s^{j*}\}_{j\in J}, \{c^{j*}\}_{j\in J}\}$ *given by*:

$$1+r^*=1/\beta\,,\;\; f'(k^*)=1+r^*,\;\; w^*=\frac{f(k^*)-f'(k^*)k^*}{f'(k^*)}\,,$$

$$s^{1*}=k^*+w^*,\;\; c^{1*}=w^*/|J|+r^*(k^*+w^*)\,,$$

$$s^{j*}=0,\;\; c^{j*}=w^*/|J|,\;\; j\neq 1\,.$$

4. Main result

Now we can prove the main result of this note, which says that any equilibrium path converges to the steady-state equilibrium *regardless* of the economy's elasticity of substitution in production.

Theorem. *For any equilibrium path* $\{r_t^*, w_t^*, k_t^*, \{s_t^{j\,*}\}_{j\in J}, \{c_t^{j\,*}\}_{j\in J}\}_{t=0,1,...}$ *starting from a non-degenerate initial state, i*) $k_t^* \to k^*$, *ii*) $c_t^{1\,*} \to c^{1*}$ *and iii*) *for all* $j\neq 1$ *and for sufficiently large t,* $c_t^{j\,*} = w_t^*/|J|,\;\; s_t^{j\,*}=0$.

Proof. Let $\{r_t^*, w_t^*, k_t^*, \{s_t^{j\,*}\}_{j\in J}, \{c_t^{j\,*}\}_{j\in J}\}_{t=0,1,...}$ be an equilibrium path starting from a non-degenerate initial state.

Lemma 1. *If* $\lim_{t\to\infty} k_t^* = k_\infty$ *exists, then* $k_\infty = k^*$, *and* $s_t^{j\,*}=0$ and $c_t^{j*} = w_t^*/|J|$ *for all* $j\neq 1$ *and t large enough.*

Lemma 2. *If* $k_t^* \le k^*$ *for all t large enough, then* $\lim_{t\to\infty} k_t^* = k_\infty$ *exists*.

Lemma 1 and Lemma 2 are the counterparts of Becker and Foias (1987, Propositions 4 and 5) and can be proved in just the same way.

Lemma 3. *Either*

$$k_0^* \ge k_1^* \ge k_2^* \ge \ldots \ge k^* \tag{4}$$

or there exists a τ *such that*

$$k_t^* < k^*, \; t > \tau. \qquad (5)$$

This lemma is the key to understanding the difference between the *ex post* and the *ex ante* models. It is a close analogue to Becker and Foias (1987, Lemma 2) and is proved in essentially the same way, but if Becker and Foias (1987, Lemma 2) is based on the assumption that $f'(k)k$ is always increasing in k, Lemma 3 is not based on such an assumption.

Proof of Lemma 3. Suppose that $k_\tau^* > k^*$ and $k_\tau^* \geq k_{\tau-1}^*$ for some τ. Let $J(\tau)=\{j\in J \mid x_\tau^{j*} > 0\}$. Since $1+r_\tau^* < 1+r^*$, we have

$$\beta_j(1+r_\tau^*) < \beta_j(1+r^*) \leq 1, \; j \in J.$$

For each $j\in J$, $\{s_t^{j*}, c_t^{j*}\}_{t=0,1,\ldots}$ is a solution to (3). It satisfies the following first-order conditions:

$$\beta_j(1+r_t^*)u'(c_{t+1}^{j*}) \leq u'(c_t^{j*}) \;\; (= \text{ if } \; s_t^{j*} > 0), \; t=0,1,\ldots.$$

Therefore,

$$u'(c_{\tau+1}^{j*}) > u'(c_\tau^{j*}), \; j \in J(\tau).$$

and hence

$$c_{\tau+1}^{j*} < c_\tau^{j*}, \; j \in J(\tau). \qquad (6)$$

We have

$$\sum_{j\in J(\tau)}(1+r_\tau^*)\, s_\tau^{j*} = f(k_\tau^*) \geq f(k_{\tau-1}^*)$$

$$\geq \sum_{j\in J(\tau)}(1+r_{\tau-1}^*)\, s_{\tau-1}^{j*}$$

Because of (6), we get

$$\sum_{j\in J(\tau)}[(1+r_\tau^*)\, s_\tau^{j*} - c_{\tau+1}^{j*}] > \sum_{j\in J(\tau)}[(1+r_{\tau-1}^*)\, s_{\tau-1}^{j*} - c_\tau^{j*}] \qquad (7)$$

Now suppose that

$$k^*_{\tau+1} < k^*_{\tau} \tag{8}$$

and hence $w^*_{\tau+1} < w^*_{\tau}$. We have

$$k^*_{\tau} + w^*_{\tau} = \sum_{j \in J(\tau)} s^{j\,*}_{\tau}$$

$$= \sum_{j \in J(\tau)} [(1 + r^*_{\tau-1})\, s^{j\,*}_{\tau-1} + w^*_{\tau}/|J| - c^{j*}_{\tau}].$$

Therefore,

$$k^*_{\tau} = \sum_{j \in J(\tau)} [(1 + r^*_{\tau-1})\, s^{j\,*}_{\tau-1} - c^{j*}_{\tau}] - \frac{|J \setminus J(\tau)|}{|J|}\, w^*_{\tau}.$$

At the same time

$$k^*_{\tau+1} + w^*_{\tau+1} \geq \sum_{j \in J(\tau)} s^{j\,*}_{\tau+1}$$

$$= \sum_{j \in J(\tau)} [(1 + r^*_{\tau})\, s^{j\,*}_{\tau} + w^*_{\tau+1}/|J| - c^{j*}_{\tau+1}]$$

and hence

$$k^*_{\tau+1} \geq \sum_{j \in J(\tau)} [(1 + r^*_{\tau})\, s^{j\,*}_{\tau} - c^{j*}_{\tau+1}] - \frac{|J \setminus J(\tau)|}{|J|}\, w^*_{\tau+1}.$$

Since, by assumption,

$$\frac{|J \setminus J(\tau)|}{|J|}\, w^*_{\tau+1} < \frac{|J \setminus J(\tau)|}{|J|}\, w^*_{\tau},$$

it follows from (7) that $k^*_{\tau+1} > k^*_{\tau}$, which contradicts (8).

This contradiction shows that if

$$k^*_{\tau} > k^* \text{ and } k^*_{\tau} \geq k^*_{\tau-1}, \tag{9}$$

then $k^*_{\tau+1} \geq k^*_\tau$ and hence $\lim k^*_t$ exists and is larger than k^*. This implies that (9) cannot hold for any $\tau \geq 1$. Thus, if $k^*_t \geq k^*$ for all t=0,1,..., then (4) must hold, and if there exists t such that $k^*_t < k^*$, then (5) must hold. □

If alternative (4) holds, then Theorem follows from Lemma 1. If alternative (5) holds, then it follows from Lemmas 1 and 2. □

Acknowledgement

Financial support from Russian Foundation for Basic Research (grant No. 11-06-00183-a) is gratefully acknowledged.

References

Becker, R. A. (1980) On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households, *Quarterly Journal of Economics*, 95 (1980), pp. 375-382.

Becker, R. A. (2006) Equilibrium dynamics with many agents. In: Dana R.-A., Le Van, C., Mitra, T. and Nishimura, K. (Eds.) *Handbook of Optimal Growth* 1. Discrete Time. Springer, 2006.

Becker, R. A. and C. Foias (1987) A Characterization of Ramsey Equilibrium, *Journal of Economic Theory*, 41, pp. 173-184.

Becker, R.A., and C. Foias, (1994). The local bifurcation of Ramsey equilibrium. *Economic Theory*, 4, pp. 719–744.

Becker, R. A., J. H. Boyd III, and C. Foias (1991) The Existence of Ramsey Equilibrium, *Econometrica*, 59, pp. 441-460.

Bosi, S. and L. Ragot (2010) Time, Bifurcations and Economic Applications, *Thema Working Paper* n°2010-01, Université de Cergy Pontoise, France.

Bosi, S. and T. Seegmuller (2010) On the Ramsey equilibrium with heterogeneous consumers and endogenous labor supply, *Journal of Mathematical Economics*, 46, pp. 475–492.

Harris, D. J. (1981) On the timing of wage payments. *Cambridge Journal of Economics*, 1981, 5, pp. 369-391.

Sorger, G. (1994) On the structure of Ramsey equilibrium: cycles, indeterminacy, and sunspots, *Economic Theory*, 4, pp. 745–764.