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Estimation of Multivariate

Stochastic Volatility Models:

A Comparative Monte Carlo Study

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JEL Classification: C32

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Estimation of Multivariate Stochastic Volatility Models: A Comparative Monte Carlo Study

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Preliminary version

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1 Introduction

In financial time series literature, it is already established that the volatilities of asset returns are changing over time. Moreover, they are likely to be serially correlated. To illustrate this stylized fact with an example, in Figure 1 we present the indices and returns (100 x $\log(P_t/P_{t-1})$) of FTSE-100 and DAX stock markets between dates 4/1/2005 and 4/11/2011. We also plot the squared returns, as a proxy of volatilities, and a rolling window estimate of correlations, with a window of 60 days. It is observed that the volatilities are changing over time. Moreover, the volatilities are clustered; *i.e.* higher (lower) values of volatilities are followed by higher (lower) values, which implies that the volatilities are serially correlated. To capture this kind of a dynamic volatility effect, the generalized autoregressive conditional heteroskedasticity (GARCH) models have been proposed by Engle (1982) and Bollerslev (1986). In GARCH models the time varying volatility is modelled as a deterministic function of squared previous day returns and previous day volatilities; therefore in GARCH approach the volatilities are observation driven. Currently a wide range of GARCH models are available in the literature and are well documented in the surveys: see Bollerslev *et al.* (1992) for univariate and Bauwens *et al.* (2006), Silvennoinen *et al.* (2009) for multivariate models.

An alternative approach to modelling time varying volatility is to consider it as an unobserved component and let the logarithm of it follow an autoregressive process. Therefore in this approach, the volatilities are parameter driven. Models of this kind are named as stochastic volatility (SV) models in the literature. The SV approach is attractive because of its similarity to the models used in financial theory to describe the behavior of prices; see Hull and White (1987), Taylor (1986, 1994), and Shephard and Andersen (2008) for the origins of SV models. Moreover it has been shown that the SV models describe the behavior of volatilities more accurately compared to GARCH models (see for example Danielsson (1994), Kim *et al.* (1998), and Carnero *et al.* (2004)). Given the way the SV models are set up, their statistical properties are easy to derive from the process that the volatilities follow. However, although statistically more attractive than GARCH models, SV models have the disadvantage in terms of estimation because their exact likelihoods are difficult to evaluate. The following survey papers are available about the univariate and multivariate SV models and estimation methods: Broto and Ruiz (2004), Asai *et al.* (2006), Chib *et al.* (2009), Ghysels *et al.* (1996), Yu and Meyer (2004), Maasoumi and McAleer (2006).

Several methods have been proposed for estimating SV models. A relatively easy approach is the quasi-maximum likelihood estimation (QML) proposed independently by Nelson (1988) and Harvey *et al.* (1994). In this approach, the log-squared returns are modelled as a linear state space form where the transformed innovations are assumed to follow a Gaussian distribution although in fact the true distribution is based on $\ln \chi_1^2$ (see Sandman and Koopman (1998) for the univariate and Asai and McAleer (2006) for the multivariate case). Ruiz (1994) showed that the QML estimators are consistent and asymptotically normal. However due to the Gaussianity assumption, QML approach is an estimation based on approximations and therefore, as noted by several papers as Jacquier *et al.* (1994), Breidt and Carriquiry (1996) and Sandmann and Koopman (1998), QML estimator is inefficient.

The evaluation of exact likelihood requires high dimensional integration which could be based on evaluating these integrals with simulation methods and then maximizing the resulting likelihood function. This class of estimation approaches include the accelerated importance sampling (AGIS) approach developed in Danielsson and Richard (1993) and efficient importance sampling (EIS) approach proposed by Liesenfeld and Richard (2003, 2006), and the Monte Carlo likelihood (MCL) approach proposed by Sandman and Koopman (1998). Different from the QML estimation, the MCL method of Sandman and Koopman (1998) used log-squared transformation of returns taking into account the true distribution of the errors and therefore modelling the log-squared returns via a linear non-Gaussian state space model. A review of these importance sampling methods could be found again in Asai *et al.* (2006).

The MCL method considered in this paper is the one proposed by Jungbacker and Koopman (2006) that extended the theoretical results of Shephard and Pitt (1997), Durbin and Koopman (1997), and Jungbacker and Koopman (2005). In this method, the returns are modelled without the log-squared transformation. Durbin and Koopman (1997) showed that the loglikelihood of the state space models with non-Gaussian errors can be written as a sum of the loglikelihood of the approximating Gaussian model and a correction for the departures from the Gaussian assumptions with respect to the true model. This form of likelihood has the advantage that the simulations are only required for the departures of the likelihood of the true model from the Gaussian likelihood, rather than for the likelihood itself. Jungbacker and Koopman (2006) used this approach to estimate three multivariate stochastic volatility (MSV) models: the stochastic time varying scaling factor model, where the variance matrix of the returns are scaled by the log-volatilities, the constant correlation MSV model of Harvey *et al.* (1994) and a time varying correlation MSV model based on Cholesky decomposition. In the latter set up, the correlation dynamics is driven by the volatilities and a correlation parameter. Tsay (2005) adopted a Cholesky decomposition based approach to ensure the positive definiteness of the covariance matrix. The MSV model he proposed is basically the same time varying correlation MSV model as considered in Jungbacker and Koopman (2006) with the correlation parameter following a stochastic autoregressive process.

Finally, the Monte Carlo Markov Chain (MCMC) methods are receiving much attention since they provide the most efficient estimation tools (see Andersen *et al.* (1999)). For a survey on MCMC methods and MCMC estimation of several MSV models, see Asai *et al.* (2006), Meyer and Yu (2000), Chib *et al.* (2009). MCMC method will be outside the scope of this paper.

When fitting an MSV model to a financial time series, researchers are ultimately interested in estimating the underlying volatilities and correlations. Therefore, when making a comparison of performances between different estimators, one should also consider looking at their relative performances in estimating the in-sample volatilities and correlations. In this respect, we employ several Monte Carlo (MC) experiments where the performances of QML and MCL methods in estimating the parameters, volatilities and correlations are compared. It is already known that MCL methods have better small sample properties compared to QML methods in parameter estimation. However, in the literature there is a need for Monte Carlo simulation studies comparing QML and MCL methods in terms of in-sample volatility and correlation estimations in a multivariate setup and for different parameter sets. In this paper, we attempt to fill this gap with a number of MC experiments for several models.

For our MC experiments, we first consider the Constant Correlation MSV model of Harvey *et al.* (1994). As pointed out by Tsui and Yu (1999), the correlations do not have to be constant for certain assets. This is also observed in Figure 1 that the estimated correlations are changing over time. For this reason, we also consider the Time Varying Correlation MSV model discussed in Jungbacker and Koopman (2006). Another stylized fact is the so called leverage effect which refers to the negative relation between the current returns and future volatilities. Black (1976) and Christie (1982) found that there is a negative relation between the ex-post volatility of the return rates on assets and the current value of the asset. One way to explain this is that decreasing prices of the assets (negative returns) imply an increased leverage of the firms which is believed to increase uncertainty and hence volatility. (See Gyhsels *et al.* (1996)). As an example, in Figure 1, we see that on average the volatilities between $t = 800$ and $t = 1100$, where the indices are in general falling, are much higher than the volatilities of the period between $t = 1100$ and $t = 1600$, where the indices are in general rising. Jungbacker and Koopman (2005) proposed a univariate SV model with leverage and discussed how to estimate it via MCL method. In our paper we propose a direct multivariate generalization of this model and refer to it as MSV with diagonal leverage, where the correlations between the innovations of returns and volatilities are diagonal. A similar but more restrictive model has been proposed, but not estimated, by Danielsson (1998), where these correlations are modelled as a function of the variances of the innovations in the volatility equations. Asai and McAleer (2006) estimated the MSV with leverage model of Danielsson (1998) via MCL method of Sandman and Koopman (1998) and they provided the log-squared transformation of the model necessary to implement this estimation. Using the transformations they provided, it is also possible to estimate MSV with leverage model of Danielsson (1998) with QML method. Furthermore, we propose the MSV with non-diagonal leverage model where the correlations between the innovations of returns and volatilities are non-diagonal; *i.e.* the innovations of the volatility of series i is correlated with the innovations of the returns of series j. We also provide the necessary transformations to estimate these two MSV with leverage models via MCL method which are derived based on the univariate estimation in Jungbacker and Koopman (2006). We adapt the transformations of Asai and McAleer (2006) for estimating our two MSV models with leverages via QML method.

The results obtained in this paper confirm that QML estimator has lower small sample performance than MCL estimator. When the correlations are constant, the QML estimator is performing closer to the MCL estimator especially when the true value of the underlying correlation is high and/or if the variances of the SV processes are high. Also, when the correlations are let to vary over time, the performance of the QML estimator approaches to that of the MCL estimator even with lower correlations. On the other hand, with low constant correlations and low variances of the SV processes, the efficiency of the QML estimator is relatively lower. When leverage is allowed in the model, the performance of QML estimator is worse in estimating the underlying correlations compared to its performance in the model without leverage. Higher values in the true leverage matrix decreased the performance of QML estimator of the correlations even more. From our results, we conclude that the QML estimator could be used when the series are expected to have high correlations (whether constant or time varying) and when the variances of the SV processes are high. Particularly in the case of MSV models with leverage we do not recommend the use of QML estimator. On the other hand, when it is of interest to estimate models with high number of series, the implementation of QML estimator is easier and more feasible than that of MCL estimator. Moreover, the analytical derivatives needed for the MCL estimation are harder to obtain with large cross-sections. One could choose to use numerical derivatives, but the derivatives obtained by numerical approximation for large state vectors could be very time consuming and numerically unstable. Therefore we come to the conclusion that when estimating MSV models for several series, such as modelling the returns of international stock markets, MCL method should be preferred for all the models considered in this paper. The QML method could be used for the estimation of models with medium-to-large number of series, such as the returns of a high number of assets in a stock market, especially when the series are expected to be highly correlated with high variances in the SV processes.

The paper is organized as follows: in Section 2.2 we discuss briefly the Constant Correlation MSV model, Time Varying Correlation MSV model and the two MSV models with leverage we propose and later provide information on how these models can be estimated via Quasi Maximum Likelihood and Monte Carlo Likelihood methods. In Section 2.3 we explain the set up of our Monte Carlo experiments and discuss the results. In section 2.4, we estimate a trivariate MSV model with leverage for the returns on three major European stock markets. Finally in section 2.5, we discuss further topics for research and conclude.

2 Multivariate Stochastic Volatility (MSV) Models

2.1 The Basic Model

The univariate SV model was proposed by, among others, Taylor (1982, 1986). Harvey *et al.* (1994) extended this univariate SV model to a multivariate context, proposing the first multivariate SV (MSV) model. If we let $y_t = (y_{1t}, y_{2t}, ..., y_{kt})'$ be a kx1 vector of observations at time t and $h_t = (h_{1t}, h_{2t}, ..., h_{kt})'$ be the corresponding log-volatilities, then this model is defined as:

$$
\begin{aligned}
y_t &= H_t^{1/2} \varepsilon_t \\
H_t &= diag\left\{\exp(h_{1t}), \exp(h_{2t}), ..., \exp(h_{kt})\right\} = diag\left\{\exp(h_t)\right\}
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
h_{t+1} &= \Gamma + \Phi h_t + \eta_t \\
h_1 &\backsim N\left((I_k - \Phi)^{-1}\Gamma, \Sigma_0\right)
\end{aligned}
\tag{2}
$$

$$
\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \backsim N\left(0, \begin{bmatrix} P_\varepsilon & 0 \\ 0 & Q_\eta \end{bmatrix}\right)
\tag{3}
$$

where Γ is a kx1 vector of, and Φ is a kxk matrix of parameters. I_k denotes a kxk identity matrix. The covariance matrices P_ε and Q_η are of the corresponding errors ε_t and η_t. The diagonal elements of P_ε are restricted to be equal to one for identification purposes, therefore P_ε is a correlation matrix. For simplicity, we do not consider volatility spillovers, *i.e.* Φ is a diagonal matrix. However, the volatilities h_t are still dependent on each other via Q_η matrix. Finally, the (i, j) element of Σ_0 is the (i, j) element of Q_η divided by $(1 - \Phi_{ii}\Phi_{jj})$.[1] By construction, this model assumes constant correlations, therefore following Yu and Meyer (2006), we will refer to this model as Constant Correlation MSV (CCMSV) model. In our analysis, we focus on the parameters, in order: $\Psi = (vecl(P_\varepsilon)', \Gamma', diag(\Phi)', vech(Q_\eta)')'$.[2] In this model there are $k^2 + 2k$ parameters to estimate.

[1] That is, Σ_0 satisfies the stationarity condition: $\Sigma_0 = \Phi\Sigma_0\Phi + Q_\eta$. Therefore the elements of Σ_0 can be obtained by: $vec(\Sigma_0) = (I_{k^2} - \Phi \otimes \Phi)^{-1} vec(Q_\eta)$, where *vec* is the operator that stacks the columns of a matrix and $\otimes$ is a Kronecker product.

[2] The operator *vec* stacks all columns of a matrix, while *vech* stacks the columns of the lower triangular part of a matrix and *vecl* stacks the columns of the strict lower triangular (exluding the leading diagonal from the lower triangular matrix) part of a matrix.

2.2 Time Varying Correlation MSV

The Time Varying Correlation MSV model considered in our paper is the one mentioned in Jungbacker and Koopman (2006). We will refer to this model as TVCMSV. Following the notation above, the observation equation (1) is modified as:

$$y_t = DH_t^{1/2}\varepsilon_t \tag{4}$$
$$\varepsilon_t \backsim N(0, I_k)$$

where D is a lower unity triangular matrix. The idea is to decompose the conditional variance of y_t, $Var(y_t|h_t) = V_t = DH_tD'$, and therefore having a stochastic dynamics behind the variances and correlations implied by V_t. If we would call $g_{ii,t} = \exp(h_{i,t})$ and $D = \{q_{ij} \neq 0 \ when \ i > j, \ 0 \ otherwise\}$, then the implied correlations by the model are given by:

$$\sigma_{ii,t} = \sum_{s=1}^{i} q_{is}^2 g_{ss,t}, \ \ i = 1, 2, ..., k$$

$$\sigma_{ij,t} = \sum_{s=1}^{j} q_{is}q_{js}g_{ss,t}, \ \ i > j, i = 2, 3, ..., k$$

$$p_{ij,t} = \frac{\sigma_{ij,t}}{\sqrt{\sigma_{ii,t}\sigma_{jj,t}}} = \frac{\sum_{s=1}^{j} q_{is}q_{js}g_{ss,t}}{\sqrt{\sum_{s=1}^{i} q_{is}^2 g_{ss,t} \sum_{s=1}^{j} q_{js}^2 g_{ss,t}}}$$

This model is also a special case of factor MSV models proposed by Shephard (1996) and further studied in Aguilar and West (2000) and Chib *et.al.* (2006) with the number of factors being equal to the number of series. A shortcoming of this model is that the driving forces underlying the volatility and correlation dynamics are the same; $g_{ii,t}$ and q_{ij}. The model parameters are $\Psi = (vecl(D)', \Gamma', diag(\Phi)', vech(Q_\eta)')'$. The number of parameters to be estimated in this model is also given by $k^2 + 2k$.

Tsay (2005) let the correlation parameters to be dynamic in the sense that the unity lower triangular matrix D becomes $D_t = \{q_{ijt} \neq 0 \ when \ i > j, \ 0 \ otherwise\}$ where q_{ijt} follows a Gaussian AR(1) process. Then the equation (4) becomes:

$$y_t = D_tH_t^{1/2}\varepsilon_t, \tag{5}$$

where the kx1 vector q_t evolves with the equation:

$$q_{t+1} = \beta + \Psi q_t + v_t$$
$$q_1 \backsim N\left((I_k - \Psi)^{-1}\beta, \Lambda_0\right)$$

such that:

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \\ v_t \end{pmatrix} \backsim N\left(0, \begin{bmatrix} I_k & 0 & 0 \\ 0 & Q_\eta & 0 \\ 0 & 0 & \Lambda_v \end{bmatrix}\right)$$

where Λ_0 is defined similar to Σ_0. We can put this model to a state space form as follows: let $\alpha_t = (h_t', q_t{}')', \omega_t = ((\eta_t)', (v_t)')',\ \bar{\Gamma} = (\Gamma', \beta')'$ such that:

$$\alpha_{t+1} = \bar{\Gamma} + \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix} \alpha_t + \omega_t, \text{where } \omega_t \backsim N\left(0, \begin{bmatrix} Q_\eta & 0 \\ 0 & \Lambda_v \end{bmatrix}\right) \tag{6}$$
$$\alpha_1 \backsim N\left(\begin{pmatrix} (I_k - \Phi)^{-1}\Gamma \\ (I_k - \Psi)^{-1}\beta \end{pmatrix}, \begin{bmatrix} \Sigma_0 & 0 \\ 0 & \Lambda_0 \end{bmatrix}\right)$$

How to estimate the TVCMSV model defined via (5) and (6) via QML and MCL method is left for future research. The model parameters are $\Psi = (\beta', diag(\Psi)', \Gamma', diag(\Phi)',\ vech(Q_\eta)')'$ and the number of parameters to estimate in this model is $k^2 + 5k$. In our MC experiments we only consider the TVCMSV model of Jungbacker and Koopman (2006).

2.3 MSV with Leverage Effect

The first MSV model with diagonal leverage we propose here is a direct generalization of the univariate model considered in Jungbacker and Koopman (2005). Changing the definition of the errors slightly, we could rewrite the equations (1), (2) and (3) of CCMSV model as follows:

$$y_t = H_t^{1/2} P_\varepsilon^* \varepsilon_t \tag{7}$$
$$h_{t+1} = \Gamma + \Phi h_t + Q_\eta^* \eta_t$$

with the following modification is made the CCMSV model:

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \backsim N\left(0, \begin{bmatrix} I_k & L \\ L & I_k \end{bmatrix}\right) \tag{8}$$

where $L = \{\lambda_{ii},\ i = 1...k : \lambda_{ii}\ \epsilon\ [-1, 1]\}$ is assumed to be a diagonal matrix. Therefore by construction, the MSV with diagonal leverage model defined by equations (7) and (8) implies constant correlations. A transformation similar to the one in Jungbacker and Koopman (2005) could be then adapted to write this model in a state space form:

$$\begin{aligned} y_t &= H_t^{1/2} P_\varepsilon^* \{\ \varepsilon_t^* + S\eta_{2t}\} \\ h_{t+1} &= \Gamma + \Phi h_t + Q_\eta^* \{\eta_{1t} + \eta_{2t}\} \end{aligned} \tag{9}$$

$$\begin{pmatrix} \varepsilon_t^* \\ \eta_{1t} \\ \eta_{2t} \end{pmatrix} \backsim N\left(0, \begin{bmatrix} I_k - |L| & 0 & 0 \\ 0 & I_k - |L| & 0 \\ 0 & 0 & |L| \end{bmatrix}\right)$$

where S matrix is a diagonal matrix of the signs of each element of L while $|L|$ is the absolute value of (the elements of) L matrix. (Therefore $S|L| = L$). P_ε^* and Q_η^* are obtained via Cholesky defactorization of P_ε and Q_η, respectively. The errors are all mutually and serially independent. It can be shown that the transformed model in equation (9) is consistent with the MSV model with leverage defined by equations (7) and (8).

Defining the state and signal vectors as $\alpha_t = (h_t', (Q_\eta^*\eta_{2,t})')'$, $\eta_t = ((Q_\eta^*\eta_{1,t})', (Q_\eta^*\eta_{2,t+1})')'$ and $\bar{\Gamma} = (\Gamma', 0_k)'$, we have the transformed model ready for MCL estimation:

$$y_t = H_t^{1/2} P_\varepsilon^* \{\ \varepsilon_t^* + S\eta_{2t}\} \tag{10}$$

$$\alpha_{t+1} = \bar{\Gamma} + \begin{pmatrix} \Phi & I_k \\ 0 & 0 \end{pmatrix} \alpha_t + \eta_t, \text{ where } \eta_t \backsim N\left(0, \begin{bmatrix} Q_\eta^*(I_k - |L|)Q_\eta^{*\prime} & 0 \\ 0 & Q_\eta^*|L|Q_\eta^{*\prime} \end{bmatrix}\right) \tag{11}$$

$$\alpha_1 \backsim N\left(\begin{pmatrix} (I_k - \Phi)^{-1}\bar{\Gamma} \\ 0 \end{pmatrix}, \Omega_0\right)$$

$$vec(\Omega_0) = \left[I_{4k^2} - \begin{pmatrix} \Phi & I_k \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \Phi & I_k \\ 0 & 0 \end{pmatrix}\right]^{-1} vec \begin{bmatrix} Q_\eta^*(I_k - |L|)Q_\eta^{*\prime} & 0 \\ 0 & Q_\eta^*|L|Q_\eta^{*\prime} \end{bmatrix}$$

The parameter vector to be estimated is therefore $\Psi = (vecl(P_\varepsilon), \Gamma', diag(\Phi)', vech(Q_\eta)', diag(L)')'$ and the number of parameters to estimate in this model is $k^2 + 3k$. A similar but more restricted model is considered in Danielsson (1998) and estimated in Asai and McAleer (2006) where $\bar{L} = diag(\lambda_1 \sigma_{11}^{1/2}, \lambda_2 \sigma_{22}^{1/2}, ..., \lambda_k \sigma_{kk}^{1/2})$ and $Q_\eta = \{\sigma_{\eta,ij}\}$. It should be noted that in relation to our model, $\bar{L} = Q_\eta^* L P_\varepsilon^*$.

It could also be the case that the L matrix is non-diagonal in the sense that the errors in the observation equation of series i are correlated with the errors in the volatility equation of series j. Then the transformation above should be modified. Assuming that L matrix is symmetric and (positive or negative) semi-definite, we can define a scalar s which takes a value 1 (-1) if the L matrix is positive (negative) semi-definite. Therefore replacing the S matrix with the scalar s and $|L|$ with sL in the equations above would provide us with the necessary transformation.

$$y_t = H_t^{1/2} P_\varepsilon^* \{ \varepsilon_t^* + s\eta_{2t} \}$$

$$\alpha_{t+1} = \bar{\Gamma} + \begin{pmatrix} \Phi & I_k \\ 0_k & 0_k \end{pmatrix} \alpha_t + \eta_t, \text{ where } \eta_t \backsim N\left(0, \begin{bmatrix} Q_\eta^*(I_k - sL)Q_\eta^{*\prime} & 0 \\ 0 & Q_\eta^*(sL)Q_\eta^{*\prime} \end{bmatrix}\right)$$

$$\alpha_1 \backsim N\left(\begin{pmatrix} (I_k - \Phi)^{-1}\bar{\Gamma} \\ 0 \end{pmatrix}, \Omega_0\right)$$

$$vec(\Omega_0) = \left[I_{4k^2} - \begin{pmatrix} \Phi & I_k \\ 0_k & 0_k \end{pmatrix} \otimes \begin{pmatrix} \Phi & I_k \\ 0_k & 0_k \end{pmatrix}\right]^{-1} vec \begin{bmatrix} Q_\eta^*(I_k - sL)Q_\eta^{*\prime} & 0 \\ 0 & Q_\eta^*(sL)Q_\eta^{*\prime} \end{bmatrix}$$

where $\bar{\Gamma}$ is defined as above. The parameter vector in this case is $\Psi = (vecl(P_\varepsilon), \Gamma', diag(\Phi)', vech(Q_\eta)', vec(L)')'$ which has $k^2 + 3k + k(k-1)/2$ parameters to estimate.

The estimation of these MSV with leverage models via QML could be done by adopting the transformations in Asai and McAleer (2006) and is discussed in section 2.4.1. We assume throughout the paper for simplicity that whenever the true L matrix is non-diagonal, it is symmetric. In reality, this is not necessarily the case. Moreover, the symmetricity assumption is not needed for QML estimation but is required for MCL estimation along with the assumption that L is positive or negative semi-definite.

2.4 Estimating the MSV Models

The estimation methods considered in this paper are the Quasi-maximum Likelihood (QML) method of Harvey *et al.* (1994) and Monte Carlo Likelihood (MCL) method of Jungbacker and Koopman (2006). These estimation methods are briefly explained below. Originally the CCMSV model proposed in Harvey *et al.* (1994) was estimated by Quasi-maximum Likelihood approach while Jungbacker and Koopman (2006) estimated this model by Monte Carlo

Likelihood approach. The TVCMSV model with deterministic correlation parameter in Jungbacker and Koopman (2006) was estimated via MCL approach. The univariate MSV model with leverage in Jungbacker and Koopman (2005) was estimated using MCL method while Asai and McAleer (2006) estimated the restricted model of Danielsson (1998) by the MCL approach of Sandman and Koopman (1998). In this paper, we estimate all the models mentioned by both QML approach of Harvey *et al.* (1994) and MCL approach of Jungbacker and Koopman (2006).

2.4.1 Quasi-maximum Likelihood (QML) Estimation

In this estimation method, the multivariate return vector y_t is put through a log-squared transformation in order to obtain a state space formation (SSF) of the model. For the CCMSV model, the observation equation and the state equation are given as:

$$\log(y_t^2) = h_t + \log(\varepsilon_t^2) = -1.2703\iota + h_t + \xi_t$$
$$h_{t+1} = \Gamma + \Phi h_t + \eta_t$$

where ι is a vector of ones and the mean of $\log(\varepsilon_{it}^2)$ is known to be -1.2703, and its variance is $\pi^2/2$. In fact, the distribution of $\log(\varepsilon_{it}^2)$ is based on a $\ln \chi_1^2$ distribution. (See for Sandman and Koopman (1998) for the univariate and Asai and McAleer (2006) for the multivariate model). We can replace $\log(\varepsilon_t^2) + 1.2703\iota$ with ξ_t whose mean is therefore a vector of zeros and covariance matrix is given by P_ξ, which is defined below. QML method approximates the distribution of ξ_t with $N(0, P_\xi)$. The estimation procedure is relatively easy: Kalman filter is applied to the log-squared returns and afterwards, the one-step ahead prediction errors and their variances are used to obtain the likelihood function. However, this estimation only yields minimum mean square linear estimators because Kalman filter is a linear filter. How to improve the performance of QML estimators in a multivariate setting using a nonlinear filter is an interesting topic for future research.[3] Taking into account the non-Gaussian distribution of ξ_t, the asymptotic standard errors can be obtained following Dunsmuir (1979). Harvey (1989, pp 212-3) notes that these asymptotic standard errors can not be used for testing if the parameters in the matrix Q_η are significantly different from zero. On the other hand usual quasi-maximum likelihood theory applies and the Bollerslev-Wooldridge robust standard errors can be used. To estimate the in-sample estimates of volatilities and correlations, a Kalman smoothing algorithm is employed.

[3] Watanabe (1999) used a nonlinear filtering to improve the performance of QML estimators in a univariate setting.

Although, the QML method provides consistent estimators, because of the Gaussian approximation, it is likely to have poor small sample properties. Breidt and Carriquiry (1996) and Sandman and Koopman (1998) are some of the papers that document the inefficiency of QML estimation.

It was shown in Harvey *et al.* (1994) that the ij-th element of the covariance matrix P_ξ is given by $(\pi^2/2)p^*_{ij}$, where $p^*_{ii} = 1$ and:

$$p^*_{ij} = \frac{2}{\pi^2} \sum_{s=1}^{\infty} \frac{(s-1)!}{(1/2)_s s} p_{ij}^{2s}$$

where $(x)_s = x(x+1)...(x+s-1)$. In our implementation, we first maximize with respect to the variable p^2_{ij}, and then after the maximization we obtain $|p_{ij}|$. The sign of p_{ij} can be recovered from the sign of the product of corresponding pair of observations, *i.e.* $y_i y_j$. If more than half of the multiplications $y_i y_j$ is positive, then the sign of p_{ij} is positive.

One problem with the QML estimation is the existence of inliers, *i.e.* due to missing data or simply by chance some returns will be zero or very close to zero. Therefore a log-squared transformation of this return will explode. To take care of this, several methods are used in the literature. Kim *et al.* (1998) considered a transformation such as $\log(y_t^2 + c)$ where $c = 0.001$, while Fuller (1996) assumed a data driven transformation. We follow here the transformation discussed in Sandman and Koopman (1998) where the values of $\log(y_t^2)$ which are less than -20 is set equal to -20.

Ruiz (1994) and Harvey *et al.* (1994) suggest that the intercept of the SV process could be obtained directly from the observations via a moment estimator, and the loglikelihood is optimized for the rest of the parameters. This could prove useful when the cross section is large. In fact, this approach could also be used for the MCL estimation when the errors are assumed to be Gaussian as in QML estimation. However, in this paper we preferred to estimate all parameters by maximizing the loglikelihood.

The estimation of TVCMSV model via QML method is very similar to the estimation of CCMSV model. It is only required that in the estimation, the log-squared transformation should be applied to $D^{-1}y_t$ and the resulting loglikelihood function contains an additional term: $T\log(\det(D))$. Given that in TVCMSV set up in our paper the D matrix is lower unity triangular, its determinant is one and therefore this additional term is equal to zero. Alternatively the D matrix could have been defined as a lower triangular matrix, with nonzero values in the leading diagonal and the intercept term in the volatility equation, Γ, is a vector of zeros. Then the additional term in the loglikelihood would be different than zero. See Jungbacker and Koopman (2006) for details.

For estimating the MSV model with (diagonal or nondiagonal) leverage via QML method, the log-squared transformation as discussed in Asai and McAleer (2006) can be applied to the model:

$$\log(y_t^2) = h_t + \log(\varepsilon_t^2) = h_t + \xi_t$$
$$h_{t+1} = \Gamma + \Gamma_t^* + \Phi h_t + \eta_t^*,\ \eta_t^* \backsim N\left(0, \Sigma_{\eta,t}^*\right)$$
$$E(\eta_t^* \xi_t') = \bar{L}_t^*$$
$$\Sigma_{\eta,t}^* = \Sigma_{\eta,t} - \bar{L}P_\varepsilon^{-1}\bar{L} + \bar{L}P_\varepsilon^{-1}\left[\left\{P_{|\varepsilon|} - \tfrac{2}{\pi}\iota\iota'\right\} \circ (s_t s_t')\right] P_\varepsilon^{-1}\bar{L}$$
$$\Gamma_t^* = \sqrt{\tfrac{2}{\pi}}\bar{L}P_\varepsilon^{-1}s_t$$
$$\bar{L}_t^* = \bar{L}P_\varepsilon^{-1}\left[\left\{R_{|\varepsilon|} - c\sqrt{\tfrac{2}{\pi}}\right\} \circ (s_t\iota')\right]$$

where s_t is a vector constructed from the signs of the returns in y_t vector, $c = -1.2703$, and the expressions for $P_{|\varepsilon|}$ and $R_{|\varepsilon|}$ can be found in the appendix of Asai and McAleer (2006). It should be noted once again that $\bar{L} = Q_\eta^* L P_\varepsilon^{*\prime}$ in relation to the construction of our leverage model. As expected, when the parameter values in L matrix are equal to zero, the state space form representation in CCMSV is obtained. Using this transformation, it is straightforward to estimate the MSV models with leverage by QML method by using a properly constructed Kalman filtering.

2.4.2 Monte Carlo Likelihood (MCL) Estimation

Proposed by Durbin and Koopman (1997) and Shephard and Pitt (1997), this estimation method is based on constructing the likelihood function for general state space models using Monte Carlo techniques. Sandman and Koopman (1998) put the log-squared transformed returns to a linear non-Gaussian state space form and proceeds with the estimation taking into account the true distribution of the log-squared transformed errors. What we refer to as the MCL method in this paper is the one proposed by Jungbacker and Koopman (2006), which extended the method in Durbin and Koopman (1997) for the observation vector without the log-squared transformation. Other simulated maximum likelihood methods are considered by Danielsson and Richard (1993), Liesenfeld and Richard (2003). In MCL method, the loglikelihood function is approximated as a sum of a Gaussian part, constructed via Kalman filter, and a minor remainder part which is evaluated using simulations. Therefore it only needs a small number of simulations to achieve the desirable accuracy for empirical analysis.

After some manipulations Durbin and Koopman (1997) showed that the likelihood function for the non-Gaussian model based on importance sampling can be written by:

$$p(y) = p_G(y) \int \frac{p(y|h)p(h)}{p_G(y|h)p_G(h)} p_G(h|y)dh$$

where $p_G(y)$ represents the Gaussian likelihood function of the approximating model which is defined by:

$$\tilde{y}_t = h_t + v_t \text{ where } v_t \sim N(0, G_t) \text{ for } t = 1...N$$

and h_t is defined as before. If we would define $\dot{p}(y_t|\hat{h}_t) = \frac{\partial \log p(y_t|h_t)}{\partial h_t}$ and $\ddot{p}(y_t|h_t) = \frac{\partial^2 \log p(y_t|h_t)}{\partial h_t \partial h_t'}$, then $G_t = -\ddot{p}(y_t|\hat{h}_t)^{-1}$ for t=1...N and $\hat{h}$ is the mode of $p(h|y)$. In this paper, the models considered have $p(h) = p_G(h)$, therefore further simplification can be done on the likelihood.

By defining $\tilde{y}_t = \hat{h}_t + G_t\dot{p}(y_t|\hat{h}_t)$, it can be shown that the first and second derivatives of $\log p(h|y)$ and $\log p_G(h|\tilde{y})$ agree in the mode $\hat{h}$. Using the algorithm in Jungbacker and Koopman (2006) based on Kalman filtering and smoothing, one can compute this mode. (See Jungbacker and Koopman (2006) for an illustration with a univariate SV model) Later the Monte Carlo estimator of the likelihood is then given by:

$$\hat{p}(y) = p_G(\tilde{y})M^{-1} \sum_{m=1}^{M} w(\alpha^m) \text{ where } w(\alpha^m) = \frac{p(y|h)}{p_G(\tilde{y}|h)} \text{ and } \alpha^m \sim p_G(h|\tilde{y})$$

where M is the number of samples to be generated from $p_G(h|\tilde{y})$ using the simulation smoother algorithm of Jong and Shephard (1995) or Durbin and Koopman (2002). However, it was noted in Jungbacker and Koopman (2005) that, when $G_t = -\ddot{p}(y_t|\hat{h}_t)^{-1}$ is not positive definite, the simulation smoothing method of Durbin and Koopman (2002) cannot be used. In our estimations we take the number of draws $M = 200$.

In the case of CCMSV model, first and second derivatives $\dot{p}(y_t|h_t)$ and $\ddot{p}(y_t|h_t)$ can be obtained from the conditional density:

$$\log p(y_t|h_t) = -0.5k \log(2\pi) - 0.5 \sum_{i=1}^{k} h_{it} - 0.5 \log(\det(P_\varepsilon)) - 0.5 d_t' P_\varepsilon^{-1} d_t \text{ for } t = 1...T$$

where $d_t = H_t^{-1/2} y_t$. The possible existence of an indefinite matrix for $\ddot{p}(y_t|\hat{h}_t)$ requires the approach of Jungbacker and Koopman (2005). As Jungbacker and Koopman (2006) suggested, when the model gets too complicated or when explicit expressions for $\dot{p}(y_t|h_t)$ and $\ddot{p}(y_t|h_t)$ can not be obtained analytically, as a last resort numerical approximations can be used. For the CCMSV model the analytical derivatives are provided by Jungbacker and Koopman (2006) and these can also be used to obtain the derivatives for TVCMSV. In our estimations, we used analytical derivatives also for the MSV with leverage models and we provide them in the appendix.

Finally, the in-sample estimates of the underlying volatilities can be obtained from the smoothed estimate of the state vector α (which is just volatilities in case of CCMSV and TVCMSV models but a larger vector in MSV models with leverage) which can be computed from:

$$\hat{\alpha} = \frac{\sum_{i=1}^{M} \alpha^i w(\alpha^i)}{\sum_{i=1}^{M} \alpha^i}$$

where α^i is a draw from the conditional density $p_G(\alpha|y)$ for the approximating Gaussian linear model. When making these draws, the simulation device mentioned in Jungbacker and Koopman (2005) is used to increase the computational efficiency. This device is based on an unconditional draw from $p(\alpha)$ and on a conditional mean adjustment. (See Jungbacker and Koopman (2005) for details.)

In our experience, the computational time required for MCL estimation turned out to be very high compared to that of the QML estimation. Especially when the sample size or the cross-section size is increased, it takes our code much more time to converge than it does for QML estimation. When the cross-section size is large, it is not that obvious to write the analytical derivatives and if instead one considers numerical derivatives in this case, then the derivatives calculated with respect to large state vectors could be very time consuming and numerically unstable. The QML method on the other hand is much more flexible.

3 Monte Carlo Experiments

In this section, we report the results of our MC experiments in order to compare the performance of QML and MCL methods when estimating the models considered in the paper for several different parameter sets. For each model and parameter set, we generated $B = 100$ time series vectors of dimension $k = 2$ with sample size $T = 500$. For comparison purposes, we look at the performances in parameter estimation as well as in in-sample smoothed volatility and correlation estimations. The estimation results are reported in terms of MC means of parameter estimates, corresponding MC standard deviations and root mean squared error for each parameter estimate as a measure of efficiency.[4] On the other hand, the kernel density estimates of the mean deviations and mean absolute deviations of estimated log-volatilities,

[4]For comparison purposes, in case of MSV with leverage models as well we report the results for the parameters in P_ε and Q_η matrices, instead of reporting the results for the Cholesky factors in their formulation (see section 2.3).

$\widehat{h}_{it}$, and correlations, $\widehat{p}_t$, from their true values are provided.[5] These deviations are calculated for each series over the number of simulations B; *i.e.* for each t:

$$\Delta\widehat{h_{it}} = \tfrac{1}{B}\textstyle\sum_{b=1}^{B}\left(\widehat{h}_{it,b} - h_{it,b}\right) \quad \text{and } |\Delta|\widehat{h_{it}} = \frac{1}{B}\sum_{b=1}^{B}\left|\widehat{h}_{it,b} - h_{it,b}\right| \qquad (12)$$

$$\Delta\widehat{p_t} = \tfrac{1}{B}\textstyle\sum_{b=1}^{B}\left(\frac{\widehat{p}_{t,b}-p_{t,b}}{p_{t,b}}\right) \quad \text{and } |\Delta|\widehat{p_t} = \frac{1}{B}\sum_{b=1}^{B}\left|\frac{\widehat{p}_{t,b} - p_{t,b}}{p_{t,b}}\right|$$

where $|.|$ is an absolute value. Given that in the case of Constant Correlation MSV (CCMSV) and MSV models with leverage the correlations are constant (the correlation estimate is actually a parameter estimate), the kernel density estimate of the deviations of B different estimates of the correlation parameter from the true correlation parameter will be plotted. However, for the Time Varying Correlation MSV (TVCMSV), as in the case of volatilities, the kernel density estimates of $\Delta\widehat{p_t}$ and $|\Delta|\widehat{p_t}$ are plotted. Finally mean absolute errors (MAE) and root mean squared errors (RMSE) of the volatility and correlation estimates are also reported in tables. In this case, the mean absolute error and the root mean squared error are calculated as:

$$MAE = \frac{1}{T}\frac{1}{B}\sum_{t=1}^{T}\sum_{b=1}^{B}\left|\widehat{h}_{it,b} - h_{it,b}\right| \text{ and } RMSE = \frac{1}{T}\frac{1}{B}\sum_{t=1}^{T}\sum_{b=1}^{B}\left(\widehat{h}_{it,b} - h_{it,b}\right)^2 \qquad (13)$$

$$MAE = \frac{1}{T}\frac{1}{B}\sum_{t=1}^{T}\sum_{b=1}^{B}\left|\frac{\widehat{p}_{t,b} - p_{t,b}}{p_{t,b}}\right| \text{ and } RMSE = \frac{1}{T}\frac{1}{B}\sum_{t=1}^{T}\sum_{b=1}^{B}\left(\frac{\widehat{p}_{t,b} - p_{t,b}}{p_{t,b}}\right)^2$$

For the CCMSV model, the true values of the parameters $\Psi = (vecl(P_\varepsilon)', \Gamma', diag(\Phi)', vech(Q_\eta)')'$ and parameter estimation results are given in Tables 1 and 2. These results for the CCMSV model confirm the previous results in the literature that the small sample performance of MCL is better than that of QML; the QML method is less efficient. The efficiency of QML estimator of the correlation parameter increases as the two series become more correlated ($p = 0.8$). When the series are less correlated ($p = 0.2$), the QML doesn't estimate the correlation parameter very accurately: even though the mean is more or less around the true value, we observe a relatively high variance. Also when the variance of the SV processes are higher (comparing Exp 1 and Exp 3) the QML estimator gains efficiency in estimating the autoregressive coefficients, Φ. The same can be said also for the MCL estimator of Φ that the RMSE is smaller when the SV processes have more variance. Comparing Exp 1 and Exp 5, we can say that when the true value of p is high ($p = 0.8$), QML and MCL estimates of this

[5] It should be noted that $\widehat{h}_{it,b} - h_{it,b}$ is a log-difference, while $\widehat{p}_{t,b} - p_{t,b}$ is only a difference. Therefore the latter is divided by $p_{t,b}$ to have the same sense of percentage deviation.

parameter have less MC standard deviation. It is also noticed that overall the performance of MCL estimator improves consistently for all the parameters when p increases. When the variance of SV process is higher, it is seen that the estimation performance of both QML and MCL estimators for autoregressive parameters increase while there are slight changes in the RMSE of the correlation estimates. (Comparing Exp 2 and Exp 5, Exp 6 and Exp 8)

Figures 2 and 4 show the kernel density estimates of the deviations of volatility and correlation estimates from the true values. From these figures we could visually confirm that MCL estimators of the volatilities are more efficient compared to QML estimators. The high variance of the QML correlation estimates can be noticed in the third column; especially in the experiments where the true correlation parameter value is 0.2. In fact, it is observed that the QML correlation estimate in Exp 1 is hitting to 0 most of the time. While higher variance of the SV process errors brings with it an increase in the variance of the estimated volatilities for both QML and MCL estimators (comparing Exp 1 with Exp 3 and Exp 6 with Exp 8), when the series are highly correlated both estimators seem to perform better in estimating the underlying volatilities and correlations (comparing for instance Exp 1, 2, 3 with Exp 5). From figures 3 and 5 we arrive to the same conclusions that with lower variance in the stochastic volatility process, the QML volatility estimates are closer to the MCL volatility estimates. On the other hand when the true correlation is high, QML correlation estimates are closer to the MCL correlation estimates. It is seen in Figure 3 that the average absolute deviations of the QML correlation estimates are concentrated around 1, when $p = 0.2$. The reason is that the maximization of the QML log-likelihood is done with respect to the squared correlations (see Section 2.4.1.) and it seems that the QML estimates for the squared correlations were hitting to zero most of the time when the true correlation is low. This can also be observed from Figure 2.

In Table 3, the RMSE of the volatility and correlation estimates of QML and MCL estimators are given. From this table the inefficiency of QML estimation in estimating the correlation parameter when the true value is low can be seen clearly: when the correlation parameter value is increased from 0.2 to 0.8, the relative RMSE of QML correlation estimates improves twofolds. Looking at this table, it can be said that QML performs closest to MCL estimator in estimating the volatilities in the experiments where the second autoregressive parameter and the variance of the SV processes are high (Exp 4 and Exp 6). On the other hand, QML estimator of the correlation parameter performs closer to MCL estimator when the correlation parameter is high. (Exp 6) Our conclusion from these experiments is that MCL estimation should be preferred to QML estimation. QML estimation could be used when the series are expected to be highly correlated, the SV processes behind the series have higher variance and the sample size is large.

For the experiments with TVCMSV model, the true values for the parameters (except the correlation parameter) is chosen from the experiment 1 of CCMSV model. The correlation parameter values 0.2041 and 1.3333 are chosen such that the correlation between the volatility adjusted series are 0.2 and 0.8, respectively. The parameter estimation results in Table 4 suggest that QML estimator performs better in estimating the correlation parameter as well as the underlying correlations with TVCMSV model than with CCMSV model. Also, it is observed that when the correlations are higher (in Exp 2 relative to Exp 1), the MC standard deviations and RMSE of all QML estimates are less; while the performance of MCL estimator seems to be similar in these two experiments. Figure 6 shows the kernel density estimates of the deviations and absolute deviations of QML and MCL volatility and correlation estimates from the true values for TVCMSV model. The underlying correlations are estimated with less variance by both QML and MCL methods when the correlations between the series are high. Looking at Table 5, we can see that while the performance of QML and MCL in estimating the underlying volatilities is more or less the same as in corresponding CCMSV experiments (Exp 1 and Exp 5), the performance QML estimator in estimating the underlying correlations increased relative to the MCL correlation estimator. Therefore compared to the CCMSV model, we have less concerns in estimating the TVCMSV model with QML estimator rather than MCL estimator, while we still suggest that MCL estimator should be used.

For the MSV with diagonal leverage model, all the parameter values are taken from experiment 1 of CCMSV model. For the additional parameters that control for the leverage, we chose $L = diag\{-0.2000, -0.2500\}$ and $L = diag\{-0.5500, -0.6000\}$. In Table 6, we report these true values of the parameters as well as the results of the QML and MCL estimations when the data was generated by an MSV model with diagonal leverage. It is observed that compared with the Exp 1 of CCMSV model, the performance of QML estimator has decreased when two more parameters were included to control for the leverage while the performance of MCL estimator seems to remain similar. When the leverage effect is higher the QML estimates of the autoregressive parameters have less standard deviation and RMSE. It is also observed that, the performance of QML estimator in estimating the correlations of this model is lower compared to the experiments with the models without leverage. On the other hand MCL estimator of the leverage parameters, although having less standard deviation, seems to be deviating from the true values relatively more compared to QML estimator. Some of these results can be confirmed visually from Figure 7 where the kernel density estimates of the deviations and absolute deviations of QML and MCL volatility and correlation estimates from the true values are plotted. For instance in the third column the kernel density estimates for the QML correlation estimates have very high variance. In practice this means that for a given data, the QML estimate of the correlation parameter could possibly have a value far from the true value. Finally the MAE and RMSE of the volatility and correlation estimates are reported in Table 7. It is observed that for both estimators, the volatility estimates have higher MAE and RMSE compared to the Exp 1 of CCMSV model and they increase with the strength of leverage. The correlation estimates obtained via QML estimator have 5 to 7 times higher MAE and RMSE than the MCL estimator. Including leverage effects to the model doesn't seem to have that much effect on the performance of the MCL correlation estimator.

In two other MC experiments we consider the MSV model with non-diagonal leverage. The MCL estimation of this model requires that the assumption that the leverage matrix is symmetric and positive or negative semi-definite. In the first experiment (Exp 1) we consider a true leverage matrix that is symmetric but indefinite, therefore the restriction of the MCL estimation is binding. In the second experiment (Exp 2), we consider a leverage matrix that is symmetric negative definite. The true values of the parameters, except the off-diagonal parameter of the leverage matrix, are taken from Exp 1 of the MSV model with diagonal leverage. The QML estimation does not require symmetricity or positive or negative semi-definiteness assumptions although we assume that the leverage matrix is symmetric. Moreover, in the first experiment, we also estimate the same data with QML method imposing the restrictions on the leverage matrix. The parameter estimation results are given in Table 8 along with the true parameter values. In the first experiment comparing unrestricted and restricted QML estimation results, we see that restricted QML estimate of the off-diagonal parameter of the leverage matrix is lower and the restricted QML estimates of the leading diagonal of the leverage matrix are higher compared to the corresponding unrestricted QML estimates. This result confirms that the restriction was binding. Both unrestricted and restricted QML estimates of the correlation parameter are far from the true value. While having uniformly less bias than the unrestricted QML estimates, MCL estimator does its best to capture the off-diagonal element of the leverage matrix while the MCL estimates for the leading diagonal of the leverage matrix have more or less the same value as the corresponding unrestricted QML ones. It is observed that MCL correlation estimator has similar performance in this experiment compared to Exp 1 of the CCMSV model while the unrestricted QML estimates of the correlation parameter are far from the true value with higher RMSE compared to Exp 1 of the CCMSV model.

In the second experiment (Exp 2) only the unrestricted QML estimation results are reported along with the MCL estimation results. When the off-diagonal element of the leverage matrix was decreased (from Exp 1 to Exp 2), in general less bias and RMSE were obtained for the MCL estimates. As it was in the first experiment, the QML estimate of the correlation parameter has very high standard deviation. Figure 8 reports the kernel density estimates of the deviations and absolute deviations of unrestricted QML, restricted QML (for Exp. 1 only) and MCL volatility and correlation estimates from the true values. Both for volatilities or correlations, the kernel densities corresponding to unrestricted and restricted QML estimators seem to be very close. For both series the mode of the kernel density estimate corresponding to the MCL volatility estimates slightly deviate from zero in the first experiment while this deviation is very small or non-existent in the second experiment. This could be the result of the restriction imposed or simply due to randomness because in the first experiment both unrestricted and restricted QML estimates of the volatility of second series also seem to be underestimating the true volatility. In the third column we see that the QML correlation estimates have very high variance as in Exp. 1 of CCMSV model, while the MCL correlation estimates have much less variance and are concentrated around the true value of the correlation parameter. When the off-diagonal element of the leverage matrix has less magnitude, the MCL correlation estimates are more dense around the true value, while the QML correlation estimates seem to have a similar distribution as in the first experiment.

In Table 9, we provide the MAE and RMSE of the volatility and correlation estimates. Compared to the Exp. 1 of the MSV model with diagonal leverage, in the first experiment the MAE and RMSE of the QML volatility estimates are higher while the MAE and RMSE of the MCL volatility estimates are slightly lower. In the second experiment, the MAE and RMSE of the MCL volatility and correlation estimates are lower relative to the corresponding numbers in Exp 2 of the MSV model with diagonal leverage. Overall, the restricted QML estimator seems to perform closer to the MCL estimator given that the same restriction is imposed. It is also observed that the relative MAE and RMSE (QML/MCL) of the correlation estimates increased from the first experiment to second experiment. Finally, it should be noted that both QML and MCL estimators are more efficient in estimating the volatilities and correlations when the true value of the off-diagonal of the leverage matrix is lower, while the improvement is larger for the MCL estimator. Also, MCL estimator of the volatilities and correlations in the first experiment with non-diagonal indefinite leverage matrix perform similar to the first experiment with diagonal leverage matrix in Table 7. Therefore although the restriction imposed in the MCL method could cause underestimation of the off-diagonal element of the leverage matrix as seen in Table 8, the volatilities and correlations are estimated by MCL with less RMSE compared to the corresponding QML estimator.

Looking at the results of the experiments with MSV models with (diagonal and non-diagonal) leverage and considering the high MAE and RMSE of the QML correlation estimates, we suggest using the MCL method for estimating these models rather than using QML estimation. It could be that the performance of QML estimator improves with higher correlation in the data or higher variance of the SV processes but it is not expected to be better than the cases considered in the experiments with CCMSV model.

4 An Empirical Application

In this section our aim is to find empirical evidence supporting the MSV with non-diagonal leverage model, i.e. the return shocks of one series is correlated with the volatility shocks of another series. For this estimation, a trivariate series of length 1717 is obtained from the returns of IBEX 35, FTSE 100 and DAX stock markets for the period between 4/1/2005 and 4/11/2011. The returns are calculated as: 100 x $\log(P_t/P_{t-1})$. The descriptive statistics of the data is provided in Table 10. It is observed that the IBEX 35 and DAX returns are skewed right while FTSE-100 is skewed left. On the other hand, as expected, all series have high kurtosis. We also report the Box-Ljung statistics for serial correlation to 10 lags for the returns and its squared and log-squared transformations. Box-Ljung statistic for the return series, y_t, suggests that the data may not be random walks, more likely in the case of FTSE-100. On the other hand, there is strong evidence of nonlinearity in the squared returns and log-squared returns; suggesting that there is autocorrelation in these series.

A univariate SV model with leverage is fit for each of the series. The QML and MCL estimation results for the univariate model is given for each series in Table 11. From the results of the univariate estimation, we see that the MCL estimates imply more persistent SV processes compared to QML estimates. MCL estimates of the autoregressive parameter suggest that these SV processes are even close to random walk. Also it is noted that MCL estimates of the leverage coefficients, the elements of L, are higher compared to QML estimates.

The estimation of the MSV model with leverage requires the restriction that the L matrix is symmetric and (positive or negative) semidefinite. This latter restriction is not required by the QML estimation. However for comparison purposes, we also estimated the model via QML assuming this restriction. The estimation results for the MSV model with leverage are given in Tables 12 and 13. If we compare the results of the multivariate estimation with the results of the univariate estimation in Table 11, we see that the QML estimates of the intercept, of the autoregressive parameter and of the variance of the SV process are more or less the same in both cases while the self-leverage of each series, that is the diagonal of the estimated L matrix, is estimated to be less in magnitude for FTSE-100 and DAX indices compared to the univariate results.

The MCL estimates of the autoregressive parameters are higher in the univariate estimation compared to the multivariate estimation while the estimates of the self-leverage of each series are lower in the multivariate estimation. When comparing the unrestricted QML and MCL estimation results, we see that the correlation estimates obtained by these two methods are more or less the same. While the MCL estimates of the autoregressive parameters are higher, the MCL estimates of elements of the variance matrix of the SV process are lower; this is due to the fact that the estimation tries to match the unconditional variance in the data and when the estimates of the autoregressive parameters are high, the variance matrix of the SV process is pushed downwards.

When we look at the leverage matrix estimates, we see that MCL estimates of the diagonal elements of L matrix are higher compared to the QML estimates. The MCL leverage parameter estimates are statistically significant. Moreover, the likelihood ratio test to compare the MCL estimation results of CCMSV and MSV-NDL models suggest that the data is explained better by the latter model.[6] Figure 9 shows for each series the absolute values of the returns plotted along with the QML and MCL smooth estimates of the standard deviations. It is observed that QML overestimated the volatilities of the period after the big volatility shocks. (for example around $t = 1000$) Finally, the MCL estimates of the standard deviations follow the absolute values of the returns closely while QML estimates are experiencing some jumps when volatility of the data is increasing.

Table 14 shows the results of univariate and multivariate Box-Ljung tests for serial correlation to ten lags on $\hat{v}_t = \hat{H}_t^{-1/2} \hat{P}_\varepsilon^{*^{-1}} y_t$, the residuals standardized by the estimated standard deviations and decorrelated by the estimated constant correlations, and on $\hat{v}_t^2$ and $\log \hat{v}_t^2$. We can see that QML method is not able to estimate the underlying volatilities well, as in the univariate tests the null of no autocorrelation is rejected for the $\hat{v}_t^2$ and $\log \hat{v}_t^2$. For the MCL method, when estimating MSV with non-diagonal leverage and constant correlation MSV models, this null is not rejected at 5% and in some cases at 1% significance levels, with the exception for DAX for the $\hat{v}_t^2$. If we look at the multivariate Box-Ljung test results, we see that for QML estimation the null is rejected for all $\hat{v}_t^2$ and $\log \hat{v}_t^2$. Given the no rejections in the univariate Box-Ljung test results for the MCL method, the rejections of the null for $\hat{v}_t^2$ and $\log \hat{v}_t^2$ in the multivariate case would mean that the MSV with non-diagonal leverage model and the constant correlation MSV model may not be able to explain well the underlying correlation dynamics. This implies that the underlying correlations could be time varying.

5 Conclusions

In this paper, via Monte Carlo (MC) experiments, we compare the performance of Quasi Maximum Likelihood estimation method of Harvey et al. (1994) and Monte Carlo Likelihood estimation method of Jungbacker and Koopman (2006) in estimating the parameters as well as in estimating the underlying volatilities and correlations. With these methods, we estimate the Constant Correlation MSV model of Harvey et al. (1994), the Time Varying Correlation MSV model of Jungbacker and Koopman (2006). Moreover, we propose two MSV models with leverage which are new to the literature. The first MSV model with leverage is a direct generalization of the univariate model in Jungbacker and Koopman (2005). In this model, each series has its own leverage effect: i.e. the return shocks of series i is correlated with the

[6] The likelihood ratio test can't be used with the QML estimation because it is based on approximations.

volatility shocks of series i while the correlation of the return shocks of series i and the volatility shocks of series j is zero. Therefore in this model the leverage matrix is diagonal, hence we refer to it as MSV model with diagonal leverage. In the second MSV model with leverage, we relax this assumption and let the off-diagonals of the leverage matrix to be non-zero. We refer this model as MSV with non-diagonal leverage.

The estimation of CCMSV model via QML and MCL are discussed in Harvey et al. (1994) and Jungbacker and Koopman (2006), respectively. Jungbacker and Koopman (2006) also provides the estimation procedure for the TVCMSV model which follows from a small modification of the estimation procedure of CCMSV model. This modification can be applied to the QML method in order to estimate the TVCMSV model. For the estimation of MSV models with (diagonal and non-diagonal) leverage, we adopt the transformations discussed in Asai and McAleer (2006). On the other hand, the transformations of the MSV models with leverage given in this paper are new to the literature and are based on the univariate transformation in Jungbacker and Koopman (2006).

We considered eight different parameter sets for the CCMSV model in our MC experiments. The results confirm the previous findings in the literature that QML estimator is inefficient in terms of parameter estimation. It is observed that when the true value of the correlation parameter is low, the QML estimator of this parameter has very high variance. Therefore, when estimating a model with real data, if the underlying correlation parameter is low, the QML estimate will not be very informative. We also observed that the performance of the QML estimator increases as the series become more correlated and when the SV processes have higher variance. In estimating the underlying volatilities and correlations, the performance of MCL estimator was superior to that of QML estimator in all parameter sets, although the QML estimator was performing closer to MCL estimator in the experiments where the correlations were higher or SV processes had higher variance.

For the TVCMSV model, we considered two experiments; one with low correlation and another one with high correlation. It appeared that the performance of the QML estimator relative to the MCL estimator was much better compared to the experiments of CCMSV model. With time varying correlations, the QML estimator was able to perform close to the MCL estimator even when the correlations were low.

For the MSV models with diagonal leverage we considered two experiments, one with low leverage and another one with high leverage. Our results showed that relative to the experiments with CCMSV, the inefficiency of the QML estimator increased while the performance of MCL stayed the same when leverage is introduced. Increasing the true values of the leverage parameters further decreases the performance of the QML estimator. The correlation estimates of the QML model had very high root mean squared error (five to seven times the ones of MCL estimator). For the MSV model with non-diagonal leverage, we also considered two experiments; one where the leverage matrix was indefinite and another one where it was negative definite. In the first experiment, the restriction that "the leverage matrix should be symmetric and positive or negative semi-definite" was binding, while in the latter it was not. Our results confirm that in the first experiment, even though the MCL method underestimated the off-diagonal leverage parameter, it was able to capture the underlying volatilities and correlations almost as good as in the case of CCMSV model. On the other hand in both MSV models with leverage, QML correlation estimates had high bias and high standard deviation such that its performance was worse than in the corresponding case of CCMSV model.

Based on our results, we conclude that even though in the case of TVCMSV, QML estimator performs close to MCL estimator, the latter is always preferred. We do not recommend using QML estimators for the models with leverage. Although QML method can be implemented much easier than MCL and the estimation time is much less in QML estimation; we suggest its use if it is expected that the series have high and/or time varying correlation and the SV processes have higher variance. Given the results in the literature on the inefficiency of QML estimator in small samples, it would be also a plus if the sample size is large, when using QML method. On the other hand the implementation of MCL estimation is relatively more complicated than the QML estimation. Therefore MCL estimation requires much more time to converge. When the cross-section size is large, the analytical derivatives for the MCL estimation are harder to obtain and if one would like to use numerical derivatives in this case, then the derivatives calculated for large state vectors could be very time consuming and numerically unstable. QML is not as much affected by the large cross-sections or large sample sizes. Therefore based on our experience, we would suggest using MCL method in the estimation of MSV models for several series, as for modelling the returns of international stock market indices, and QML method could be used for the estimation with medium-to-large number of series from a stock market.

The inefficiency of QML method could be improved partially by employing a nonlinear filter instead of Kalman filter. The latter is a linear filter and therefore leads to minimum mean square linear estimators rather than minimum mean squared estimators. Watanabe (1999) provides a nonlinear filter for QML estimation for the univariate SV model and extending it to a multivariate setup would be an interesting topic.

Another point to consider would be to introduce a correlation between the SV process errors and the stochastic correlation parameter errors in the Tsay (2005) model. The intuition behind this extra parameter would be that the volatility shocks are correlated with the correlation shocks, meaning that when the series are more volatile, they are expected to be more correlated. As we have seen in the recent crisis, the markets tend to move more closely when there are bad news, while their recoveries from these falls might not be as correlated.

6 Appendix

Following Jungbacker and Koopman (2006) and Lutkepohl (1996), we obtained the derivatives for the bivariate MSV model with diagonal leverage needed for deriving the approximating linear model. For the nondiagonal leverage model, it can be easily modified. On the other hand, these derivatives are extendable to cases with more than $k = 2$ series; in the empirical estimation part these derivatives are used for $k = 3$ case.

$$y_t = H_t^{1/2} P_\varepsilon^* \varepsilon_t \Longrightarrow \varepsilon_t = P_\varepsilon^{*^{-1}} H_t^{-1/2} y_t$$

H_t and P_ε^* as defined in (1) and (3). Then using (9) we can write:

$$d_t = P_\varepsilon^{*^{-1}} H_t^{-1/2} y_t - S Q_\eta^{*^{-1}} \alpha_{2,t}$$

α defined as in (10). If we let $X = I_2 - SL$ where I_2 is a 2x2 identity matrix and $\alpha_{1,t}$ be the volatilities part of α_t, then the loglikelihood for (10) would be given by:

$$\log p(y_t|h_t) = -0.5k\log(2\pi) - 0.5\sum_i \alpha_{1,t,i} - 0.5\log(\det(P_\varepsilon^* X P_\varepsilon^{*\prime})) - 0.5 d_t' X^{-1} d_t$$

Then the first derivatives with respect to the state vector α_t would be given by:

$$\frac{\partial l_t}{\partial \alpha_t} = -0.5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 0.5 \frac{\partial d_t'}{\partial \alpha_t} \left(X^{-1} + X^{-1\prime}\right) d_t$$

$$\frac{\partial d_t'}{\partial \alpha_t} = \left(\frac{\partial d_t}{\partial \alpha_t}\right)' = \begin{pmatrix} \left\{-0.5 P_\varepsilon^{*^{-1}} diag(H_t^{-1/2} y_t)\right\}' \\ -\left\{S Q_\eta^{*^{-1}}\right\}' \end{pmatrix}$$

The second derivatives are obtained from:

$$\frac{\partial^2 l_t}{\partial \alpha_t \partial \alpha_t'} = -0.5 \left\{ \frac{\partial d_t'}{\partial \alpha_t} \left(X^{-1} + X^{-1\prime}\right) \frac{\partial d_t}{\partial \alpha_t'} + \left[d_t' \left(X^{-1} + X^{-1\prime}\right) \otimes I_4\right] \frac{\partial vec\left(\frac{\partial d_t'}{\partial \alpha_t}\right)}{\partial \alpha_t'} \right\}$$

where I_4 is a 4x4 identity matrix and $\otimes$ is a Kronecker product. The last expression in the equation is equal to:

$$\frac{\partial vec\left(\frac{\partial d_t'}{\partial \alpha_t}\right)}{\partial \alpha_t'} = 0.25Z$$

where $Z_{1,1} = \left\{P_\varepsilon^{*^{-1}} diag(H_t^{-1/2} y_t)\right\}_{1,1}$, $Z_{5,1} = \left\{P_\varepsilon^{*^{-1}} diag(H_t^{-1/2} y_t)\right\}_{2,1}$, and $Z_{6,2} = \left\{P_\varepsilon^{*^{-1}} diag(H_t^{-1/2} y_t)\right\}_{2,2}$ while the rest of the entries are zeros.

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Figure 1: Indices, returns, squared returns (as a proxy for volatilities) and correlations (estimated by a rolling window of 60 days) of FTSE-100 and DAX stock markets between 4/1/2005 - 4/11/2011. Source: Yahoo Finance.

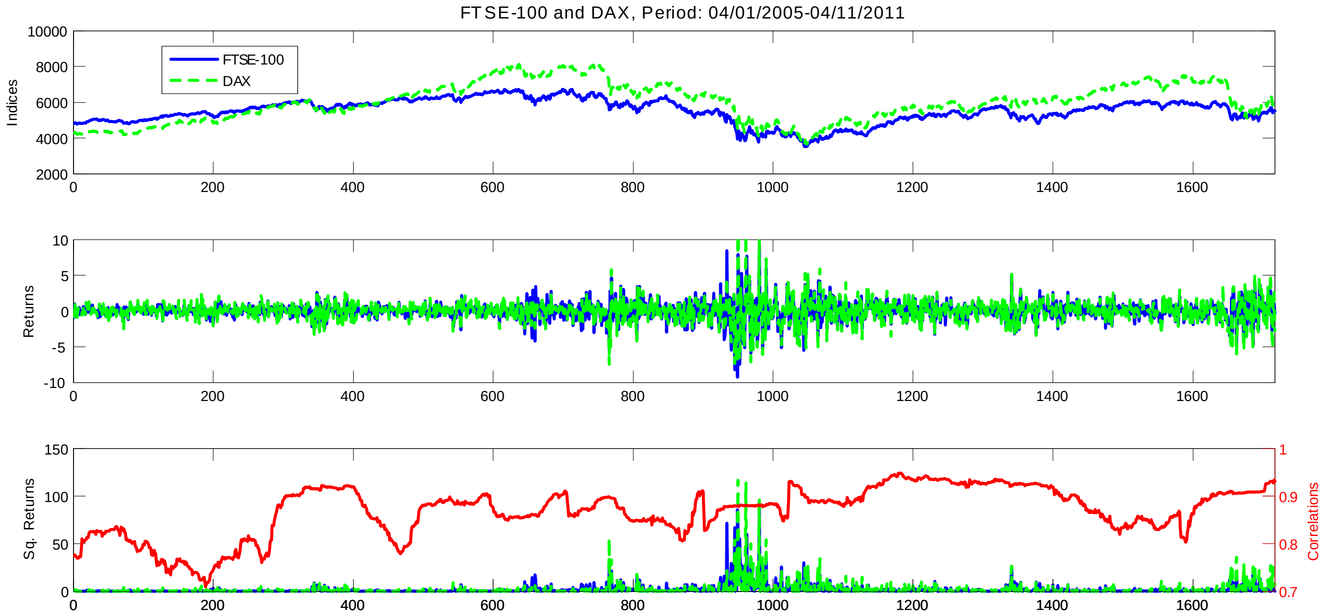

Figure 2: Kernel density estimates of the deviations of MCL and QML volatility and correlation estimates from the true ones, for the CCMSV model. Experiments 1 to 4.

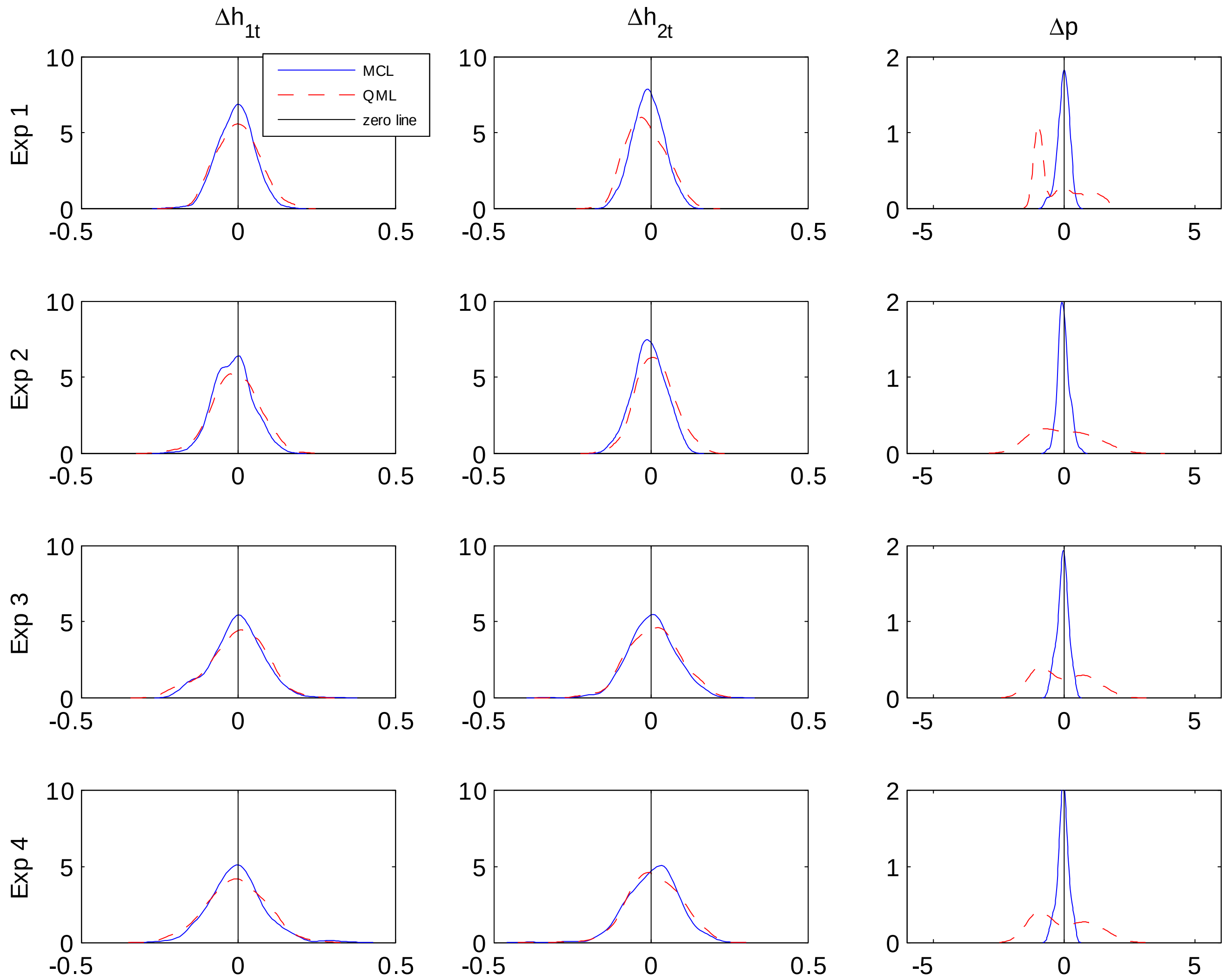

Figure 3: Kernel density estimates of the absolute deviations of MCL and QML volatility and correlation estimates from the true ones, for the CCMSV model. Experiments 1 to 4.

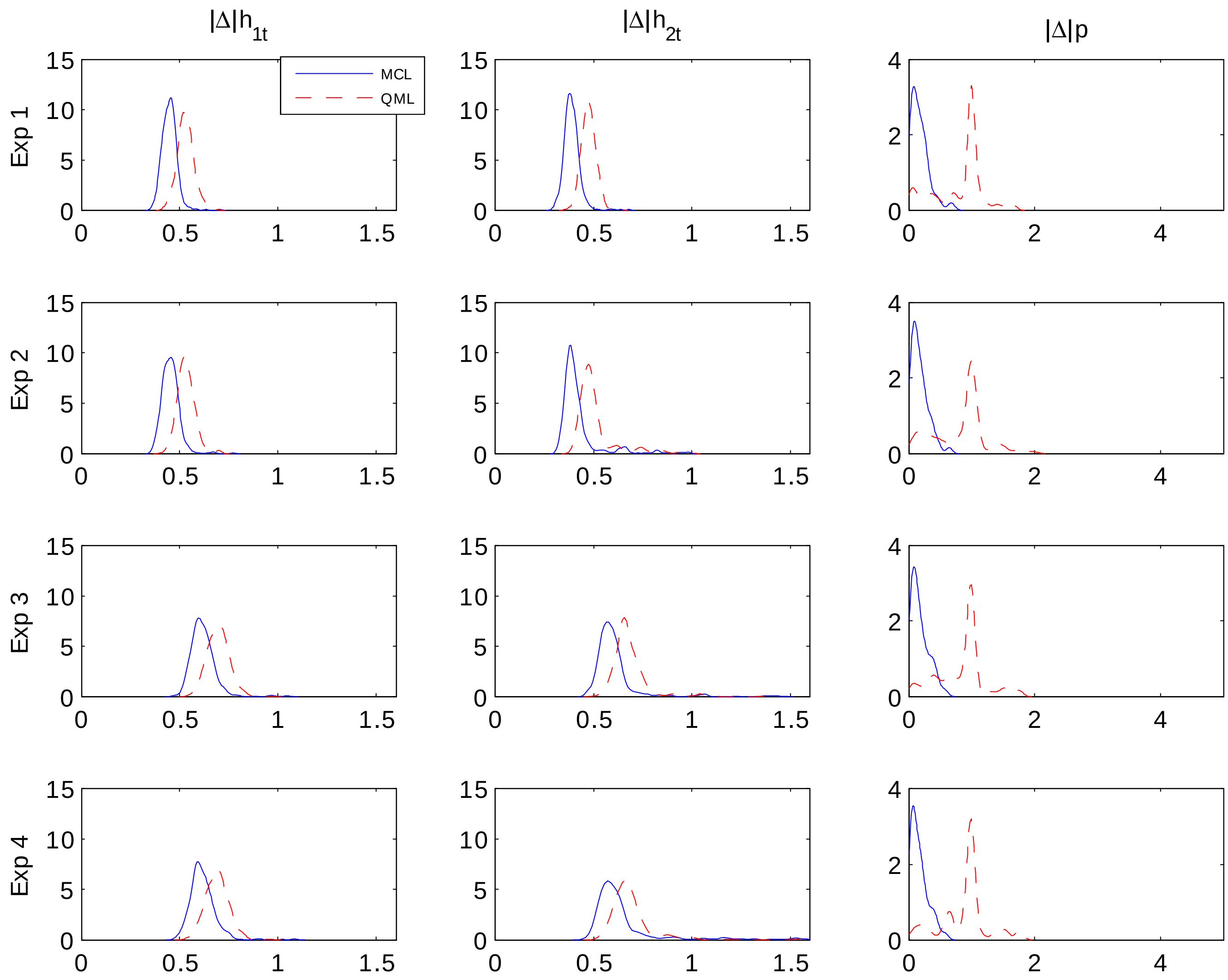

Figure 4: Kernel density estimates of the deviations of MCL and QML volatility and correlation estimates from the true ones, for the CCMSV model. Experiments 5 to 8.

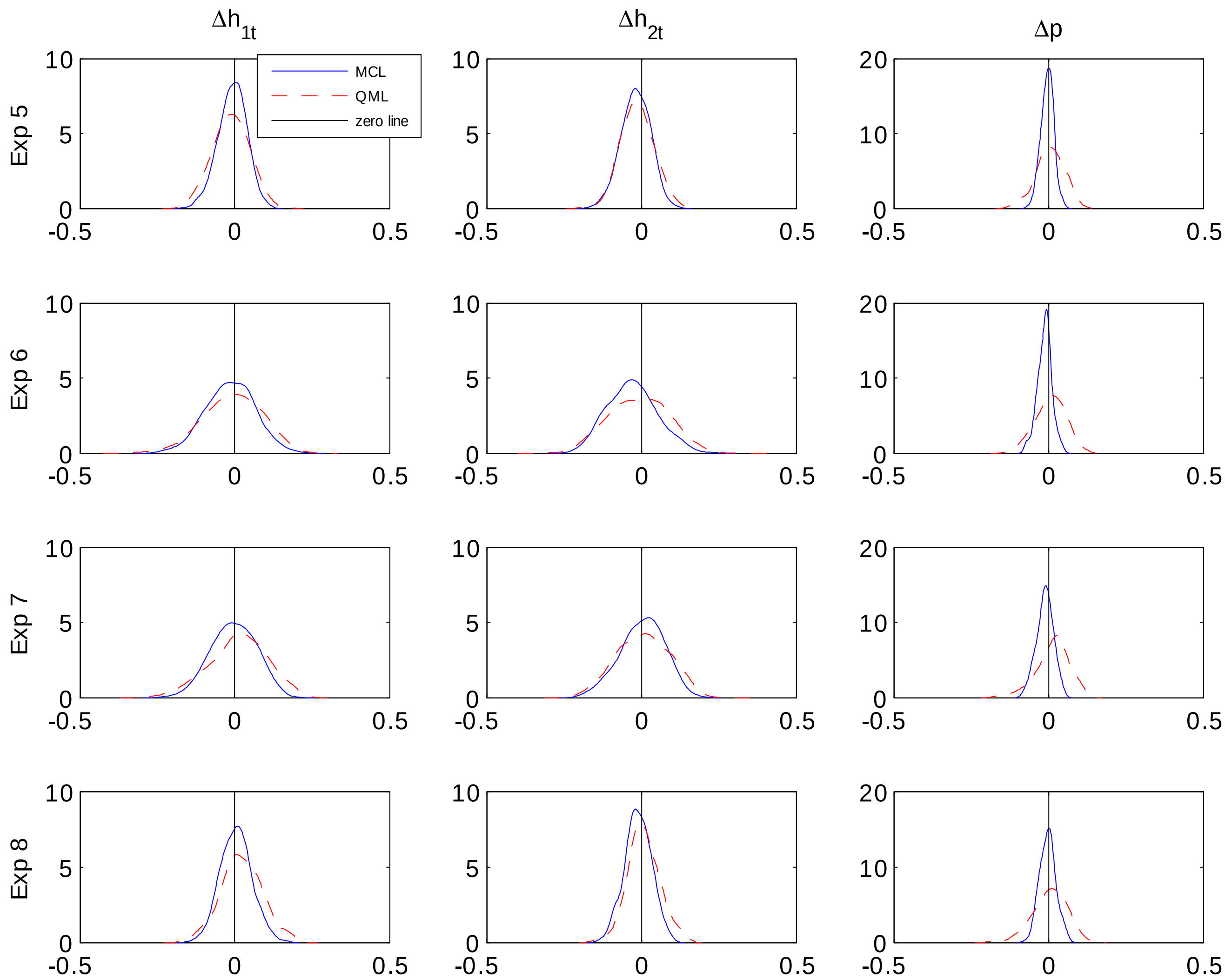

Figure 5: Kernel density estimates of the absolute deviations of MCL and QML volatility and correlation estimates from the true ones, for the CCMSV model. Experiments 5 to 8.

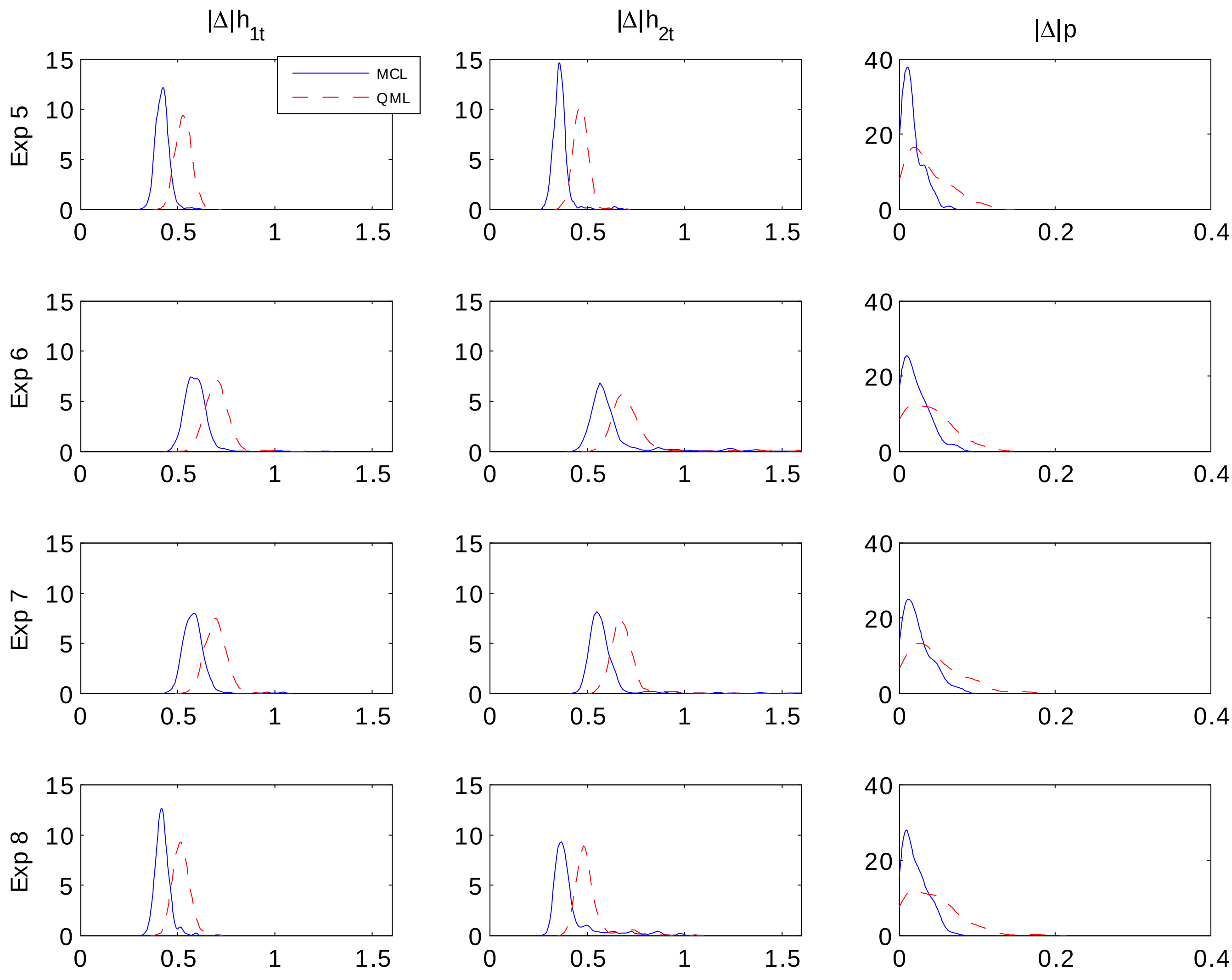

Figure 6: Kernel density estimates of the deviations and absolute deviations of MCL and QML volatility and correlation estimates from the true ones, for the TVCMSV model.

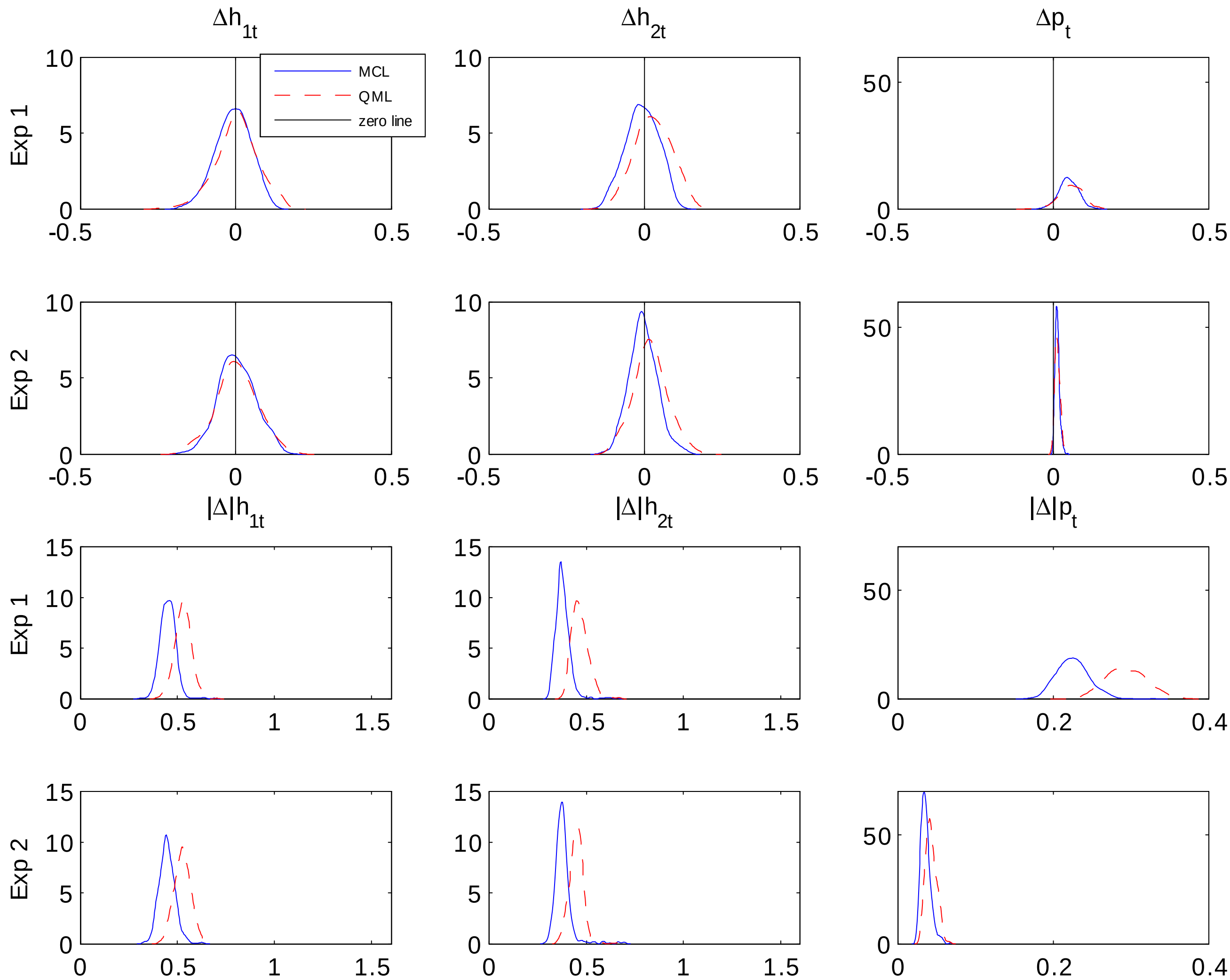

Figure 7: Kernel density estimates of the deviations and absolute deviations of MCL and QML volatility and correlation estimates from the true ones, for the MSV model with diagonal leverage.

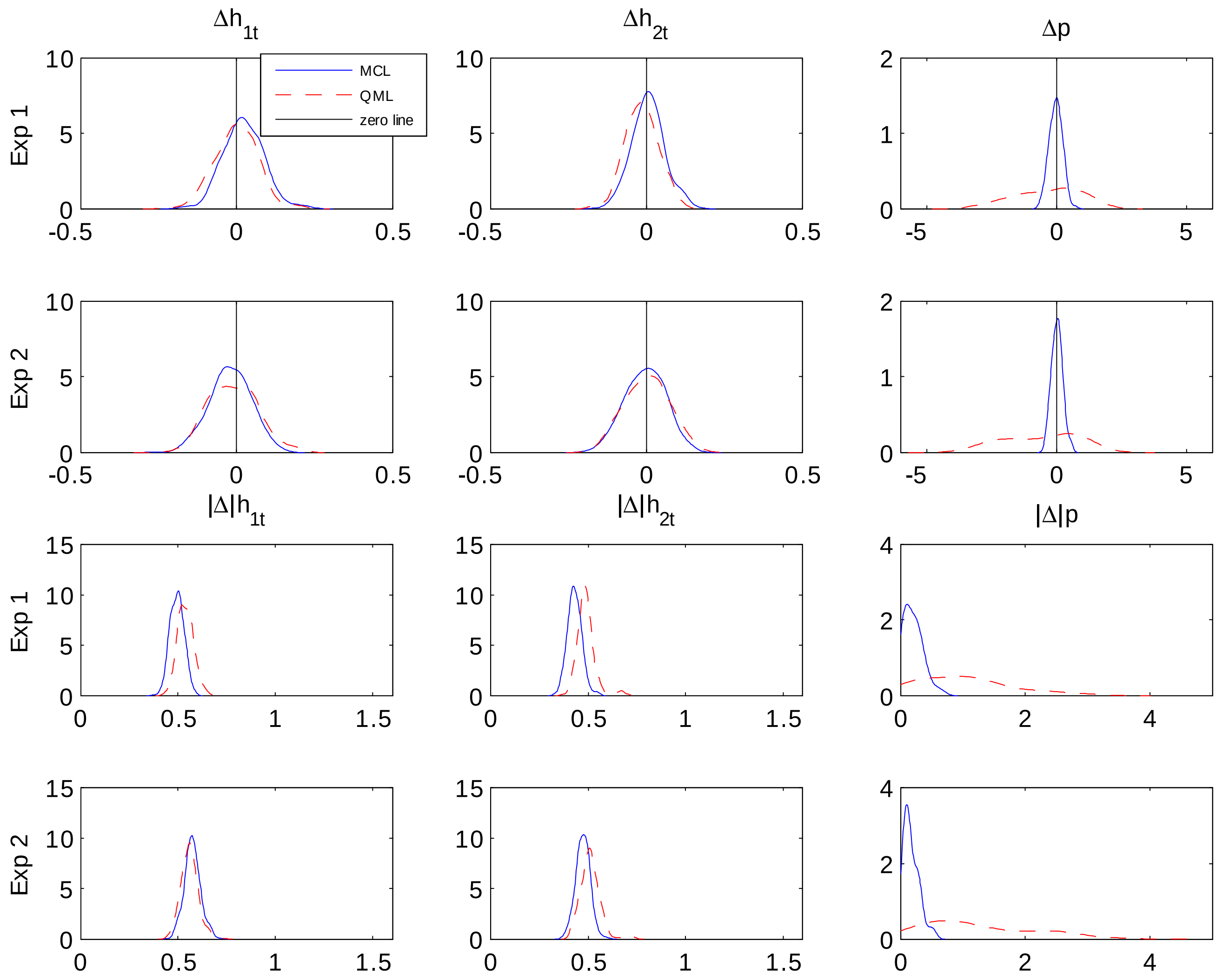

Figure 8: Kernel density estimates of the deviations and absolute deviations of MCL and QML volatility and correlation estimates from the true ones, for the MSV model with non-diagonal leverage. Exp 1: true leverage matrix is indefinite. Exp 2: true leverage matrix is negative definite.

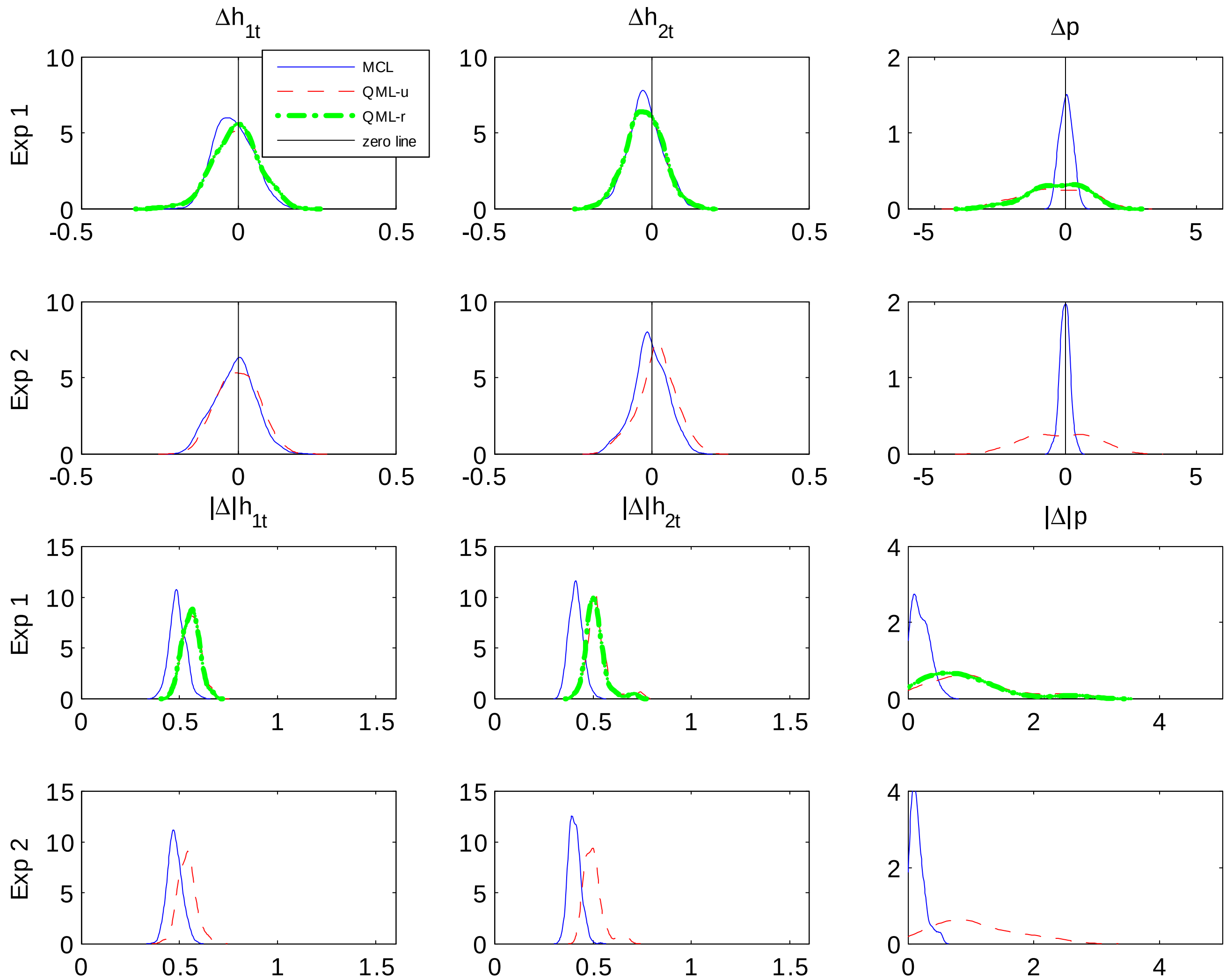

Figure 9: Absolute values of the returns and the MCL and QML smooth estimates of the standard deviations for IBEX 35, FTSE 100 and DAX stock markets between 4/1/2005 - 4/11/2011. Data source: Yahoo Finance

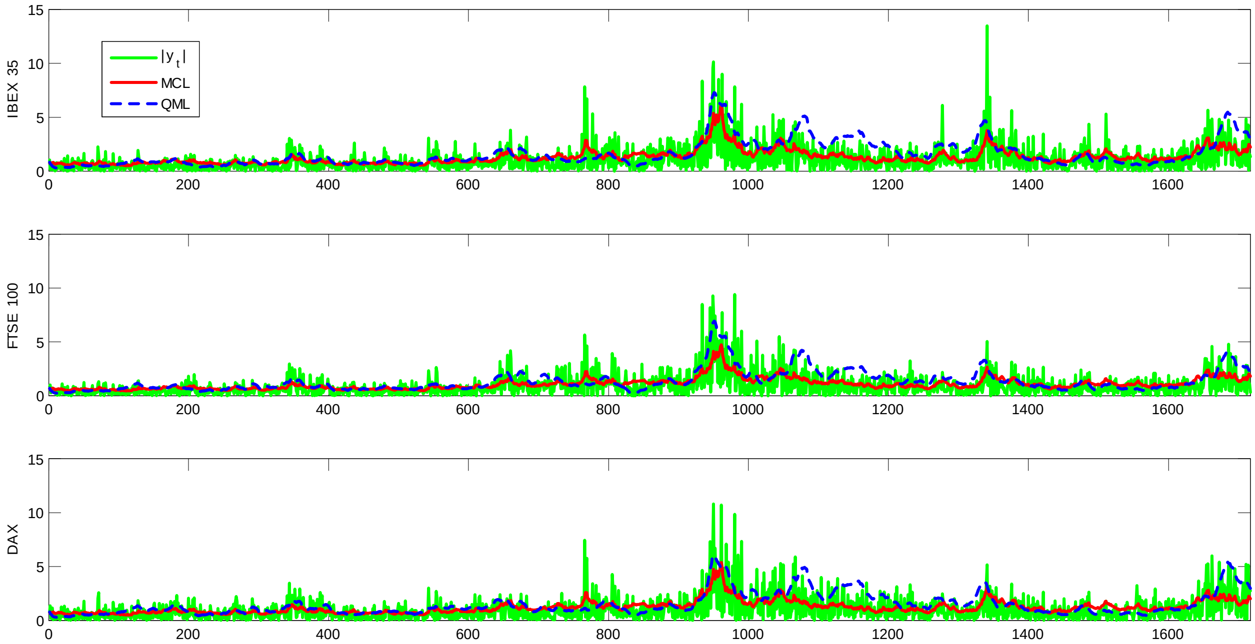

Table 1: The parameter estimation results of the simulations where the data is generated by a CC-MSV model and estimated via QML and MCL methods.

Estim.\Param.	$\{P_\varepsilon\}_{21}$	Γ_{11}	Γ_{21}	Φ_{11}	Φ_{22}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{22}$
Exp 1 - True	0.2000	-0.1000	-0.1300	0.9000	0.9500	0.1500	0.0400	0.0800
QML	0.1509	-0.1368	-0.2770	0.8646	0.8948	0.2373	0.0539	0.1824
	(0.1751)	(0.1092)	(0.4187)	(0.0971)	(0.1488)	(0.2731)	(0.0601)	(0.2454)
	[0.1819]	[0.1153]	[0.4437]	[0.1033]	[0.1587]	[0.2867]	[0.0617]	[0.2659]
MCL	0.1943	-0.1197	-0.1673	0.8811	0.9354	0.1649	0.0390	0.0862
	(0.0444)	(0.0617)	(0.0965)	(0.0499)	(0.0350)	(0.0695)	(0.0231)	(0.0363)
	[0.0447]	[0.0647]	[0.1034]	[0.0534]	[0.0380]	[0.0711]	[0.0231]	[0.0368]
Exp 2 - True	0.2000	-0.1000	-0.1300	0.9000	0.9800	0.1500	0.0400	0.0800
QML	0.1919	-0.1821	-0.2573	0.8241	0.9604	0.2593	0.0380	0.1015
	(0.1858)	(0.2093)	(0.5369)	(0.1835)	(0.0798)	(0.3124)	(0.0532)	(0.1333)
	[0.1860]	[0.2248]	[0.5518]	[0.1986]	[0.0821]	[0.3310]	[0.0533]	[0.1350]
MCL	0.1957	-0.1376	-0.1822	0.8649	0.9716	0.1797	0.0392	0.0809
	(0.0427)	(0.0727)	(0.1018)	(0.0641)	(0.0157)	(0.0782)	(0.0288)	(0.0279)
	[0.0429]	[0.0819]	[0.1144]	[0.0731]	[0.0178]	[0.0836]	[0.0288]	[0.0279]
Exp 3 - True	0.2000	-0.1000	-0.1300	0.9000	0.9500	0.4000	0.1500	0.3500
QML	0.1880	-0.1265	-0.1526	0.8707	0.9404	0.4954	0.1618	0.3649
	(0.1833)	(0.0790)	(0.0647)	(0.0577)	(0.0214)	(0.2785)	(0.0732)	(0.1333)
	[0.1837]	[0.0833]	[0.0686]	[0.0647]	[0.0234]	[0.2944]	[0.0742]	[0.1341]
MCL	0.1899	-0.1119	-0.1402	0.8867	0.9459	0.3974	0.1491	0.3325
	(0.0428)	(0.0536)	(0.0499)	(0.0346)	(0.0171)	(0.0881)	(0.0498)	(0.0850)
	[0.0440]	[0.0549]	[0.0510]	[0.0370]	[0.0179]	[0.0882]	[0.0498]	[0.0868]
Exp 4 - True	0.2000	-0.1000	-0.1300	0.9000	0.9800	0.4000	0.1500	0.3500
QML	0.1876	-0.1322	-0.1873	0.8676	0.9712	0.5114	0.1554	0.3439
	(0.1940)	(0.0832)	(0.0814)	(0.0598)	(0.0111)	(0.2813)	(0.0651)	(0.0929)
	[0.1944]	[0.0892]	[0.0995]	[0.0680]	[0.0142]	[0.3025]	[0.0653]	[0.0931]
MCL	0.1918	-0.1150	-0.1727	0.8852	0.9733	0.4054	0.1464	0.3314
	(0.0410)	(0.0521)	(0.0760)	(0.0334)	(0.0105)	(0.0841)	(0.0447)	(0.0750)
	[0.0418]	[0.0542]	[0.0872]	[0.0366]	[0.0124]	[0.0842]	[0.0449]	[0.0772]

Note: For each experiment, the true parameter values are reported in the first row. Then for each estimation method, MC mean, standard deviation (in parantheses) and root mean squared error (in square brackets) of the parameter estimates are reported, respectively. Experiments 1-4.

Table 2: The parameter estimation results of the simulations where the data is generated by a CC-MSV model and estimated via QML and MCL methods.

Estim.\Param.	$\{P_\varepsilon\}_{21}$	Γ_{11}	Γ_{21}	Φ_{11}	Φ_{22}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{22}$
Exp 5 - True	0.8000	-0.1000	-0.1300	0.9000	0.9500	0.1500	0.0400	0.0800
QML	0.8034	-0.1613	-0.2117	0.8363	0.9197	0.2747	0.0567	0.1230
	(0.0365)	(0.1398)	(0.2018)	(0.1377)	(0.0660)	(0.3111)	(0.0568)	(0.1578)
	[0.0366]	[0.1527]	[0.2177]	[0.1517]	[0.0726]	[0.3352]	[0.0593]	[0.1635]
MCL	0.7961	-0.1130	-0.1692	0.8854	0.9349	0.1559	0.0477	0.0841
	(0.0160)	(0.0476)	(0.0880)	(0.0384)	(0.0333)	(0.0500)	(0.0227)	(0.0326)
	[0.0165]	[0.0494]	[0.0964]	[0.0411]	[0.0366]	[0.0503]	[0.0240]	[0.0328]
Exp 6 - True	0.8000	-0.1000	-0.1300	0.9000	0.9800	0.4000	0.1500	0.3500
QML	0.8047	-0.1304	-0.1747	0.8714	0.9725	0.4666	0.1504	0.3383
	(0.0376)	(0.0763)	(0.0875)	(0.0622)	(0.0132)	(0.2286)	(0.0852)	(0.0997)
	[0.0379]	[0.0821]	[0.0983]	[0.0684]	[0.0152]	[0.2381]	[0.0852]	[0.1004]
MCL	0.7893	-0.1107	-0.1549	0.8906	0.9756	0.3703	0.1482	0.3250
	(0.0183)	(0.0403)	(0.0779)	(0.0285)	(0.0117)	(0.0821)	(0.0608)	(0.0655)
	[0.0212]	[0.0417]	[0.0818]	[0.0300]	[0.0125]	[0.0873]	[0.0608]	[0.0701]
Exp 7 - True	0.8000	-0.1000	-0.1300	0.9000	0.9500	0.4000	0.1500	0.3500
QML	0.8097	-0.1171	-0.1726	0.8822	0.9353	0.4297	0.1439	0.3852
	(0.0430)	(0.0561)	(0.0970)	(0.0505)	(0.0307)	(0.2127)	(0.1008)	(0.1633)
	[0.0441]	[0.0586]	[0.1059]	[0.0536]	[0.0340]	[0.2147]	[0.1010]	[0.1671]
MCL	0.7901	-0.1042	-0.1412	0.8978	0.9469	0.3520	0.1422	0.3193
	(0.0211)	(0.0373)	(0.0543)	(0.0247)	(0.0175)	(0.0763)	(0.0560)	(0.0725)
	[0.0233]	[0.0376]	[0.0555]	[0.0248]	[0.0178]	[0.0901]	[0.0565]	[0.0787]
Exp 8 - True	0.8000	-0.1000	-0.1300	0.9000	0.9800	0.1500	0.0400	0.0800
QML	0.8030	-0.1294	-0.2093	0.8650	0.9676	0.2212	0.0525	0.1035
	(0.0424)	(0.1174)	(0.1583)	(0.1079)	(0.0245)	(0.2554)	(0.0468)	(0.0662)
	[0.0426]	[0.1210]	[0.1771]	[0.1134]	[0.0275]	[0.2652]	[0.0484]	[0.0702]
MCL	0.7971	-0.1097	-0.1693	0.8854	0.9737	0.1469	0.0440	0.0826
	(0.0201)	(0.0429)	(0.0765)	(0.0392)	(0.0122)	(0.0444)	(0.0248)	(0.0233)
	[0.0203]	[0.0440]	[0.0860]	[0.0419]	[0.0137]	[0.0445]	[0.0251]	[0.0235]

Note: For each experiment, the true parameter values are reported in the first row. Then for each estimation method, MC mean, standard deviation (in parantheses) and root mean squared error (in square brackets) of the parameter estimates are reported, respectively. Experiments 5-8.

Table 3: Mean absolute error (MAE) and root mean squared error (RMSE) of the QML and MCL volatility and correlation estimates for CC-MSV model

MAE	Method	$\|\Delta\|h_{1t}$	$\|\Delta\|h_{2t}$	$\|\Delta\|p$	RMSE	Method	Δh_{1t}	Δh_{2t}	Δp
Exp 1	QML	0.5304	0.4748	0.8122	Exp 1	QML	0.6409	0.5459	0.9095
	MCL	0.4482	0.3885	0.1694		MCL	0.5407	0.4951	0.2235
	QML/MCL	1.1834	1.2222	4.7945		QML/MCL	1.1853	1.1026	4.0694
Exp2	QML	0.5335	0.5076	0.8245	Exp 2	QML	0.6334	0.6336	0.9300
	MCL	0.4539	0.4236	0.1668		MCL	0.5611	0.4676	0.2145
	QML/MCL	1.1753	1.1983	4.9430		QML/MCL	1.1288	1.3550	4.3357
Exp 3	QML	0.6873	0.6856	0.8349	Exp 3	QML	0.9383	0.8748	0.9185
	MCL	0.6177	0.6058	0.1695		MCL	0.8885	0.8137	0.2200
	QML/MCL	1.1126	1.1317	4.9256		QML/MCL	1.0560	1.0751	4.1750
Exp 4	QML	0.6912	0.7417	0.8905	Exp 4	QML	1.0889	0.8983	0.9720
	MCL	0.6168	0.6646	0.1583		MCL	1.0506	0.8286	0.2090
	QML/MCL	1.1206	1.1160	5.6254		QML/MCL	1.0364	1.0842	4.6507
Exp 5	QML	0.5305	0.4679	0.0365	Exp 5	QML	0.6829	0.6164	0.0457
	MCL	0.4212	0.3610	0.0161		MCL	0.5038	0.5015	0.0206
	QML/MCL	1.2595	1.2961	2.2671		QML/MCL	1.3555	1.2291	2.2182
Exp 6	QML	0.7040	0.7652	0.0378	Exp 6	QML	0.8746	0.9181	0.0474
	MCL	0.5943	0.6774	0.0202		MCL	0.7893	0.7105	0.0265
	QML/MCL	1.1846	1.1296	1.8713		QML/MCL	1.1081	1.2922	1.7877
Exp 7	QML	0.6967	0.6905	0.0438	Exp 7	QML	0.9336	0.9068	0.0551
	MCL	0.5838	0.5849	0.0228		MCL	0.8309	0.8759	0.0291
	QML/MCL	1.1934	1.1805	1.9211		QML/MCL	1.1236	1.0353	2.1724
Exp 8	QML	0.5194	0.5146	0.0421	Exp 8	QML	0.5563	0.8516	0.0532
	MCL	0.4223	0.4170	0.0201		MCL	0.5217	0.6874	0.0254
	QML/MCL	1.2299	1.2341	2.0945		QML/MCL	1.0663	1.2388	2.0985

Table 4: The parameter estimation results of the simulations where the data is generated by a TVC-MSV model and estimated via QML and MCL methods.

Estim.\Param.	$\{D\}_{21}$	Γ_{11}	Γ_{21}	Φ_{11}	Φ_{22}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{22}$
Exp 1 - True	0.2041	-0.1000	-0.1300	0.9000	0.9500	0.1500	0.0400	0.0800
QML	0.2034	-0.1852	-0.2068	0.8205	0.9178	0.2742	0.0648	0.0897
	(0.0439)	(0.2449)	(0.3879)	(0.2109)	(0.1540)	(0.3603)	(0.0815)	(0.1079)
	[0.0439]	[0.2593]	[0.3954]	[0.2254]	[0.1574]	[0.3811]	[0.0852]	[0.1083]
MCL	0.2039	-0.1322	-0.1691	0.8684	0.9352	0.1783	0.0501	0.0903
	(0.0174)	(0.0703)	(0.0759)	(0.0670)	(0.0286)	(0.0817)	(0.0315)	(0.0348)
	[0.0174]	[0.0773]	[0.0854]	[0.0741]	[0.0322]	[0.0865]	[0.0330]	[0.0363]
Exp 2 - True	1.3333	-0.1000	-0.1300	0.9000	0.9500	0.1500	0.0400	0.0800
QML	1.3275	-0.1541	-0.2205	0.8459	0.9151	0.2712	0.0597	0.0984
	(0.0250)	(0.1413)	(0.3675)	(0.1315)	(0.1338)	(0.3438)	(0.0774)	(0.0945)
	[0.0257]	[0.1513]	[0.3785]	[0.1421]	[0.1383]	[0.3646]	[0.0798]	[0.0962]
MCL	1.3359	-0.1358	-0.1677	0.8623	0.9358	0.1759	0.0441	0.0898
	(0.0164)	(0.0829)	(0.0809)	(0.0780)	(0.0307)	(0.0757)	(0.0286)	(0.0392)
	[0.0166]	[0.0903]	[0.0892]	[0.0867]	[0.0338]	[0.0800]	[0.0289]	[0.0404]

Note: For each experiment, the true parameter values are reported in the first row. Then for each estimation method, MC mean, standard deviation (in paranthesis) and root mean squared error (in square brackets) of the parameter estimates are reported, respectively.

Table 5: Mean absolute error (MAE) and root mean squared error (RMSE) of the QML and MCL volatility and correlation estimates for the TVCMSV model

MAE	Method	$\|\Delta\|h_{1t}$	$\|\Delta\|h_{2t}$	$\|\Delta\|p_t$	RMSE	Method	Δh_{1t}	Δh_{2t}	Δp_t
Exp 1	QML	0.5297	0.4679	0.2938	Exp 1	QML	0.6045	0.7038	0.4206
	MCL	0.4518	0.3789	0.2251		MCL	0.5176	0.4678	0.3058
	QML/MCL	1.1724	1.2349	1.3052		QML/MCL	1.1679	1.5044	1.3754
Exp 2	QML	0.5263	0.4502	0.0425	Exp 2	QML	0.6834	0.5564	0.0752
	MCL	0.4473	0.3788	0.0355		MCL	0.5148	0.5185	0.0628
	QML/MCL	1.1766	1.1885	1.1972		QML/MCL	1.3275	1.0730	1.1975

Table 6: The parameter estimation results of the simulations where the data is generated by an MSV model with diagonal leverage and estimated via QML and MCL methods.

Estim.\Param.	$\{P_\varepsilon\}_{21}$	Γ_{11}	Γ_{21}	Φ_{11}	Φ_{22}	L_{11}	L_{22}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{22}$
Exp 1 - True	0.2000	-0.1000	-0.1300	0.9000	0.9500	-0.2000	-0.2500	0.1500	0.0400	0.0800
QML	0.1289	-0.1650	-0.2379	0.8431	0.9093	-0.1543	-0.2431	0.2529	0.0475	0.1306
	(0.2438)	(0.1745)	(0.3002)	(0.1472)	(0.1125)	(0.2192)	(0.3691)	(0.2728)	(0.0578)	(0.1547)
	[0.2539]	[0.1862]	[0.3190]	[0.1579]	[0.1197]	[0.2240]	[0.3692]	[0.2916]	[0.0583]	[0.1628]
MCL	0.1935	-0.1272	-0.1712	0.8760	0.9350	-0.1335	-0.1822	0.1650	0.0413	0.0842
	(0.0499)	(0.0603)	(0.0864)	(0.0508)	(0.0322)	(0.1526)	(0.2087)	(0.0681)	(0.0277)	(0.0347)
	[0.0503]	[0.0662]	[0.0957]	[0.0562]	[0.0355]	[0.1665]	[0.2195]	[0.0697]	[0.0277]	[0.0349]
Exp 2 - True	0.2000	-0.1000	-0.1300	0.9000	0.9500	-0.5500	-0.6000	0.1500	0.0400	0.0800
QML	0.0751	-0.1798	-0.2235	0.8274	0.9162	-0.3822	-0.5342	0.2520	0.0404	0.1071
	(0.2748)	(0.1595)	(0.2049)	(0.1285)	(0.0700)	(0.2371)	(0.2939)	(0.2723)	(0.0504)	(0.1214)
	[0.3018]	[0.1784]	[0.2252]	[0.1476]	[0.0777]	[0.2905]	[0.3012]	[0.2908]	[0.0504]	[0.1243]
MCL	0.2029	-0.1648	-0.1976	0.8452	0.9273	-0.2952	-0.4543	0.2033	0.0446	0.0879
	(0.0416)	(0.0696)	(0.0958)	(0.0616)	(0.0373)	(0.1380)	(0.1675)	(0.0733)	(0.0276)	(0.0329)
	(0.0417)	[0.0951]	[0.1173]	[0.0824]	[0.0437]	[0.2898]	(0.2220)	(0.0906)	(0.0280)	(0.0338)

Note: For each experiment, the true parameter values are reported in the first row. Then for each estimation method, MC mean, standard deviation (in paranthesis) root mean squared error (in square brackets) of the parameter estimates are reported, respectively.

Table 7: Mean absolute error (MAE) and root mean squared error (RMSE) of the QML and MCL volatility and correlation estimates for the MSV model with diagonal leverage

MAE	Method	$\|\Delta\|h_{1t}$	$\|\Delta\|h_{2t}$	$\|\Delta\|p$	RMSE	Method	Δh_{1t}	Δh_{2t}	Δp
Exp 1	QML	0.5403	0.4903	1.0304	Exp 1	QML	0.6821	0.6206	1.2695
	MCL	0.4953	0.4304	0.1990		MCL	0.6208	0.5414	0.2515
	QML/MCL	1.0909	1.1392	5.1779		QML/MCL	1.0987	1.1463	5.0477
Exp 2	QML	0.5598	0.5125	1.2375	Exp 2	QML	0.7054	0.6480	1.5090
	MCL	0.5747	0.4729	0.1668		MCL	0.7231	0.5976	0.2085
	QML/MCL	0.9741	1.0837	7.4191		QML/MCL	0.9755	1.0843	7.2374

Table 8: The parameter estimation results of the simulations where the data is generated by an MSV model with non-diagonal leverage and estimated via unrestricted QML, restricted QML and MCL methods.

Estim.\Param.	$\{P_\varepsilon\}_{21}$	Γ_{11}	Γ_{21}	Φ_{11}	Φ_{22}	L_{11}	L_{21}	L_{22}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{22}$
Exp 1 - True	0.2000	-0.1000	-0.1300	0.9000	0.9500	-0.2000	-0.2300	-0.2500	0.1500	0.0400	0.0800
QML - unrestricted	0.1124	-0.1811	-0.2374	0.8292	0.9099	-0.1689	-0.1539	-0.2280	0.2844	0.0514	0.1511
	(0.2428)	(0.1616)	(0.2617)	(0.1430)	(0.0984)	(0.2474)	(0.1795)	(0.3349)	(0.3023)	(0.0548)	(0.2548)
	[0.2581]	[0.1808]	[0.2829]	[0.1596]	[0.1062]	[0.2494]	[0.1950]	[0.3357]	[0.3309]	[0.0560]	[0.2645]
QML - restricted	0.1323	-0.1964	-0.2716	0.8126	0.8970	-0.2009	-0.0951	-0.2637	0.2969	0.0672	0.1671
	(0.2081)	(0.1506)	(0.2498)	(0.1407)	(0.0934)	(0.2035)	(0.1251)	(0.2612)	(0.2475)	(0.0678)	(0.2053)
	[0.2189]	[0.1788]	[0.2871]	[0.1656]	[0.1074]	[0.2035]	[0.1330]	[0.2615]	[0.2878]	[0.0731]	[0.2231]
MCL	0.2045	-0.1550	-0.2307	0.8560	0.9141	-0.1530	-0.1236	-0.2342	0.1971	0.0447	0.1099
	(0.0474)	(0.0830)	(0.1517)	(0.0707)	(0.0561)	(0.1008)	(0.0800)	(0.1373)	(0.0863)	(0.0393)	(0.0552)
	[0.0477]	[0.0996]	[0.1821]	[0.0833]	[0.0666]	[0.1112]	[0.1087]	[0.1382]	[0.0984]	[0.0396]	[0.0628]
Exp 2 - True	0.2000	-0.1000	-0.1300	0.9000	0.9500	-0.2000	-0.0500	-0.2500	0.1500	0.0400	0.0800
QML - unrestricted	0.1535	-0.1560	-0.2669	0.8428	0.8957	-0.1441	-0.0106	-0.1813	0.2049	0.0426	0.1630
	(0.2405)	(0.1436)	(0.3407)	(0.1843)	(0.1388)	(0.2530)	(0.1958)	(0.3825)	(0.1696)	(0.0602)	(0.2345)
	[0.2450]	[0.1541]	[0.3672]	[0.1929]	[0.1490]	[0.2591]	[0.1997]	[0.3886]	[0.1783]	[0.0602]	[0.2488]
MCL	0.1960	-0.1356	-0.2012	0.8702	0.9232	-0.1452	-0.0159	-0.1995	0.1726	0.0445	0.1046
	(0.0370)	(0.0515)	(0.1082)	(0.0441)	(0.0413)	(0.1036)	(0.0849)	(0.1384)	(0.0658)	(0.0306)	(0.0493)
	[0.0372]	[0.0626]	[0.1296]	[0.0532]	[0.0492]	[0.1172]	[0.0915]	[0.1473]	[0.0696]	[0.0309]	[0.0551]

Note: Experiment 1 refers to the case where the leverage matrix, L, is indefinite while in Experiment 2 it is (negative) definite. The restriction that was imposed to the QML estimation is the one that is required for MCL estimation, namely, the L matrix is positive or negative semidefinite. For each experiment, the true parameter values are reported in the first row. Then for each estimation method, MC mean, standard deviation (in parenthesis) and root mean squared error (in square brackets) of the parameter estimates are reported, respectively.

Table 9: Mean absolute error (MAE) and root mean squared error (RMSE) of the QML and MCL volatility and correlation estimates for the MSV model with non-diagonal leverage.

MAE	Method	$\|\Delta\|h_{1t}$	$\|\Delta\|h_{2t}$	$\|\Delta\|p$	RMSE	Method	Δh_{1t}	Δh_{2t}	Δp
Exp 1	QML-u	0.5641	0.5245	1.0643	Exp 1	QML-u	0.7125	0.6760	1.2905
	QML-r	0.5587	0.5137	0.8786		QML-r	0.7047	0.6576	1.0945
	MCL	0.4900	0.4119	0.1944		MCL	0.6164	0.5206	0.2385
	QML-u/MCL	1.1512	1.0210	5.4748		QML-u/MCL	1.1559	1.2985	5.4109
	QML-r/MCL	1.1402	1.2471	4.5195		QML-r/MCL	1.1433	1.2632	4.5891
Exp 2	QML-u	0.5390	0.5008	1.0504	Exp 2	QML-u	0.6787	0.6406	1.2250
	MCL	0.4767	0.4053	0.1468		MCL	0.5979	0.5098	0.1860
	QML-u/MCL	1.1307	1.2356	7.1553		QML-u/MCL	1.1351	1.2566	6.5860

Note: In Experiment 1 the leverage matrix, L, is indefinite while in Experiment 2 it is (negative) definite.

Table 10: Descriptive statistics of the returns

Statistics \ Series	IBEX-35	FTSE-100	DAX
Mean	-0.0034	0.0076	0.0192
SD	1.5981	1.3667	1.5163
Skewness	0.1504	-0.1385	0.0346
Kurtosis	10.7492	10.4146	9.7788
Maximum	13.4836	9.3842	10.7975
Minimum	-9.5859	-9.2646	-7.4335
Box-Ljung test for autocorrelation, p_values			
Q(10), y_t	0.0179**	0.0000***	0.0152**
Q(10), y_t^2	0.0000***	0.0000***	0.0000***
Q(10), $\log y_t^2$	0.0000***	0.0000***	0.0000***

Note: *Significant at 10%, **Significant at 5%, ***Significant at 1%

Table 11: The empirical estimation results for the univariate SV model with leverage.

Estim.	Series	Γ	Φ	L	Q_η	Log-like	AIC	BIC
QML	IBEX 35	0.0137	0.9636	-0.5262	0.0693	-3890.6	7789.2	7811.0
		(0.0012)	(0.0055)	(0.0944)	(0.0080)			
	FTSE 100	0.0003	0.9625	-0.4964	0.0747	-3911.7	7831.4	7853.1
		(0.0038)	(0.0057)	(0.0457)	(0.0066)			
	DAX	0.0202	0.9480	-0.6653	0.0801	-3902.8	7813.5	7835.3
		(0.0049)	(0.0076)	(0.0974)	(0.0080)			
MCL	IBEX 35	0.0025	0.9957	-0.6574	0.0049	-2456.7	4921.4	4943.2
		(0.0002)	(0.0012)	(0.0007)	(0.0002)			
	FTSE 100	-0.0001	0.9965	-0.6022	0.0038	-2149.1	4306.2	4328.0
		(0.0001)	(0.0014)	(0.0312)	(0.0015)			
	DAX	0.0026	0.9942	-0.8328	0.0044	-2412.3	4832.5	4854.3
		(0.0002)	(0.0014)	(0.0113)	(0.0003)			

Table 12: The empirical estimation results obtained by QML method for the MSV with non-diagonal leverage model.

QML - unrestricted	$\{P_\varepsilon\}_{21}$	$\{P_\varepsilon\}_{31}$	$\{P_\varepsilon\}_{32}$	Γ_{11}	Γ_{21}	Γ_{31}	Φ_{11}	Φ_{22}	Φ_{33}	L_{11}	L_{21}
Log-like: -11179	0.8068	0.8720	0.8743	0.0147	0.0046	0.0131	0.9612	0.9548	0.9581	-0.5793	-0.1573
AIC: 22400	(0.0232)	(0.0201)	(0.0150)	(0.0015)	(0.0014)	(0.0017)	(0.0057)	(0.0065)	(0.0068)	(0.0284)	(0.0218)
BIC: 22515	L_{31}	L_{22}	L_{32}	L_{33}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{31}$	$\{Q_\eta\}_{22}$	$\{Q_\eta\}_{32}$	$\{Q_\eta\}_{33}$	
	-0.0522	-0.0851	-0.1835	-0.2173	0.0797	0.0788	0.0718	0.0949	0.0754	0.0770	
	(0.0024)	(0.0027)	(0.0110)	(0.0124)	(0.0069)	(0.0034)	(0.0038)	(0.0058)	(0.0048)	(0.0081)	
QML - restricted	$\{P_\varepsilon\}_{21}$	$\{P_\varepsilon\}_{31}$	$\{P_\varepsilon\}_{32}$	Γ_{11}	Γ_{21}	Γ_{31}	Φ_{11}	Φ_{22}	Φ_{33}	L_{11}	L_{21}
Log-like: -11181	0.8090	0.8730	0.8747	0.0157	0.0061	0.0161	0.9609	0.9506	0.9522	-0.6219	-0.1738
AIC: 22403	(0.0226)	(0.0181)	(0.0161)	(0.0010)	(0.0028)	(0.0024)	(0.0050)	(0.0071)	(0.0083)	(0.1125)	(0.0144)
BIC: 22518	L_{31}	L_{22}	L_{32}	L_{33}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{31}$	$\{Q_\eta\}_{22}$	$\{Q_\eta\}_{32}$	$\{Q_\eta\}_{33}$	
	-0.0725	-0.1245	-0.1453	-0.2143	0.0851	0.0883	0.0839	0.1101	0.0899	0.0981	
	(0.0143)	(0.0173)	(0.0299)	(0.0177)	(0.0041)	(0.0036)	(0.0077)	(0.0080)	(0.0067)	(0.0168)	

Note: The data is obtained from the returns of IBEX 35, FTSE 100 and DAX stock markets (in order 1^{st}, 2^{nd} and 3^{rd} series). Bollerslev-Wooldridge robust standard errors are obtained for the QML estimates. Restricted QML estimation is the one where the restriction needed for MCL estimation is also employed in the QML estimation only for comparison reasons.

Table 13: The empirical estimation results obtained by MCL method for the MSV with non-diagonal leverage model.

MCL	$\{P_\varepsilon\}_{21}$	$\{P_\varepsilon\}_{31}$	$\{P_\varepsilon\}_{32}$	Γ_{11}	Γ_{22}	Γ_{33}	Φ_{11}	Φ_{22}	Φ_{33}	L_{11}	L_{21}
Log-like: -4751	0.8212	0.8297	0.8542	0.0070	-0.0008	0.0043	0.9751	0.9778	0.9755	-0.6465	-0.2151
AIC: 9543	(0.0001)	(0.0002)	(0.0001)	(0.0002)	(0.0001)	(0.0003)	(0.0002)	(0.0002)	(0.0004)	(0.0001)	(0.0001)
BIC: 9658	L_{31}	L_{22}	L_{32}	L_{33}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{31}$	$\{Q_\eta\}_{22}$	$\{Q_\eta\}_{32}$	$\{Q_\eta\}_{33}$	
p_val/CCMSV: 0.00	-0.1052	-0.1769	-0.1387	-0.2657	0.0397	0.0347	0.0373	0.0305	0.0327	0.0351	
	(0.0019)	(0.0195)	(0.0224)	(0.0377)	(0.0001)	(0.0001)	(0.0002)	(0.0001)	(0.0002)	(0.0001)	
MCL-CCMSV	$\{P_\varepsilon\}_{21}$	$\{P_\varepsilon\}_{31}$	$\{P_\varepsilon\}_{32}$	Γ_{11}	Γ_{22}	Γ_{33}	Φ_{11}	Φ_{22}	Φ_{33}	L_{11}	L_{21}
Log-like: -5588	0.8413	0.8750	0.8835	0.0053	-0.0006	0.0030	0.9812	0.9833	09827	-	-
AIC: 11207	(0.0001)	(0.0002)	(0.0002)	(0.0004)	(0.0002)	(0.0003)	(0.0003)	(0.0002)	(0.0005)	-	-
BIC: 11289	L_{31}	L_{22}	L_{32}	L_{33}	$\{Q_\eta\}_{11}$	$\{Q_\eta\}_{21}$	$\{Q_\eta\}_{31}$	$\{Q_\eta\}_{22}$	$\{Q_\eta\}_{32}$	$\{Q_\eta\}_{33}$	
	-	-	-	-	0.0409	0.0339	0.0331	0.0328	0.0298	0.0327	
	-	-	-	-	(0.0002)	(0.0002)	(0.0003)	(0.0002)	(0.0003)	(0.0003)	

Note: The data is obtained from the returns of IBEX 35, FTSE 100 and DAX stock markets (in order 1^{st}, 2^{nd} and 3^{rd} series). The standard errors of MCL estimates are obtained from the numerical approximation to the Hessian.

Table 14: Post-estimation diagnostics for the standardized decorrelated residuals.

	MSV-NDL									CCMSV		
	QML - unrestricted			QML - restricted			MCL			MCL		
Test \ p_values	IBEX-35	FTSE-100	DAX	IBEX-35	FTSE-100	DAX	IBEX-35	FTSE-100	DAX	IBEX-35	FTSE-100	DAX
Q(10), v_t	0.9817	0.0000***	0.0025***	0.9376	0.0000***	0.0000***	0.8935	0.8525	0.1396	0.9330	0.4738	0.0138**
Q(10), v_t^2	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0136**	0.0680*	0.0035***	0.0304**	0.7390	0.0144**
Q(10), $\log v_t^2$	0.0004***	0.0000***	0.0000***	0.0000***	0.0000***	0.0000***	0.0245**	0.0125**	0.5659	0.0290**	0.7708	0.1680
$Q_3(10)$ v_t		0.0000***			0.0000***			0.3931			0.2476	
$Q_3(10)$, v_t^2		0.0000***			0.0000***			0.0000***			0.0000***	
$Q_3(10)$, $\log v_t^2$		0.0000***			0.0000***			0.0008***			0.3146	

Note: *Significant at 10%, **Significant at 5%, ***Significant at 1%