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Working paper Ec-02/11

Department of Economics

St. Petersburg
2011

Alexeev A.G., Sokolov M.V. *A note on indices of return*. – European University at St. Petersburg, Department of Economics. Working Paper Ec-02/11. 20 p.

February 21, 2011

Abstract:
Axiomatic derivations of the rate of return and its generalizations to non-exponential discounting and in the presence of inflation are given by elementary methods.

Keywords: rate of return, yield, investment project, axiomatic approach
JEL classification: G11

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The work was supported by Russian Foundation for Basic Research (RFBR). Project 10-06-00130.

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A note on indices of return[*]

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Introduction

There are several measures of return for an investment project's cash flow: profitability index, the internal rate of return, modified internal rate of return, as well as various modifications of those (e.g., see Arrow and Levhari, 1969; Cantor and Lipman, 1983; Promislow, 1997; Teichroew et al., 1965a, 1965b, to mention just a few). Yet the fact that there are different methods to solve a problem (the problem of project evaluation, in this particular case) often means that each of them is of limited applicability. This is partly justified by Promislow (1997) who used an axiomatic approach to show that a set of investment projects cannot be partitioned into "acceptable" (with respect to the obtained rate of return) and "unacceptable" parts when the set is "rich" enough. In particular, this means that it is impossible to define a preference relation on a set of investment projects that would satisfy some natural properties. However, such a relation can be unambiguously defined if one either relaxes the requirement that all projects are to be classified (the completeness axiom) (Promislow, 1997) or restricts oneself to a subset from the set of all possible projects (Promislow and Spring, 1996). The task becomes much easier if one deals with the simplest investment projects of the form: an initial investment of x at time t and return x' at time t' $(>t)$. In this case most of rate of return measures are reduced to the continuously compound growth rate

$$I_0(x,t;x',t') = \frac{\ln(x'/x)}{t'-t} \qquad (1)$$

or to order preserving transformation of (1). However, even in this simple case, there are generalizations of the measure I_0 to non-exponential discounting, the presence of inflation, dependence on the amount being invested, etc. (e.g., see Promislow, 1997; Vilenskii and Smolyak, 1998).

Theory of investment projects can hardly produce a superior measure of return (typical examples from the related fields are the secular "NPV–IRR" debate and an

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index number formula problem). But theory is able to define the limits of applicability for the existed measures. This can be done by using an axiomatic approach. This approach is used successfully in investment analysis, intertemporal choice, and related areas (e.g., see Bleichrodt et al., 2008; Castillo et al., 2005, chapter 12.4; Ebert, 1984; Eichhorn, 1978; Gray and Dewar, 1971; Promislow and Spring, 1996; Vilenskii and Smolyak, 1998).

In this paper axiomatic characterizations of I_0 and a few of its generalizations are given by elementary methods. Some of the axioms are adopted from Vilenskii and Smolyak (1998), who obtained a natural characterization of the internal rate of return and its generalization. A number of alternative axiomatizations of I_0 and its modifications are presented by Ebert (1984), Gray and Dewar (1971), Promislow and Spring (1996), Vilenskii and Smolyak (1998). The paper is organized as follows. In Section 1, we introduce several modifications and generalizations of the measure I_0 and list their properties. Section 2 presents axiomatic derivations of the introduced measures on the basis of the cited properties. In Section 3, we attempt to generalize some of the results from Section 2 to investment projects under risk.

1 Indices of return and their properties

Denote by R, R_+, and R_-, respectively, the set of real numbers, the set of positive real numbers, and the set of negative real numbers. Let $\mathrm{V} = \{(x,t;x',t') \in (\mathrm{R}_+ \times \mathrm{R})^2 : t < t'\}$. An element of V is denoted by $\boldsymbol{v} = (x,t;x',t')$ and interpreted as a simple investment project: investing an amount x at time t and getting x' at time t' (one can think of x \$ (x' \$) as a portfolio position at time t (t')). This type of financial operations (along with the opposite one – a financial liability of receiving an amount x at time t and returning an amount x' at time t') is one of the most common on financial markets.

A weak order (a complete and transitive binary relation) $(\mathrm{V}, \succeq)$ is assumed to be defined on the set V. If $\boldsymbol{v} \succeq \boldsymbol{v}'$, $\boldsymbol{v}, \boldsymbol{v}' \in \mathrm{V}$, we say that $\boldsymbol{v}$ is more profitable than $\boldsymbol{v}'$. Symmetric (the "equally profitable" relation) and asymmetric (the "strictly more profitable" relation) parts of $(\mathrm{V}, \succeq)$ are denoted by $(\mathrm{V}, \sim)$ and $(\mathrm{V}, \succ)$, respectively. A function $I : \mathrm{V} \to \mathrm{R}$ is said to be an *index* (*of return*) if I represents $(\mathrm{V}, \succeq)$:

$\boldsymbol{v} \succeq \boldsymbol{v}' \Leftrightarrow I(\boldsymbol{v}) \ge I(\boldsymbol{v}')$.

A well-known example of an index of return is given by (1). Clearly, if I is an index, then $u \circ I$, where u is an order preserving transformation, is an index as well. A wide range of transformations u of the index I_0 are applied in financial analysis of investments to obtain a number of return measures: $u(x) = x - r$, where r is the inflation rate, is the real effective continuously compounded (logarithmic) return, $u(x) = e^x - 1$ is the compound growth rate, etc.

The index I_0 can be generalized in various directions. The cost of capital usually depends on the period of the investment; a generalization of I_0 under changing cost of capital is given by

$$I_1(\boldsymbol{v}) = \frac{\ln(x'/x)}{\phi(t') - \phi(t)}, \qquad (2)$$

where the function ϕ is strictly increasing and onto R. A possible intuitive meaning for the function ϕ is a transformation of the time axis onto itself (time change) that induces a constant cost of capital. Another interpretation for the function ϕ will be given below.

Some properties of the relation $(\mathrm{V}, \succeq)$ induced by index I_1 are listed below.

Monotonicity in payments:

(i) (a) $(y,t;x',t') \succ (x,t;x',t') \succ (x,t;y',t')$ for all $(x,t;x',t') \in \mathrm{V}$, $y \in (0,x)$, $y' \in (0,x')$; (b) given $\boldsymbol{v}, (y,\tau;y',\tau') \in \mathrm{V}$, there exist y_1, y_2, y_1', y_2' such that $(y_1,\tau;y,\tau') \succeq \boldsymbol{v} \succeq (y_2,\tau;y,\tau')$ and $(y,\tau;y_1',\tau') \succeq \boldsymbol{v} \succeq (y,\tau;y_2',\tau')$.

Monotonicity in period:

(ii) if $x < x'$, then $(x,\tau;x',t') \succ (x,t;x',t') \succ (x,t;x',\tau')$ for any $t < \tau < t' < \tau'$; if $x > x'$, then $(x,\tau;x',t') \succ (x,t;x',t') \succ (x,t;x',\tau')$ for any $\tau < t < \tau' < t'$; $(x,t;x,t') \sim (y,\tau;y,\tau')$ for all x, y, $t < t'$, $\tau < \tau'$; for any $(x,t;x',t'), (y,\tau;y',\tau') \in \mathrm{V}$ with $(x'-x)(y'-y) > 0$, there exist $\tau_1, \tau_2, \tau_1', \tau_2'$ such that $(y,\tau_1;y',\tau') \succeq (x,t;x',t') \succeq (y,\tau_2;y',\tau')$ and $(y,\tau;y',\tau_1') \succeq (x,t;x',t') \succeq (y,\tau;y',\tau_2')$.

Invariance with respect to a subsequent realization of projects:

(iii) $(x,t;x',t') \sim (x',t';x'',t'') \Rightarrow (x,t;x',t') \sim (x,t;x'',t'')$.

Invariance with respect to a simultaneous realization of projects:

(iv) $(x,t;x',t') \sim (y,t;y',t') \Rightarrow (x,t;x',t') \sim (x+y,t;x'+y',t')$.

Scale invariance:

(v) $(x,t;x',t') \sim (\alpha x,t;\alpha x',t')$ for any $\alpha > 0$.

Time-shift invariance:

(vi) there exists a strictly increasing function ϕ from R onto R such that $(x,t;x',t') \sim (x,\phi^{-1}(\phi(t)+\tau);x',\phi^{-1}(\phi(t')+\tau))$ for any $\tau \in \mathrm{R}$.

Covariance with respect to a change of time scale:

(vii) there exists a strictly increasing function ϕ from R onto R such that $(x,t;x',t') \succeq (x,\tau;x',\tau') \Rightarrow (x,\phi^{-1}(a\phi(t)+b);x',\phi^{-1}(a\phi(t')+b)) \succeq (x,\phi^{-1}(a\phi(\tau)+b);x',\phi^{-1}(a\phi(\tau')+b))$ for any $a > 0$, b.

Continuity:

(viii) given $\boldsymbol{v} \in \mathrm{V}$, the sets $\{\boldsymbol{v}' \in \mathrm{V} : \boldsymbol{v}' \succeq \boldsymbol{v}\}$ and $\{\boldsymbol{v}' \in \mathrm{V} : \boldsymbol{v} \succeq \boldsymbol{v}'\}$ are closed in V.

Interpretations of properties (i)–(viii) are as follows. Projects with higher returns are more profitable (i). Property (ii) assures monotonic dependence of profitability on the length investment period. By property (iii), if projects over two subsequent periods are equally profitable, the profitability over the two periods will be the same. Similarly, if two simultaneous projects are equally profitable, profitability of the consolidated project will be the same (iv). The last two properties allow to decompose the decision process into evaluation of individual investment projects (Vilenskii and Smolyak, 1998). Indeed, under (i) and (iv) if $(y,t;y',t') \succeq \boldsymbol{v}$ and $(z,t;z',t') \succeq \boldsymbol{v}$, then $(y+z,t;y'+z',t') \succeq \boldsymbol{v}$. Similarly, under (ii) and (iii) if $(y,t;y',t') \succeq \boldsymbol{v}$ and $(y',t;y'',t') \succeq \boldsymbol{v}$, then $(y,t;y'',t') \succeq \boldsymbol{v}$. Most of the known measures of return are independent of investment volume (v). There is a transformation of the time axis (time change) in which profitability is invariant with respect to a time shift (vi). There is a transformation of the time axis (time change) in which time is measured on an interval scale (vii). The function ϕ in (vi) and (vii) has the same intuitive meaning as in the definition of I_1. Another interpretation for the function ϕ will be given below. Finally, there exists a continuous index of return that represents $(\mathrm{V},\succeq)$ (viii) (Debreu, 1954).

A further generalization of I_0 to non-exponential types of discounting is given by

$$I_2(\boldsymbol{v}) = \frac{u(x'/x)}{\phi(t') - \phi(t)}, \tag{3}$$

where the function $u:\mathrm{R}_+ \to \mathrm{R}$ is strictly increasing and $\operatorname{sgn} u(x) = \operatorname{sgn}(\ln x)$. Here u determines the type of discounting: exponential discounting (compound interest) corresponds to $u(x) = \ln x$, hyperbolic discounting (simple interest) corresponds to $u(x) = x - 1$, generalized hyperbolic discounting (Loewenstein and Prelec, 1992) corresponds to $u(x) = (x^\alpha - 1)/\alpha$, $\alpha \neq 0$.

The rate of return may depend on the amount being invested (e.g., the interest rate is usually dependent on the amount being invested); a generalization of (3) to the case of such dependence is given by

$$I_3(\boldsymbol{v}) = \frac{u(x, x')}{\phi(t') - \phi(t)}, \tag{4}$$

where $u(x,\cdot)$ is strictly increasing, $u(\cdot, x')$ is strictly decreasing, and $\operatorname{sgn} u(x, x') = \operatorname{sgn}(x' - x)$.

A number of other generalizations of (1) can be obtained from the fact that $d = I_0(\boldsymbol{v})$ is a solution of the equation

$$e^{-dt} x = e^{-dt'} x', \tag{5}$$

i.e., is the internal rate of return of the investment project and can be interpreted as the discount rate that equalizes the present value of the costs x with that of future benefits x'.

Eq. (5) assumes exponential discounting and no inflation. A generalization of I_0 to other widespread types of discounting (Bleichrodt et al., 2009; Loewenstein and Prelec, 1992; Rachlin, 2006) and in the presence of inflation is given by the function that takes each $(x, t; x', t') \in \mathrm{V}$ to a unique solution $d \in \mathrm{R}$ of the equation

$$x f(d;t) = x' f(d;t'), \tag{6}$$

where $f:\mathrm{R}^2 \to \mathrm{R}_+$, $f(\cdot;t)$ is strictly increasing if $t < 0$, strictly decreasing if $t > 0$, and $f(\cdot;0) \equiv 1$. Here $f(d;t)$ is the discount factor at time t and discount rate d. A natural axiomatization of (6) (more precisely, its generalization to a large class of cash flows) is given by Vilenskii and Smolyak (1998).

Clearly, $f(d;t) = e^{-d\phi(t)}$ corresponds to I_1. This helps to interpret the denominator in Eq. (2). To simplify the interpretation, assume ϕ to be absolutely continuous, then

$\phi(t') - \phi(t) = \int_t^{t'} \rho(\tau) d\tau$ for some nonnegative (almost everywhere) Lebesgue integrable function ρ. ρ mimics the relative change in the cost of capital; thus, the difference $\phi(t') - \phi(t)$ is proportional to the continuously compounded cost of capital during the period $[t, t']$. To see this note that $\rho(t)$ is proportional (almost everywhere) to the instantaneous discount rate $-\frac{\partial \ln f(d;t)}{\partial t}$.

Using an axiomatic approach, Castillo et al. (2005, p. 80, example 5.3) argue that the function w that maps x at time t to

$$w(x,t) = g^{-1}(g(x) - \delta t), \tag{7}$$

serves as a generalization of the present value concept. Here the function g is strictly increasing and onto R, $\delta \in \{-1, 0, 1\}$. Exponential discounting at a rate $d \neq 0$ corresponds to $g(x) = |d|^{-1} \ln x$, $\delta = \mathrm{sgn}(d)$. Eq. (7) is obtained from the assumption that the present value $w(x, t+\tau)$ of an amount x at time $t+\tau$ depends only on τ and on the present value $w(x,t)$ of x at time t:

$$w(x, t+\tau) = w\big(w(x,t), \tau\big), \; w(x, 0) = x. \tag{8}$$

Under weak regularity assumptions (Moszner, 1989, théorème 3) the general continuous solution of the system (8) is given by (7). This approach induces the index of return that assigns to each $(x,t;x',t') \in \mathrm{V}$ a unique solution d of the equation

$$g(d;x) - \delta(d)t = g(d;x') - \delta(d)t', \tag{9}$$

where the function g is strictly increasing with respect to the second argument, $\delta(d) \in \{-1, 0, 1\}$. The difference $g(d;x') - g(d;x)$ with $\delta(d) \neq 0$ can be interpreted as a time interval τ such that the present value of x' at time τ (at the discount rate d) equals x, i.e. $g(d;x') - g(d;x)$ is the time value of money given the discount rate d.

Finally, the following index of return generalizes most of the examples given; it assigns to each project $(x,t;x',t') \in \mathrm{V}$ a unique solution d of the equation

$$h(d;x,t) = h(d;x',t'), \tag{10}$$

where h is continuous and strictly increasing with respect to the second argument; $h(\cdot;x,t)$ is continuous and strictly increasing if $t < 0$, decreasing if $t > 0$, and

$h(\cdot;x,0)\equiv x$. $h(d;x,t)$ is interpreted as the present value of an amount x at time t given the discount rate d. For a fixed d, a number of possible functional forms for $h(d;\cdot,\cdot)$ are axiomatized by Castillo et. al. (2005, chapter 12.4) and Eichhorn (1978).

To motivate Eq. (10) consider a generalization of the system (8) to varying cost of capital:

$$w(x,t'',t)=w\left(w(x,t'',t'),t',t\right),\ w(x,t,t)=x. \tag{11}$$

Here $w(x,t',t)$ is interpreted as the future value at time t of an amount x at time t'. The general solution of the system (11) is given by (Aczél and Gołąb, 1960, §II.1.1)

$$h\left(w(x,t',t),t\right)=h(x,t'), \tag{12}$$

where h is an arbitrary function such that for each t $h(\cdot,t)$ has the inverse. Since h in Eq. (12) is defined up to an injective transformation, it can be chosen such that $h(x,0)=x$. Hence, we interpret $h(x,t)$ as the present value of an amount x at time t: $h(x,t)=h\left(w(x,t,0),0\right)=w(x,t,0)$. These arguments justify the functional form of the index of return as a solution (10) and the given intuitive meaning for $h(d;x,t)$.

Note that generalizations (6), (9), (10) of I_0 index do not account for several important cases, in particular, hyperbolic discounting: Eq. (6) with the discount factor $f(d;t)=(1+dt)^{-1}$ has no solution for some $(x,t;x',t')\in \mathrm{V}$. In other words, there is no unambiguous analogue of the internal rate of return for hyperbolic discounting.

2 Axiomatizations

In this section axiomatizations of the introduced indices (1)–(4), (6), (9), and (10) on the bases of axioms (i)–(viii) are given. Note: axioms (i)–(viii) are not independent. In what follows we rely on Proposition 1.

Proposition 1.

(a) (iv) and (viii) $\Rightarrow$ (v); (i) (part (a)) and (v) $\Rightarrow$ (iv);

(b) (ii), (vii), and (viii) $\Rightarrow$ (vi).

Proof.

(a) Assume (iv) and (viii). With $y=x$, $y'=x'$ (iv) reduces to $(x,t;x',t') \sim (2x,t;2x',t')$. By induction,

$(x,t;x',t') \sim (nx,t;nx',t')$ for every natural n. (13)

Similarly,

$(x,t;x',t') \sim (x/m,t;x'/m,t')$ for every natural m. (14)

Combining (13) and (14), we get

$(x,t;x',t') \sim (rx,t;rx',t')$ (15)

for any positive rational number r. Now (15) holds for any $r \in \mathrm{R}_+$, since the set of positive rational numbers is everywhere dense in R_+ and (viii) holds.

Assume that conditions (i) (part (a)) and (v) are satisfied. Then (iv) holds, since $(x,t;x',t') \sim (y,t;y',t') \Leftrightarrow x'/x = y'/y$.

(b) Define a relation $(\mathrm{V},\succeq')$ by

$(x,t;x',t') \succeq (y,\tau;y',\tau') \Leftrightarrow (x,\phi(t);x',\phi(t')) \succeq' (y,\phi(\tau);y',\phi(\tau'))$ (16)

with $(\mathrm{V},\sim')$ and $(\mathrm{V},\succ')$ defined as usual. Since a strictly increasing function ϕ from R onto R is continuous, $(\mathrm{V},\succeq')$ satisfies (ii) and (viii) if and only if so does $(\mathrm{V},\succeq)$. In this notation, (vi) and (vii) take the form of:

(vi)' $(x,t;x',t') \sim' (x,t+\tau;x',t'+\tau) \quad \forall \tau \in \mathrm{R}$,

(vii)' $(x,t;x',t') \succeq' (x,\tau;x',\tau') \Rightarrow (x,at+b;x',at'+b) \succeq (x,a\tau+b;x',a\tau'+b) \quad \forall a>0$, b.

Given $(x,t_1;x',t_1') \in \mathrm{V}$, if $x = x'$ then, by axiom (ii), (vi)' holds. In what follows, we assume that $x < x'$ (the argument in case of $x > x'$ is similar).

Assume the contrary: there exists $\tau > 0$ such that

$(x,t_1;x',t_1') \succ' (x,t_1+\tau;x',t_1'+\tau)$ (17)

(if $\tau < 0$, the arguments are the same). Denote $t_2 = t_1 + \tau$ and let t_2' be a solution of the equation $(x,t_1;x',t_1') \sim' (x,t_2;x',t_2')$. By (ii), (viii), and (17), t_2' is well defined and $0 < t_2' - t_2 < t_1' - t_1$.

Let us construct two sequences $\{t_k\}_{k=1}^{\infty}$, $\{t_k'\}_{k=1}^{\infty}$ by $t_{k+1} = at_k + b$, $t_{k+1}' = at_k' + b$, $k = 1,2,\ldots$, where $a \in (0,1)$, $b \in \mathrm{R}$ is a solution of the system of linear equations $at_1 + b = t_2$, $at_1' + b = t_2'$. Thus defined sequences are increasing and converge to $t^* = b/(1-a)$.

Let t $(< t^*)$ be a unique, by (ii) and (viii), solution of $(x,t_1;x',t_1') \sim' (x,t;x',t^*)$. Then, by (vii)' with the constants a and b, $(x,t_1;x',t_1') \sim' (x,t_k;x',t_k')$, $k = 2,3,...$ Therefore,

$$(x,t;x',t^*) \sim' (x,t_k;x',t_k'),\ k = 1,2,... \tag{18}$$

Since $\{t_k\}_{k=1}^{\infty}$ and $\{t_k'\}_{k=1}^{\infty}$ are increasing and convergent, there exists k such that $t \le t_k < t_k' < t^*$. Combining this and (18), we get a contradiction with (ii). ■

The next theorem and its corollary follow directly from Proposition 1.

Theorem 1. An axiomatization of I_3 (4).

Let $(\mathrm{V}, \succsim)$ be a weak order. The following two statements are equivalent:

(a) $(\mathrm{V}, \succsim)$ satisfies (i) (part (a)), (ii), (vii), and (viii);

(b) there exists a continuous function $u : \mathrm{R}_+^2 \to \mathrm{R}$ such that I_3 represents $(\mathrm{V}, \succsim)$; here $u(x,\cdot)$ is strictly increasing, $u(\cdot,x')$ is strictly decreasing, and $\operatorname{sgn} u(x,x') = \operatorname{sgn}(x'-x)$.

Proof.

(a) $\Rightarrow$ (b). Let $(\mathrm{V}, \succsim')$ be defined by (16). By the continuous utility representation theorem of Debreu (1954) and Proposition 1 (b), there exists a continuous function $u_1 : \mathrm{R}_+^3 \to \mathrm{R}$ such that $(x,t;x',t') \mapsto u_1(x,x';t'-t)$ represents $(\mathrm{V}, \succsim')$. By (ii), $u_1(x,x';\cdot)$ is strictly decreasing if $x < x'$, strictly increasing if $x > x'$, and constant if $x = x'$.

Define

$$u_2(t) = \begin{cases} u_1(1,2;1/t) & \text{if } t > 0 \\ u_1(1,1;1) & \text{if } t = 0 \\ u_1(2,1;-1/t) & \text{if } t < 0 \end{cases}.$$

Thus defined function u_2 from R onto $u_1(\mathrm{R}_+^3)$ is strictly increasing, by (ii). Since $u_1(\mathrm{R}_+^3)$ is an interval, u_2 is continuous. Therefore, the function $I(x,t;x',t') = u_3(x,x';t'-t)$ with $u_3 = u_2^{-1} \circ u_1$, is well defined and represents $(\mathrm{V}, \succsim')$.

Given $(x,x',\tau) \in \mathrm{R}_+^3$, let $\tau' = u_3(x,x';\tau)$. Then, from the definition of u_3 and (vii), we get

$$u_3(x,x';a\tau) = \tau'/a = u_3(x,x';\tau)/a \;\; \forall\, a > 0,$$

and

$$I(x,t;x',t') = u_3(x,x';t'-t) = \frac{u_3(x,x';1)}{t'-t}. \tag{19}$$

With the function $u(\cdot) = u_3(\cdot;1)$ (19) takes the form (4). From (i) (part (a)) and (ii) it follows that $u(x,\cdot)$ is strictly increasing, $u(\cdot,x')$ is strictly decreasing, and $\operatorname{sgn} u(x,x') = \operatorname{sgn}(x'-x)$.

(b) $\Rightarrow$ (a). Trivial. ■

Corollary 1. An axiomatization of I_2 (3).

Let $(\mathrm{V}, \succeq)$ be a weak order. The following two statements are equivalent:

(a) $(\mathrm{V}, \succeq)$ satisfies (i) (part (a)), (ii), (iv), (vii), and (viii);

(b) there exists a strictly increasing and continuous function $u: \mathrm{R}_+ \to \mathrm{R}$ with $\operatorname{sgn} u(x) = \operatorname{sgn}(\ln x)$ such that I_2 represents $(\mathrm{V}, \succeq)$.

Proof.

The result follows immediately from Theorem 1 and Proposition 1 (a). ■

Theorem 2. An axiomatization of (10).

If a weak order $(\mathrm{V}, \succeq)$ satisfies (i), (iii), and (viii), then there exists a function $h: \mathrm{R} \times \mathrm{R}_+ \times \mathrm{R} \to \mathrm{R}$ such that the map $I: \mathrm{V} \to \mathrm{R}$ that takes each $\boldsymbol{v} = (x,t;x',t') \in \mathrm{V}$ to a unique solution $I(\boldsymbol{v})$ of the equation

$$h(I(\boldsymbol{v}); x, t) = h(I(\boldsymbol{v}); x', t') \tag{20}$$

represents $(\mathrm{V}, \succeq)$.

Proof.

Let $I: \mathrm{V} \to \mathrm{R}$ be the function that takes each element $\boldsymbol{v} \in \mathrm{V}$ to a solution d of the equation

$$\boldsymbol{v} \sim (1,0;e^d,1). \tag{21}$$

By (i) and (viii), I is well defined and represents $(\mathrm{V}, \succeq)$.

Denote $\mathrm{D} = \mathrm{R} \times \mathrm{R}_+ \times \{(t,t') \in \mathrm{R}^2 : t < t'\}$. Let the function $q{:}\mathrm{D} \to \mathrm{R}_+$ take each vector $(d;x,t,t') \in \mathrm{D}$ to a solution $x' \in \mathrm{R}_+$ of the equation

$$(x,t;x',t') \sim (1,0;e^d,1).$$

By (i) and (viii), the function q is well defined, continuous, and strictly monotone with respect to the first two arguments.

By (iii), we have

$$q(d;q(d;x,t,t'),t',t'') = q(d;x,t,t''). \qquad (22)$$

Let us define a function $h : \mathrm{R} \times \mathrm{R}_+ \times \mathrm{R} \to \mathrm{R}_+$ by

$$h(d;x,t) = \begin{cases} q(d;x,t,0) & \text{if } t < 0 \\ x & \text{if } t = 0 \\ \text{a unique solution } y \text{ of the equation } q(d;y,0,t) = x & \text{if } t > 0 \end{cases}. \qquad (23)$$

Thus defined function h is continuous and strictly increasing with respect to the second argument; $h(\cdot;x,t)$ is continuous and strictly increasing if $t < 0$ and strictly decreasing if $t > 0$.

Next we prove that

$$h(d;q(d;x,t_1,t_2),t_2) = h(d;x,t_1),\ (d;x,t_1,t_2) \in \mathrm{D}. \qquad (24)$$

Given $t_1 < t_2$, consider the following four possible cases.

– $t_1 = 0$ or $t_2 = 0$. (24) holds by the definition of h.

– $t_2 < 0$. Substituting t for t_1, t' for t_2, and t'' for 0 in (22), we get (24).

– $0 \in (t_1,t_2)$. From (22) with $t = t_1$, $t' = 0$, $t'' = t_2$, we obtain

$$q(d;q(d;x,t_1,0),0,t_2) = q(d;x,t_1,t_2),$$

or, equivalently, (24).

– $0 < t_1$. From (22) with $t = 0$, $t' = t_1$, $t'' = t_2$, we get

$$q(d;q(d;x,0,t_1),t_1,t_2) = q(d;x,0,t_2)$$

or, equivalently, (24) (since $q(d;\cdot,0,t_1)$ is onto R_+).

Thus, by (21) and (24), for a given $\boldsymbol{v} = (x,t;x',t') \in \mathrm{V}$ $d = I(\boldsymbol{v})$ is a solution of the equation

$$h(d;x,t) = h(d;x',t'). \qquad (25)$$

It remains to show that the solution is unique. Assume the converse: let d' ($\neq d$) be another solution of (25):

$$h(d';x,t) = h(d';x',t'). \tag{26}$$

Let $x'' = q(d';x,t,t') \neq q(d;x,t,t') = x'$. Then, by (26) and (24), we have

$$h(d';x',t') = h(d';x,t) = h(d';q(d';x,t,t'),t') = h(d';x'',t').$$

This contradicts strict monotonicity of h with respect to the second argument. ■

It follows from the proof of Theorem 2 (see (23)) that h can be chosen such that $h > 0$, $h(d;\cdot,t)$ is continuous and strictly increasing, $h(\cdot;x,t)$ is continuous and strictly increasing if $t < 0$, strictly decreasing if $t > 0$, and $h(\cdot;x,0) = x$. Hence, we interpret $h(d;x,t)$ as the present value of the payment x at time t and discount rate d. An intuitive meaning of a solution $I(\boldsymbol{v})$ of Eq. (20) is a generalization of the internal rate of return: the discount rate at which the present value of costs equals the present value of the benefits. E.g., assuming $h(d;x,t) = e^{-d\phi(t)}x$, we obtain (2) (note: without loss of generality, we may assume $\phi(0) = 0$, since the transformation ϕ in (2) is defined up to a positive affine transformation; therefore, $h(\cdot;x,0) \equiv x$ holds).

Corollary 2. An axiomatization of (6).

If a weak order $(\mathrm{V}, \succeq)$ satisfies (i), (iii), (iv), and (viii), then there exists a function $f:\mathrm{R}^2 \to \mathrm{R}_+$ such that the map $I:\mathrm{V} \to \mathrm{R}$ that takes each $\boldsymbol{v} \in \mathrm{V}$ to a unique solution $I(\boldsymbol{v})$ of the equation

$$xf(I(\boldsymbol{v});t) = x'f(I(\boldsymbol{v});t')$$

represents $(\mathrm{V}, \succeq)$; here $f(\cdot;t)$ is continuous, strictly increasing if $t < 0$, and strictly decreasing if $t > 0$.

Proof.

Let I and h be the maps from the proof of Theorem 2. Applying Proposition 1 (a), we obtain that h is linearly homogeneous with respect to the second argument. Denote $f(d;t) = h(d;1,t)$, then $h(d;x,t) = xh(d;1,t) = xf(d;t)$. ■

The function f in Corollary 2 can be chosen so that $f(\cdot;0)=h(\cdot;1,0)\equiv 1$. Thus, we interpret $f(d;t)$ as the discount factor at time t and the discount rate d. E.g., if we take $f(d;t)=e^{-dt-\pi(t)}$, where $\pi(t)$ is the rate of inflation during the period $[0,t]$, then we get the real continuously compounded return:

$$I(\boldsymbol{v})=I_0(\boldsymbol{v})-\frac{\pi(t')-\pi(t)}{t'-t}.$$

Theorem 3. An axiomatization of (9).

If a weak order $(\mathrm{V},\succeq)$ satisfies (i), (ii), (iii), (vi), and (viii), then there exists a function $g:\mathrm{R}\times\mathrm{R}_+\to\mathrm{R}$ such that the map $I:\mathrm{V}\to\mathrm{R}$ that takes each $\boldsymbol{v}=(x,t;x',t')\in\mathrm{V}$ to a unique solution $I(\boldsymbol{v})$ of the equation

$$g(I(\boldsymbol{v});x)-\mathrm{sgn}(I(\boldsymbol{v}))\phi(t)=g(I(\boldsymbol{v});x')-\mathrm{sgn}(I(\boldsymbol{v}))\phi(t'), \qquad (27)$$

represents $(\mathrm{V},\succeq)$. g is continuous and strictly increasing with respect to the second argument.

In particular, if (vi) holds with the identity transformation ϕ, then (27) simplifies to (9) with $\delta(d)=\mathrm{sgn}(d)$.

Proof.

Let $(\mathrm{V},\succeq')$ be defined by (16) and let q and h be the maps from Theorem 2 being applied to $(\mathrm{V},\succeq')$. By (vi)' (see the proof of Proposition 1 (b)), we have

$$q(d;x,t,t')=q(d;x,t+\tau,t'+\tau) \text{ for any } \tau\in\mathrm{R}. \qquad (28)$$

For given $t<t'<t''$ put $\tau=t-t'$, $\tau'=t'-t''$. Then, by (22),

$$h(d;h(d;x,\tau),\tau')=q(d;q(d;x,t,t'),t',t'')=q(d;x,t,t'')=h(d;x,\tau+\tau'), \quad (\tau,\tau')\in\mathrm{R}_-^2. \qquad (29)$$

h is continuous and strictly increasing with respect to the second argument (Theorem 2); by (ii), $h(d;x,\cdot)$ is strictly increasing if $d<0$, strictly decreasing if $d>0$, and $h(0;x,\cdot)\equiv x$.

Eq. (29) is the translation functional equation. Under the formulated conditions, its general solution has the form (Aczel, 1966, p. 249):

$$g\big(d;h(d;x,\tau)\big)=g(d;x)+\tau,\ d<0, \qquad (30)$$

$h(0;x,\tau)=x$,

$g\left(d;h(d;x,\tau)\right)=g(d;x)-\tau$, $d>0$,

where $\tau\in\mathrm{R}_-$ and the function $g:\{\mathrm{R}\setminus\{0\}\}\times\mathrm{R}_+\to\mathrm{R}$ is continuous and strictly increasing with respect to the second argument. Using (28) and the definition of h, we conclude that (30) holds for any $\tau\in\mathrm{R}$.

Extend g to $\mathrm{R}\times\mathrm{R}_+$ by $g(0;x)=x$, then (30) takes the form of

$$g\left(d;h(d;x,t)\right)=g(d;x)-\mathrm{sgn}(d)t. \qquad (31)$$

The application of Theorem 2 yields Theorem 3. ■

Since $h(d;x,t)$ in (31) is interpreted as the present value of the payment x at time t and given the discount rate d, the intuitive meaning of the difference $g(d;x')-g(d;x)$ is a time interval τ such that the present value of the benefits x' at time τ and discount rate $d\neq 0$ equals x. E.g., let $g(d;x)=\begin{cases}|d|^{-1}\ln u(x), & d\neq 0\\ x, & d=0\end{cases}$ in Eq. (31), where u is a utility function that represents preferences at time 0, u is strictly increasing and $u(\mathrm{R}_+)=\mathrm{R}_+$. Then h takes the form of discounted utility $h(d;x,t)=e^{-dt}u(x)$.

Theorem 4. Axiomatizations of I_0 (1) and I_1 (2).

Let $(\mathrm{V},\succeq)$ be a weak order. The following statements are equivalent:

(a) $(\mathrm{V},\succeq)$ satisfies (i) (part (a)), (iii), (iv), (vi), and (viii);

(b) I_1 represents $(\mathrm{V},\succeq)$.

In particular, if (vi) holds with the identity transformation ϕ in (a), then I_0 represents $(\mathrm{V},\succeq)$.

Proof.

(a) $\Rightarrow$ (b). Let $(\mathrm{V},\succeq)$ satisfy (i) (part (a)), (iii), (iv), (vi), and (viii), and let $(\mathrm{V},\succeq')$ be defined by (16).

Applying Proposition 1 (a) and (vi)' (see the proof of Proposition 1 (b)), we get

$$(x,t;x',t') \sim' (1,0;x'/x,t'-t). \tag{32}$$

By (iii), (32) with $x'/x = x''/x' = \alpha$, $t'-t = t''-t' = \tau$ reduces to $(1,0;\alpha,\tau) \sim' (1,0;\alpha^2,2\tau)$. By induction,

$$(1,0;\alpha,\tau) \sim' (1,0;\alpha^n,n\tau) \text{ for every natural } n. \tag{33}$$

Similarly,

$$(1,0;\alpha,\tau) \sim' (1,0;\alpha^{1/m},\tau/m) \text{ for every natural } m. \tag{34}$$

Combining (33) and (34), we obtain

$$(1,0;\alpha,\tau) \sim' (1,0;\alpha^r,r\tau) \tag{35}$$

for any positive rational number r. From (viii) it follows that (35) holds for any $r \in \mathrm{R}_+$. Using (32) and (35) with $r = 1/\tau$, we obtain

$$(x,t;x',t') \sim' (1,0;x'/x,t'-t) \sim' \left(1,0;(x'/x)^{\frac{1}{t'-t}},1\right). \tag{36}$$

Now, (b) follows from (36) and (i).

(b) $\Rightarrow$ (a). Trivial. ■

As a consequence of Proposition 1, substitution of (iv) for (v) (as well as of (vi) for (ii) and (vii)) in Theorem 4 provides alternative axiomatizations of I_1 and I_0.

3 A comment on indices of return under risk

In this final section an index of return on a set of investment projects under risk is constructed. We restrict ourselves with the projects with random amounts being invested and gained; time of operations is assumed to be nonrandom.

Let $\tilde{\mathrm{R}}^2_+$ be the set of two-dimensional positive random vectors defined on the same probability space $\langle \mathrm{R}^2_+, \Box^2, \mathrm{P} \rangle$ with finite first moments. Let $\tilde{\mathrm{V}} = \{(\tilde{x},t;\tilde{x}',t') : (\tilde{x},\tilde{x}') \in \tilde{\mathrm{R}}^2_+, (t,t') \in \mathrm{R}^2, t < t'\}$; an element of the set $\tilde{\mathrm{V}}$ is denoted by $\tilde{\boldsymbol{v}} = (\tilde{x},t;\tilde{x}',t')$ and interpreted as a simple investments project under risk: investing a random amount $\tilde{x}$ at time t and gaining a random amount $\tilde{x}'$ at time t'. Define a weak order ($\tilde{\mathrm{V}}$,$\succeq_r$) such that if random vectors $(\tilde{x},\tilde{x}')$, $(\tilde{y},\tilde{y}')$ in $\tilde{\mathrm{R}}^2_+$ are equal in distribution, then $(\tilde{x},t;\tilde{x}',t') \sim_r (\tilde{y},t;\tilde{y}',t')$ (with ($\tilde{\mathrm{V}}$,$\sim_r$) defined as usual).

Adopting the idea of Vilenskii and Smolyak (1998), we show that under some natural assumptions $(\tilde{\mathrm{V}}, \succsim_r)$ reduces to a weak order on the set of non-random investment projects V with amounts invested and gained being equal to the expected values of the corresponding random variables.

Now we consider the following generalizations of axioms (iv), (v), and (viii) to the case of investment projects with random payments:

(iv)$_r$ if random vectors $(\tilde{x}, \tilde{x}')$ and $(\tilde{y}, \tilde{y}')$ are independent, then $(\tilde{x}, t; \tilde{x}', t') \sim_r (\tilde{y}, t; \tilde{y}', t')$ $\Rightarrow (\tilde{x}, t; \tilde{x}', t') \sim_r (\tilde{x} + \tilde{y}, t; \tilde{x}' + \tilde{y}', t')$;

(v)$_r$ $(\tilde{x}, t; \tilde{x}', t') \sim_r (\alpha\tilde{x}, t; \alpha\tilde{x}', t')$ for any $\alpha > 0$;

(viii)$_r$ if two sequence of random vectors $\{(\tilde{x}_n, \tilde{x}'_n)\}_{n=1}^{\infty}$ and $\{(\tilde{y}_n, \tilde{y}'_n)\}_{n=1}^{\infty}$ in $\tilde{\mathrm{R}}_+^2$ converge in distribution to random vectors $(\tilde{x}, \tilde{x}')$ and $(\tilde{y}, \tilde{y}')$ in $\tilde{\mathrm{R}}_+^2$, respectively, then $(\tilde{x}, t; \tilde{x}', t') \succsim_r (\tilde{y}, t; \tilde{y}', t')$ whenever $(\tilde{x}_n, t; \tilde{x}'_n, t') \succsim_r (\tilde{y}_n, t; \tilde{y}'_n, t')$ for all n.

Proposition 2.

If a weak order $(\tilde{\mathrm{V}}, \succsim_r)$ satisfies (iv)$_r$, (v)$_r$, and (viii)$_r$, then there exists a weak order $(\mathrm{V}, \succsim)$ such that

(a) $(\mathrm{V}, \succsim)$ satisfies (iv) and (v);

(b) $(\tilde{x}, t; \tilde{x}', t') \succsim_r (\tilde{y}, \tau; \tilde{y}', \tau') \Leftrightarrow (\mathbf{E}\tilde{x}, t; \mathbf{E}\tilde{x}', t') \succsim (\mathbf{E}\tilde{y}, \tau; \mathbf{E}\tilde{y}', \tau')$.

Proof.

For a fixed $\tilde{\boldsymbol{v}} = (\tilde{x}, t; \tilde{x}', t') \in \tilde{\mathrm{V}}$, define

$$\tilde{x}_n = \frac{1}{n}\sum_{k=1}^{n} \tilde{y}_k\,,\ \tilde{x}'_n = \frac{1}{n}\sum_{k=1}^{n} \tilde{y}'_k\,,\ \tilde{\boldsymbol{v}}_n = (\tilde{x}_n, t; \tilde{x}'_n, t'),\ n = 1, 2, \ldots,$$

where $(\tilde{y}_k, \tilde{y}'_k)$, $k = 1, 2, \ldots$ are independent copies of $(\tilde{x}, \tilde{x}')$. By the law of large numbers, the sequence $\{(\tilde{x}_n, \tilde{x}'_n)\}_{n=1}^{\infty}$ converges in probability (and, therefore, in distribution) to the constant vector $(\mathbf{E}\tilde{x}, \mathbf{E}\tilde{x}')$. By (iv)$_r$ and (v)$_r$, $\tilde{\boldsymbol{v}} \sim_r \tilde{\boldsymbol{v}}_n$ for every n. Now, Proposition 2 follows from (viii)$_r$. ■

Hence, under some natural assumptions the expected values of the corresponding random payments may be used as proxies for amounts being invested and gained to get reasonable generalizations of indices (1)–(3), and (6) in case of investment under risk.

Conclusion

This paper presents characterizations for the rate of return of a simple investment project and several generalizations of it. The characterizations are given on the bases of axioms (i)–(viii) that describe the main properties of a measure of return.

Concluding, we would like to note that various measures of return are often used to calculate the cost of borrowing. Since axioms (i)–(viii) still have economic interpretations if elements of the set V are interpreted as liabilities (getting an amount x at time t and returning an amount x' at time $t' > t$) and $(\mathrm{V}, \succeq)$ is interpreted as a relation of the cost of borrowing, then the obtained results also axiomatize (1)–(4), (6), (9), and (10) as the indices of the cost of borrowing.

We hope that obtained results can serve as an additional argument to justify the use of those indices for measuring profitability of investment projects and are able to define the limits of their applicability.

Acknowledgements

The authors are grateful to S.A. Smolyak and Y.V. Vymyatnina for valuable suggestions and corrections on earlier drafts of the manuscript.

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