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Abstract:

An axiomatic approach is developed to define the ”proportional excess” on the space of positively generated NTU games. This excess generalizes to NTU games the proportional TU excess v(S)/x(S). Five axioms are proposed, and it is shown that the proportional excess, which possess Kalai’s properties except the boundary condition (it equals 1, rather than 0), is the unique excess function satisfying the axioms. The properties of proportional excess and related solutions are studied. In particular, for the proportional (pre)nucleolus a geometric characterization, which modifies the Maschler-Peleg-Shapley geometric characterization of the standard TU nucleolus, is given.

Keywords: cooperative NTU games, excess function, nucleolus, prenucleolus, (Minkowski) gauge function

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On Proportional Excess for NTU games

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1. Introduction

It is well-known that the generalization of ordinal concepts like "domination", "objection" and "counter-objection" to games without side payments (NTU games) is straightforward. However, the generalization of concepts like the Shapley value, the nucleolus and the kernel based on cardinal notion of excess creates difficulties. In his survey M.Maschler (1992) noted that "research concerning the extension of the kernel and the nucleolus to games without side payments is still scarce, ... the main issue is to decide what the analogue of the excess function should be". Although, there have been several suggestions (see, for example, Kalai (1972, 1973, 1975), Billera (1972), Vilkov (1974), Nakayama (1983), Bondareva (1979), McLean and Postlewaite(1989), Christensen at al (1996) …), these proposals have not yet achieved the status of a general theory similar to the one that exists for the side payment case.

The concept of an excess function for the class of NTU games is very useful (Kalai (1975)): first it is a concept which is natural and interesting on its own merits; second, it can be used to characterize, or redefine the core of a game; and third, excess function allow one to define the ε-core, the kernel and the nucleolus.

E.Kalai (1975) defined a family of excess functions for cooperative NTU games. Using these excess functions he defined the ε-core, the kernel and the nucleolus of a NTU game in a way that preserves a significant portion of the structure that these concepts exhibit in the sidepayment case. These excess functions satisfy some natural conditions, which seem to be required for such functions. Unfortunately, nothing consoling can be said about the uniqueness of the excess function.

At the last time there is a growing interest to the proportional “excess” defined for any positive TU game u (i.e. $u(S) > 0$ for every coalition S) by formula

$$h_S(u,x) = \frac{u(S)}{x(S)} \ .$$

Our goal is to generalize this to NTU games[1]. The choice of the particular excess is discussed in Section 3.

We restrict our attention to the space CG_+ of all positively generated (or "normally-generated") NTU games: roughly (the formal definition will be given in Section 2), a game V belongs to CG_+ if any set $V(S)$ is compactly generated, contains $\boldsymbol{O}^S$, where $\boldsymbol{O} = (0, \dots , 0)$, as its relatively interior point and coincides with the comprehensive hull of its "positive part" $V_+(S) = V(S) \cap \mathbf{R}^S_+$.

To define a corresponding NTU excess we impose five axioms (continuity axiom, scale independence axiom, “minimum” and “maximum” axioms and TU game axiom) which describe desirable properties of an excess function. Continuity axiom asserts that the excess (of a coalition) should be continuous both in x and V. Scale independence asserts that excess does not depend on linear transformations of the game and payoff vector. MIN and MAX axioms state that the excess in the “intersection game” $V = V_1 \cap V_2$ and in the “union game” $V = V_1 \cup V_2$ should be equal to the minimum and to the maximum of two component games excesses, respectively. TU game axiom asserts that in the TU case the excess should coincide with the relative excess.

These five axioms uniquely define *the proportional excess* $h_S(V,x)$ defined by formula

$$h_S(V, x) = 1 / \gamma(V(S), x) \ ,$$

where $\gamma(W, .)$ is the *gauge* (or the *Minkowski gauge function*) of a set W:

$$\gamma(W, x) = \inf \{\lambda>0 : x \in \lambda W \}.$$

We study the properties of proportional excess and corresponding proportional nucleolus and prenucleolus. (As we mentioned above $u(S)/x(S)$ is not an excess function in Kalai’s sense. Though we can

[1] Though proportional “excess” is not an excess in Kalai’s sense - it equals 1 on the boundary of the game subset (cf. (2.3) below) - we use the term excess, too.

consider some modifications of the excess like $\frac{u(S)}{x(S)}-1$ or log ($u(S)/x(S)$) or something else, but they are equivalent from the "nucleolus point of view"). In particular, we modify the Maschler-Peleg-Shapley geometric characterization of the nucleolus in the side payment case (Maschler, Peleg and Shapley (1979)) for NTU games.

The paper is organized as follows. Section 2 provides definitions and notations. Section 3 presents the corresponding motivations. In Section 4 we describe the axioms and prove the existence and uniqueness theorem. Section 5 studies the property of the proportional excess function and the properties of corresponding nucleolus and prenucleolus. Appendix contains some properties of star-shaped sets and the (Minkowski) gauge function.

2. Definitions and Notations

Let I be a non-empty finite set of players. A *coalition* is a non-empty subset of I. A *payoff vector* for I is a vector $x \in \mathbf{R}^I$. For $z \in \mathbf{R}^I$ and $S \subset I$, z^S will denote the projection of z on $\mathbf{R}^S = \{x \in \mathbf{R}^I : x_i = 0 \text{ for } i \notin S\}$.

Let $x,y \in \mathbf{R}^I$. Write $x \geq y$ if $x_i \geq y_i$ for all $i \in I$; $x > y$ if $x_i > y_i$ for all $i \in I$. Denote

$$\mathbf{R}^I_+ = \{ x \in \mathbf{R}^I : x \geq \boldsymbol{O} \},\ \mathbf{R}^I_{++} = \{ x \in \mathbf{R}^I : x > \boldsymbol{O} \},$$

where $\boldsymbol{O} = (0, 0, \dots, 0)$. We will also write xy or (x, y) for the inner product $\sum_{i \in I} x_i y_i$. We denote the coordinate-wise product by $x*y$, i.e.

$$x*y = (x_1y_1, \dots, x_ny_n).$$

Let $A \subset \mathbf{R}^I$. If $x \in \mathbf{R}^I$ then $x + A = \{ x + a : a \in A \}$ and $\lambda A = \{ \lambda a : a \in A \}$. A is *comprehensive* if $x \in \mathbf{R}^I$ and $x \geq y$ imply $y \in \mathbf{R}^I$. A is *bounded above* if $A \cap (x + \mathbf{R}^I_+)$ is bounded for every $x \in \mathbf{R}^I$. The boundary of A is denoted by ∂A.

A *cooperative game without side payments* (or shortly *NTU game*) is a pair (I,V), where $I=\{1, 2,..., n\}$ is a set of players, and V is a set-valued map that assigns to each coalition $S \subset I$ a set $V(S)$ that satisfies:

(1) $V(S) \subset \mathbf{R}^S = \{ x \in \mathbf{R}^I : x_i = 0 \text{ for } i \notin S \}$;

(2) $V(S)$ is closed, non-empty, comprehensive and bounded above.

Let us define the game space CG_+ . A NTU game $V \in CG_+$ iff for every S

(a) $V(S)$ is positively generated, i.e. $V(S) = (V(S) \cap \mathbf{R}^S{}_+) - \mathbf{R}^S{}_+$ and $V_+(S) = V(S) \cap \mathbf{R}^S{}_+$ is a compact set, and every ray $L_x = \{ \lambda x : \lambda \geq 0 \}$, $x \neq \boldsymbol{O}$ does not intersect the boundary of $V(S)$ more than once.

(b) $\boldsymbol{O}$ is an interior point of the set $V^\wedge(S)=V(S) + \mathbf{R}^{I\setminus S}$.

A set $V(S) \subset \mathbf{R}^S$ will be called a *game subset* if it satisfies (1) and (2). The space consisting of all game subsets of $\mathbf{R}^S$ satisfying (a) and (b) will be denoted by $CG^S{}_+$. Clearly, every $V \in CG_+$ is a game in Kalai's sense (cf. Kalai (1975)).

It is clear that every $U \in CG^S{}_+$ is *star-shaped* (cf. Rubinov and Yagubov (1986)), i.e. U is closed, it contains $\boldsymbol{O}$ as an interior point (in $\mathbf{R}^S$), and every ray L_x does not intersect the boundary ∂U of U more than once. (This definition is stronger than the usual one: a star-shaped subset of a real vector space contains a distinguished member, the center, which can be connected with every other element by a line segment which is completely contained in the set). Note also that for every $V(S) \in CG^S{}_+$ and every $x \in \mathbf{R}^S{}_+$, $x \neq \boldsymbol{O}$, there is a unique $\lambda > 0$ such that $\lambda x \in \partial V(S)$.

It is clear that if $V_1, V_2 \in CG_+$ then the games $V_1 \cap V_2$ and $V_1 \cup V_2$ defined by

$(V_1 \cap V_2)(S) = V_1 (S) \cap V_2 (S)$ and $(V_1 \cup V_2)(S) = V_1 (S) \cup V_2 (S)$

also belong to CG_+ .

Next, if $V(S) \in CG^S{}_+$, $A \in \mathbf{R}^S{}_{++}$ and $A * V(S) = \{A * y : y \in V(S) \}$, then $A * V(S) \in CG^S{}_+$, too.

Remark 2.1. It should be noted that, given our goal, instead of CG_+ another space $CG^1{}_+$ can be defined in a slightly different way: the second condition in (a) can be replaced by a more traditional condition known as the non-levelness condition:

$$\text{if } x, y \in \partial V(S) \cap \mathbf{R}^S{}_+ \text{ and } x \geq y \text{, then } x = y \text{,}$$

which says that $\partial V(S) \cap \mathbf{R}^S{}_+$ has no “level” segments, i.e. segments parallel to a coordinate hyperplane; it is a familiar regularity condition in game theory (see, for example, Aumann (1985)).

Two particular cases should be mentioned.

"Sidepayments game". Let v be a positive sidepayments game, i.e. $v(S) > 0$ for every S. Then the corresponding "sidepayments game" $V \in CG_+$ can be defined by

$$V(S) = \{x \in \mathbf{R}^S_+ : x(S) \leq v(S)\} - \mathbf{R}^S_+ .$$

"Hyperplane game". Let V be a hyperplane game (see, for example, Maschler and Owen (1989)), i.e. for every $S \subset I$

$$V(S) = \left\{ x \in R^S : \sum_{i \in S} p_i^{(S)} x_i \leq r_S \right\},$$

where $p^{(S)}_i > 0$ for all $i \in S$, $S \subset I$. Then the corresponding "hyperplane game" $V_1 \in CG_+$ can be defined by

$$V_1(S) = \left\{ x \in R^S_+ : \sum_{i \in S} p_i^{(S)} x_i \leq r_S \right\} - \mathbf{R}^S_+ .$$

We will omit quotation marks in what follows (in both cases).

Let us recall some definitions from Kalai's paper (1975), adopting them to the case under consideration.

A function $E_S : CG_+ \times \mathbf{R}^I \to \mathbf{R}$ will be called *excess function* for the coalition S if it satisfies the following conditions.

(A) If $x, y \in \mathbf{R}^I$ and $x_i = y_i$ for every $i \in S$, then for every game V

$$E_S(V, x) = E_S(V, y) . \tag{2.1}$$

(B) If $x, y \in \mathbf{R}^I$ such that $x_i < y_i$ for every $i \in S$, then for every game V

$$E_S(V, x) > E_S(V, y) . \tag{2.2}$$

(C) For every game V, if

$$x \in \partial V(S) \text{ then } E_S(V, x) = 0 . \tag{2.3}$$

(D) $E_S(V, x)$ is continuous jointly in x and V.

The metric on CG^S_+ is the Hausdorff metric H_S, and for $V,W \in CG_+$,

$$\rho(V,W) = \max_S H_S(V(S), W(S)). \qquad (2.4)$$

(Recall that Hausdorff metric is defined as follows (see, for example, Castaing and Valadier (1977). Let A and B be subsets of $\mathbf{R}^S$, then

$H_S(A,B)$ = max $(e(A,B), e(B,A))$, where

$e(A,B)$ = sup { $d(x,B)$: $x \in A$}, and d is Euclidean metric).

E_S will be said to be *independent of other coalitions* if for every two games V and W such that $V(S) = W(S)$, and for any $x \in \mathbf{R}^I$,

$E_S(V, x) = E_S(W, x)$.

We restrict our attention to nonnegative vectors x only, since any reasonable solution of a game $V \in CG_+$ should be, clearly, nonnegative. We consider only those excesses which are independent of other coalitions and therefore, taking into account property (a) we will consider in what follows excess functions as those functions on $CG^S_+ \times \mathbf{R}^S_+$ (or $CG^S_+ \times \mathbf{R}^I_+$).

The following notations will be used:

$IR(V)$ = { $x \in V(I)$: $\forall\, i \in I$ $x_i \geq y_i$ for any $y \in V(i)$ } - the set of individually rational points of a game V;

GR(V) = { $x \in V(I)$: there is no $y \in V(I)$ for which $y > x$ } - the set of Pareto optimal points of a game V;

$C(V)$ = { $x \in V(I)$: there is no S , $y \in V(S)$ such that $y_i > x_i$ $\forall\, i \in S$ } - the core of a game V;

$C^s(V)$ = { $x \in GR(V)$: there is no S , $y \in V(S)$ such that $y_i \geq x_i$ $\forall\, i \in S$ } - the strict core of a game V.

For any real number ε,

$C_\varepsilon(V)$ = { $x \in GR(V) \cap IR(V)$: $\forall\, S \neq I$, $E_S(V, x) \leq \varepsilon$ }

is the ε-core of a game V. Of course, the ε-core depends on the specification of the excess functions, but we omit this in notation.

Finally, we recall the definitions of the nucleolus and the prenucleolus of a game (cf., for example, Kalai (1975), Maschler (1992)). Let $\{E_S\}_S$ be a fixed family of excess functions, and let X be a closed subset of $\mathbf{R}^I$. For each $x \in X$ and game V we define a vector $\theta(x)$ to be

$$\theta(x) = (E_{S_1}(V,x),\ldots,E_{S_{2^n}}(V,x)),$$

where various excesses of all coalitions are arranged in decreasing (nonincreasing) order. The components of $\theta(x)$ are well defined and vary continuously for "good" excess functions. We say that $\theta(x)$ is lexicographically smaller than $\theta(y)$, $\theta(x) \prec_{lex} \theta(y)$, if there is a positive integer q such that $\theta_i(x) = \theta_i(y)$ whenever $i < q$ and $\theta_q(x) < \theta_q(y)$.

The *nucleolus of* V (with respect to X and given family of excess functions $\{E_S\}_S$) - denoted by $N(X,V)$ - is the set of vectors in X whose θ's are lexicographically least, i.e.

$$N(X,V) = \{x \in X : \theta(x) \preceq_{lex} \theta(y) \quad \text{for all} \quad y \in X\}.$$

If $X = IR(V) \cap GR(V)$ then $N(X,V) := N(V)$ is called the *nucleolus of a game V*. If $X = GR(V)$ than $N(X,V) := PN(V)$ is called the *prenucleolus of a game* V.

3. Some Comments

As we have mentioned above, our aim is to generalize the proportional excess to NTU games and to study the properties of corresponding nucleolus and prenucleolus. But why do we focus on this excess?

First of all note that the proportional excess is ordinally equivalent to the relative excess $\dfrac{v(S)-x(S)}{v(S)}$: both functions *(v(S) – x(S))/v(S)* and *v(S)/x(S)* define the same ordering on the set of corresponding values, i.e.

$$\frac{v(S_1)-x(S_1)}{v(S_1)} \geq \frac{v(S_2)-x(S_2)}{v(S_2)} \quad \Leftrightarrow \quad \frac{v(S_1)}{x(S_1)} \geq \frac{v(S_2)}{x(S_2)} \text{ and} \tag{3.1}$$

$$\frac{v(S)-x(S)}{v(S)} \geq \frac{v(S)-y(S)}{v(S)} \quad \Leftrightarrow \quad \frac{v(S)}{x(S)} \geq \frac{v(S)}{y(S)}. \tag{3.2}$$

These excesses make good sense in economic environment, when the players are e.g. companies. If an excess at *x* is measure of dissatisfaction of a coalition of companies from *x* (satisfaction, if negative) it is reasonable to assume that a rich coalition will "tolerate" a large loss and a poor coalition will not "tolerate" a much smaller loss. Thus a nucleolus based on the above, and similar excess function, is a reasonable solution concept. If a poor person looses, say, $1000 he will protest strongly, whereas a large conglomerate may not bother to even rise the issue. In such cases the fact that the excess is not invariant under translation is even a merit[2].

Let us turn to the proportional value of a TU game which has been studied in a series of papers. Ortmann (2000) defined the proportional solution of a positive cooperative game using the concept of "proportional division of surplus". Yanovskaya (2001) showed that the proportional value can be defined by means of excess $\dfrac{v(S)}{x(S)}$ in the following sense: proportional value is the solution of the minimization problem on the set of positive imputations of some function, which is similar to that used for definition of the least square values, but in which the standard excess

v(S) – x(S) is replaced by some function depending exactly on $\dfrac{v(S)}{x(S)}$. The proportional excess $\dfrac{v(S)}{x(S)}$ was used by Fiagle at al (1998) to compute the

[2] I indebted Michael Maschler for this interpretation.

“nucleon” of cooperative game (i.e. the nucleolus corresponding to the proportional excess).

Feldman (1999) presents a theory of bargaining and value allocation in cooperative games based on the principle of equal proportional gain. It is in contrast to standard value theory presented by the Shapley value, which embodies the principle of equal gain. In a situation where two risk-neutral players are bargaining over the division of the proceeds of cooperation, a sum of money, the standard result is that they will split the surplus equally. The proportional value, in contrast, would have them split the surplus so each gains in equal proportion to that which could be obtained by each alone.

Proportional allocation is not a new idea. Young (1994) writes that “proportionality is deeply rooted in law and custom as a norm of distributive justice”. Thompson (1998) puts proportionality “at the heart of equity theory”. It is the commonly recognized standard of business practice.

Equal and proportional allocation are the two basic principles of distributive justice. There is, however, only very small game theoretic literature related to proportionality, and none study equal proportional gain (cf. Feldman (1999)). The relevant results on proportionality are, instead, found in the accounting and social choice literature. One reason for this omission is proportional gain outcomes change with the choice of origin of player’s utility scale, and are thus not translation covariant. This has been believed to be an unacceptable property, as it is customary to consider that translation of player’s utility scale, changing its origin, should not change outcomes. In the proportional approach, however, essential information is distorted by this process.

The following example (Lemaire (1991), cf. also Feldman (1999)) shows that the proportional approach provides a rational and practical solution to an important class of problems which are poorly served by the Shapley value.

Example 3.1. Three players have 1.8 million, 900 thousand, and 300 thousand Belgian Francs to invest. The interest rate on sums less than 1 million is 7.75 %, then for sums less than 3 million the interest rate is

10.25 %, and for sums of 3 million or grater it is12 %. If players pool their funds, they will receive the 12 % rate. Lemaire suggests that the first player should “be entitled to a higher rate on the grounds that she can achieve a yield of 10.25 % on her own, and the others only 7.75 %” (Lemaire (1991: 19)). Lemaire then construct the cooperative game (assuming 3 months of simple interest) for this problem, and illustrates different solution concepts. He shows the Shapley value of this game, expressed as a vector (51.750, 25.875, 12.375). Lemaire also reports these payoffs in annualized rates of interest, (11.5 %, 11.5 %, 16.5 %), and comments that the “allocation is much too generous” to the third player, “who takes great advantage” because he is essential to achieve the highest interest rate. Lemaire shows that the nucleolus generates results close to that of Shapley value and concludes that these solutions, “defined in an additive way, fail in this multiplicative problem” (Lemaire (1991: 37)).

The proportional value of this game is (55.022, 26.552, 8.425), and it gives returns of (12.2 %, 11.8 %, 11.2 %): an outcome consistent with the intention that larger investors should not receive smaller rates of return in the problem.

Lemaire (1991) describes a “proportional nucleolus”, where the excess of a coalition is defined in relative terms:

$$E_S(V, x) = \frac{v(S) - x(S)}{v(S)}.$$

Lemaire reports that the proportional nucleolus gives all players in Example 3.1 a 12 % return, which he writes, is the “common practice” in such situations.

In this paper we study the proportional excess. The generalization of relative excess to NTU games - the “gauge” excess function which is defined by $g_S(V,x) = 1 - \gamma(V(S), x)$ - was studied by the author in Pechersky (1997, 2000).

Note also that the appearance of Minkowski gauge function is not altogether surprising, since it (or some its relatives) has had several applications, though not in game theory but, for example, in economic theory: refer, for instance, to the Farrell Measure of Technical Efficiency and some other production-theoretic indexes (see, for example, Fisher and Shell (1998)), or the Input Distance Function (see, for example, Färe (1988))[3].

4. Proportional Excess

As we noted at the beginning, our aim is to generalize the proportional excess to NTU games. Let H_S be an excess function, i.e.

$H_S : CG^S{}_+ \times \mathbf{R}^S{}_+ \to \mathbf{R}$. We impose following axioms (we write V instead of $V(S)$).

A1 (Continuity). $H_S(V, x)$ is continuous jointly in V and $x \neq \boldsymbol{O}$.

A2 (Scale Independence). If $V \in CG^S{}_+$, $A \in \mathbf{R}^S{}_{++}$ and $A*V = \{A*y : y \in V\}$, then

$$H_S(A*V, A*x) = H_S(V, x) .$$

A3 (MIN). Let $V_1, V_2 \in CG^S{}_+$, then

$$H_S(V_1 \cap V_2 , x) = \min \{H_S(V_1 , x), H_S(V_2 , x)\}.$$

A4 (MAX). Let $V_1, V_2 \in CG^S{}_+$, then

$$H_S(V_1 \cup V_2, x) = \max \{H_S(V_1 , x), H_S(V_2 , x)\}.$$

A5 (TU games). Let $V \in CG_+$ corresponds to a positive TU game v , i.e.

$$V(S) = \{x \in \mathbf{R}^S{}_+ : x(S) \leq v(S)\} - \mathbf{R}^S{}_+ ,$$

then $H_S(V, x) = \dfrac{v(S)}{x(S)}$.

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[3] I am grateful to H.Keiding for this observation.

Continuity Axiom is self-explanatory, and there is no need to comment on it. Scale Independence seems also to be clear, but the absence of translation invariance should be stressed. Note that in the TU case the translation invariance can be justified by lack of “income effect” (see Section 3). However, in the NTU case, where the “income effect” can be of great importance, translation invariance seems to be not justified.

MIN and MAX axioms A3 and A4 seem to be natural conditions. The interpretation of games $V_1 \cap V_2$ and $V_1 \cup V_2$ is evident: corresponding set $V_1 \cap V_2$ represents the payoffs vectors feasible for coalition S in *both* games V_1 and V_2, and $V_1 \cup V_2$ is the set of payoffs vectors feasible for coalition S for *at least in one* of two given games V_1 and V_2. (Note that these two properties are trivially fulfilled in (classical) TU case for all excesses – standard $v(S) - x(S)$, relative $(v(S) - x(S))/v(S)$, and proportional “excess” $v(S)/x(S)$).

Finally, axiom A5 is the reformulation of our aim to generalize the TU proportional excess to NTU games.

Let $V \in CG_+$ be an arbitrary game. Define a function

$h_S : CG^S_+ \times \mathbf{R}^I_+ \to \mathbf{R}$ by

$$h_S(V, x) = 1/\gamma(V(S), x^S), \tag{4.1}$$

where $\gamma(W, y) = \inf \{\lambda > 0 : y \in \lambda W\}$ is the *gauge* (or the *Minkowski gauge function*) of a set W (cf. Rockafellar (1970)).

Theorem 4.1. There is a unique excess function

$$H_S : CG^S_+ \times \mathbf{R}^I_+ \to \mathbf{R}$$

satisfying axioms A1-A5, and $H_S = h_S$, where h_S is defined by (4.1).

We will prove the Theorem for $S = I$ and omit index I for simplification of notation. The proof for an arbitrary S is the same. We prove firstly some Lemmas.

Let $z \in \mathbf{R}^I_{++}$ and $\{ y \in \mathbf{R}^I : y \le z \}$, i.e. $P_z = z - \mathbf{R}^I_+$. Clearly P_z is star-shaped. Also star-shaped is every finite union of such sets, i.e. for every natural number M and any arbitrary vectors $z^m \in \mathbf{R}^I_{++}$, $m = 1,2, \dots , M$ the set $P(z^1, \dots , z^M)$ defined by

$$P(z^1,..., z^M) = \bigcup_{m=1}^{M} P_{z^m}$$

is star-shaped. Obviously $P(z^1, \dots , z^M) \in CG^I_+$, since $z^m > \boldsymbol{O}$ for every m.

Lemma 4.1. Let $V \in CG^I_+$ be an arbitrary game subset. Then for every $\varepsilon > 0$ there exist a natural number M and points $z^m \in \mathbf{R}^I_{++}$, $m = 1,2, \dots , M$ such that

1) $P(z^1, \dots , z^M) \supset V$;

2) $\rho\,(P(z^1, \dots , z^M), V) \le \varepsilon\, n^{1/2}$.

Proof. Clearly, since V is positively generated it is sufficient to consider only the "positive parts" of V and $P(z^1, \dots , z^M)$. Consider the covering of V_+ by the cubes with the edge equal to ε and vertices in the nodes of the ε-lattice in $\mathbf{R}^I$. Since V_+ is compact, it is contained in a finite union of such cubes, each having non-empty intersection with V_+. Denote the number of such cubes by M, and let $z^1, \dots , z^M$ be the maximum vertices of these cubes (i.e. vertices with coordinate-wise maximum).

Consider the set $P(z^1,..., z^M) = \bigcup_{m=1}^{M} P_{z^m}$. By construction,

$$P(z^1, \dots , z^M) \supset V .$$

Moreover, the following inclusion holds also by construction:

$$V + (\varepsilon\, n^{1/2})B \supset P(z^1, \dots , z^M) ,$$

where B is the unit ball in $\mathbf{R}^I$. Hence, $\rho\,(P(z^1, \dots , z^M), V) \le \varepsilon\, n^{1/2}$. □

Lemma 4.2. Let $V \in CG^I_+$ be a hyperplane game subset, i.e.

$$V = \left\{ z \in R^I_+ : \sum_{i \in I} p_i z_i \le r \right\} - \mathbf{R}^I_+$$

for some $p_i > 0,\ i = 1, \ldots, n$, then $H(V, x) = \dfrac{r}{\sum_{i \in I} p_i x_i}$ and it coincides with $g(V, x)$.

Proof. Let $A = (p_1, \ldots, p_n) \in \mathbf{R}^I_{++}$. Then A transforms the game subset V into the TU game subset

$$A{*}V = \{w \in \mathbf{R}^I_{+} : \sum_{i \in I} w_i \le r\} - \mathbf{R}^I_{+},$$

where $w_i = p_i z_i$. We have, by A5:

$$G(A*V, A*x) = \frac{r}{\sum w_i}.$$

By A4 , $G(V, x) = G(A{*}V, A{*}x) = r / \sum p_i x_i$.

It is easy to check that $\gamma(V, x) = \dfrac{\sum p_i x_i}{r}$. Indeed,

$$\inf\{\lambda > 0 : x \in \lambda V\} = \inf\{\lambda > 0 : \sum p_i x_i \le \lambda r\} =$$

$$= \inf\{\lambda > 0 : \sum p_i x_i / r \le \lambda\} = \sum p_i x_i / r .$$

Hence $H(V, x) = 1/\gamma(V, x) = h(V, x)$. □

Lemma 4.3. Let $z \in \mathbf{R}^I_{++}$. Then $H(P_z, x) = \min_i\{\dfrac{z_i}{x_i}\}$, and it coincides with $h(P_z, x)$.

Proof. Clearly, the faces of P_z are the sets

$$Q_i = \{ y \in \mathbf{R}^I : ye^i = ze^i,\ y \le z \},\ i = 1,2, \ldots, n,$$

where $e^i_j = \begin{cases} 1, & j = i, \\ 0, & j \ne i. \end{cases}$

Consider for some natural k a hyperplane game subset

$$V^i = \{y \in R^I_+ : y(e^i + \frac{1}{k} e^{I \setminus i}) \le z(e^i + \frac{1}{k} e^{I \setminus i})\} - \mathbf{R}^I{}_+ .$$

Then by Lemma 4.2 we have

$$H(V^i, x) = \frac{z_i + \frac{1}{k} z(I \setminus i)}{x_i + \frac{1}{k} x(I \setminus i)} .$$

Next consider a game V_k defined by $V_k = \bigcap_{i=1}^{n} V^i$. Clearly, $V_k \in CG^I{}_+$ and we have by A2

$$H(V_k, x) = \min_i \{\frac{z_i + \frac{1}{k} z(I \setminus i)}{x_i + \frac{1}{k} x(I \setminus i)}\}.$$

Besides, $H(V_k, x) = 1/ \gamma(V_k , x)$. Letting $k \to +\infty$ we get by A1

$$H(P_z, x) = \min_i \{\frac{z_i}{x_i}\} = 1 / \gamma (P_z, x). \qquad \square$$

Lemma 4.4. Let $z^1, \dots , z^M \in \mathbf{R}^I{}_{++}$, then

$$H(P(z^1, \dots , z^M), x) = \max_{m=1,\dots,M} \min_{i=1,\dots,n} (\frac{z_i^m}{x_i}).$$

Besides, $H(P(z^1, \dots , z^M), x) = 1/\gamma (P(z^1, \dots , z^M), x)$.

Proof. By A4

$$H(P(z^1, \dots , z^M), x) = \max_{m=1,\dots,M} H(P_{z^m}, x) = \max_{m=1,\dots,M} \min_i (\frac{z_i^m}{x_i}) .$$

Since

$$\gamma(V \cap V', x) = \max (\gamma(V, x), \gamma(V', x)) , \text{ and}$$

$\gamma(V \cup V', x) = \min (\gamma(V, x), \gamma(V', x))$

for any $V, V' \in CG^I_+$ (see Appendix) we have, by Lemma 4.3

$$H(P(z^1, \dots , z^M), x) = 1/ \gamma (P(z^1, \dots , z^M), x). \qquad \square$$

Proof of Theorem 4.1. The Theorem follows immediately from Lemmas 4.1 – 4.4 and continuity of H and h. □

We call this excess *proportional excess*, too (though, as we mentioned above, it is not an excess in Kalai's sense).

Now we consider some property of the proportional excess which seems to be interesting. Let us define an operation $\oplus$ on CG^S_+. For $V, V' \in CG^S_+$ define the set $V \oplus V'$ as follows:

$$V \oplus V' = cl \bigcup_{\alpha \in [0,1]} (\alpha V \cap (1-\alpha)V') \ ,$$

(it is assumed $0 \cdot V = \bigcap_{\alpha > 0} \alpha V$).

Proposition 4.1. Let $V, V' \in CG^S_+$, then $V \oplus V' \in CG^S_+$, and

$$\gamma(V \oplus V', x) = \gamma(V, x) + \gamma(V', x).$$

Proof. See Appendix.

The set $V \oplus V'$ is known as the *inverse sum* of star-shaped sets V and V' (see Rubinov and Yagubov (1986)). (It follows from the definition of operation $\oplus$ that the set CG^S_+ can be considered as a semilinear space with respect to "summation" $\oplus$ and the product $\bullet$ of set V and number $\lambda > 0$, defined by formula $\lambda \bullet V = (1/\lambda)V$ ($\lambda \bullet V = \mathbf{R}^S_+$ for $\lambda = 0$)).

The following Corollary is obvious.

Corollary 4.1. Let $V, V' \in CG^S_+$, then

$$\frac{1}{h_S(V \oplus V')} = \frac{1}{h_S(V, x)} + \frac{1}{h_S(V', x)} \ .$$

Remark 4.2. The same reasoning can be used to generalize the relative excess *(v(S)-x(S))/v(S)* (cf. Pechersky (2001)). The resulting excess is the gauge excess $g_S(V,x) = 1 - \gamma(V(S), x)$ studied in Pechersky (2000), where another approach was used. For that excess the corresponding analogues of Corollary 4.1 looks slightly differently:

$$g_S(\lambda\bullet V \oplus \lambda'\bullet V', x) = \lambda g_S(V, x) + \lambda' g_S(V', x) \qquad (4.2)$$

for any λ, $\lambda' > 0$, such that $\lambda + \lambda' = 1$ .The corresponding scheme was based in a sense on idea underlying the definition of expected utility function: a complete, transitive and continuous preference relation on the set of all pairs "game subset - payoff vector" was defined; further, using (4.2) more structure on the corresponding valuation function was imposed. The valuation function obtained was exactly the gauge excess function.

5. Properties of Gauge Excess

Though the proportional excess is not an excess function in Kalai's sense the following theorem holds (it is nothing else but simple reformulation of some assertions of Theorem 1 in Kalai (1975)).

Theorem 5.1. Let $\{ h_S \}_S$ be a family of proportional excesses and $V \in CG_+$. Then

1) $C_0(V) = C(V)$ and $C^s(V) = \{ x \in GR(V): \quad h_S(V, x) < 0 \quad \forall S \neq I \}$.

2) For every game $V \in CG_+$ there is a real number ε such that

$C_\varepsilon(V) \neq \varnothing$.

3) The kernel $K(V) \neq \varnothing$ for any game V .

4) $N(V) \subset K(V)$ for every game V .

5) For every game V , $N(V) \subset C_\varepsilon(V)$ whenever $C_\varepsilon(V) \neq \varnothing$.

6) For every game V , $N(V) \neq \varnothing$ and consists of a finite number of points.

7) For every game V , $PN(V) \neq \varnothing$ and consists of a finite number of points.

Proof. Since $h_S - 1$ is an excess function in Kalai's sense, his proof can be applied directly. □

Proposition 5.1. Let $V, V' \in CG_+$ be such NTU games that $V(I) = V'(I)$ and $V'(S) = aV(S), \forall S \neq I$ for some $a > 0$. Then

$$x \in PN(V) \Leftrightarrow x \in PN(V')$$

where $PN(V)$ is the prenucleolus of a game V.

Proof. Let us consider an arbitrary coalition $S \subset I$ and the corresponding set $V(S)$. It is clear that

$$\gamma(V'(S), x) = \gamma(aV(S), x) = \inf \{\lambda > 0 : x \in \lambda aV(S)\} =$$

$$= \inf \{\mu/a > 0 : x \in \mu V(S)\} = (1/a)\gamma(V(S), x) .$$

Then $h_S(V'(S), x) = 1/\gamma(V'(S), x) = a/\gamma(V(S), x)$. Hence,

$$h_S(V', x) = a\, h_S(V, x) \text{ and } \theta(V', x) = a(\theta(V, x)),$$

which yields the assertion. □

Note that this property is analogous to one of the characteristic property of the prenucleolus in the sidepayments case for classical excess function (see Sobolev (1975), Pechersky and Sobolev (1983, 1995)): Let v and v' be two sidepayment game such that $v(I) = v'(I)$, and for an arbitrary real number a $v'(S) = v(S) + a$ for every $S \neq I$. Then $PN(v') = PN(v)$.

In the proof of the Proposition we have used the property

$$\gamma(aV(S), x) = (1/a)\gamma(V(S), x) . \quad (5.1)$$

This property is important and we will use it to modify Maschler-Peleg-Shapley's geometric characterization of the nucleolus in side payment case (see Maschler, Peleg and Shapley (1979)) for NTU games. We start with the definition of lexicographic centers of a NTU game in Maschler-Peleg-Shapley spirit. Of course, it is more cumbersome, but the intuition behind it is transparent and very close to that by Maschler-Peleg-Shapley.

Let V be a game in CG_+ , and let $h(S, x) \equiv h_S(V, x)$. Let us define $X^0 = IR(V) \cap GR(V)$ (we suppose it is not empty) and

$$\Sigma^0 = \{S \subset I : S \neq I, \emptyset\}.$$

$$\varepsilon^1 = \min_{x\in X^0}\max_{S\in\Sigma^0} h(S,x) \qquad (5.2)$$

$$X^1 = \{\, x \in X^0 :\ \max_{S\in\Sigma^0} h(S,x) = \varepsilon^1 \,\} \qquad (5.3)$$

Both ε^1 and X^1 are well-defined, and X^1 is a compact set. Let now $x \in X^1$. Define

$$\Sigma^1(x) = \{\, S \in \Sigma^0 :\ h(S,x) = \varepsilon^1 \,\}. \qquad (5.4)$$

Since Σ^0 is finite, the set X^1 is partitioned into finite family of compact sets $X_1^1, X_2^1, \ldots, X_{r_1}^1$ such that $\Sigma^1(x') = \Sigma^1(x)$ for any $x, x' \in X_l^1$, $l = 1, \ldots, r_1$. Denote such $\Sigma^1(x)$ by Σ_{1l} , i.e.

$$\Sigma_{1l} = \{\, S \in \Sigma^0 :\ h(S,x) = \varepsilon^1 \text{ for all } x \in X_l^1 \,\}. \qquad (5.5)$$

Let $\sigma_1 = \min_{l=1,..,r_1} |\Sigma_{1l}|$ and $\tilde{\Sigma}_1 = \{\Sigma_{1l} : |\Sigma_{1l}| = \sigma_1\} \equiv \{\Sigma_{11}, \ldots, \Sigma_{1M}\}$, $M > 0$.

Consider $\Sigma^1_m = \Sigma^0 \setminus \Sigma_{1m}$, $m = 1, \ldots, M$. If $\Sigma_m^1 = \varnothing$ then we stop the process. If $\Sigma_m^1 \neq \varnothing$ we continue the process recursively for each m.

Let $\varepsilon^2_m = \min_{X^1_m}\max_{S\in\Sigma^1_m} h(S,x)$, and take only such m that

$$\varepsilon^2_m = \min\{\, \varepsilon^2_1, \ldots, \varepsilon^2_M \,\}. \qquad (5.6)$$

Denote such ε^2_m by ε^2 . Further we can define the set

$$X^2_m = \left\{ x \in X^1_m : \max_{S\in\Sigma^1_m} h(S,x) = \varepsilon^2 \right\}$$

and corresponding sets $X^2_{m1}, X^2_{m2}, \ldots, X^2_{mr_2}$ and $\Sigma_{m21}, \ldots, \Sigma_{m2r_2}$ for m satisfying (5.6).

Let $\sigma_2 = \min_m \max_{l=1,\ldots,r_2} |\Sigma_{m2l}|$ and $\tilde{\Sigma}_2 = \{\Sigma_{m2l} : |\Sigma_{m2l}| = \sigma_2\}$,

and so on.

Clearly, this process will stop after finite number of steps. It defines a finite set of points in $IR(V) \cap GR(V)$ called lexicographic centers and this set coincides with the nucleolus $N(V)$ of the game V.

The geometry of the process is akin to that in sidepayment case (see Maschler, Peleg and Shapley (1979)) and can be described in a following way. We start with any ε large enough so that the ε-core is nonempty:

$$C_\varepsilon(V) = \{\, x \in GR(V) \cap IR(V) :\ \forall S \neq I,\ h_S(V,x) \leq \varepsilon \,\} \neq \varnothing .$$

If $h_S(V,x) \leq \varepsilon$ then $1/\gamma(V(S),x) \leq \varepsilon$ and $\gamma(V(S),x) \geq 1/\varepsilon$. Let us consider the set $aV(S)$ for $a > 0$. By (5.1) $\gamma(aV(S),x) = (1/a)\gamma(V(S),x)$, hence

$$\gamma(aV(S), x) \geq 1 \;\Leftrightarrow\; \gamma(V(S), x) \geq a\ ,$$

and we can take $a = 1/\varepsilon$. So, as ε is decreasing, a is increasing.

Thus, the process can be characterized by expanding ("blowing up") the sets $V(S)$. We start with sufficiently small $a > 0$, such that

$$\{\, x \in GR(V) \cap IR(V) : \gamma(V(S), x) \geq a,\ \ \forall S \neq I \ \} =$$
$$= \{\, x \in GR(V) \cap IR(V) : x^S \not\in \text{relint}\ aV(S)\ ,\ \ \forall S \neq I \ \} \neq \varnothing. \qquad (5.7)$$

Then we begin to expand ("blow up") all sets $V(S)$ (by increasing a) "pushing up" some x from $GR(V) \cap IR(V)$. This expansion (blowing up) is performed at equal speed and is stopped either when the set (5.7) become empty or would become disjoint from $GR(V) \cap IR(V)$. This bring us to the sets $X_m^{\,1}$. Any further increase in a will render the sets $X_m^{\,1}$ empty. We therefore continue to expand only those sets $V(S)$, where $S \in \Sigma_m^{\,1}$ (there are M such $\Sigma_m^{\,1}$). These we expand so long as the corresponding modification of the set (5.7) is neither empty nor disjoint from $GR(V) \cap IR(V)$. This bring us to the sets X^2_{mr} ... The process continues in the same manner until all the sets $aV(S)$ have been expanded to their respective limits (i.e. corresponding to appropriate $1/a = \varepsilon^k$).

Remark 5.1. The same procedure can be used for the prenucleolus. We have to replace $GR(V) \cap IR(V)$ by $GR(V)$, and to consider only whether set (5.7) is empty.

Example 5.1. Let $V \in CG_+$ be a bargaining game in the sense that for some $q \in \mathbf{R}^I_{++}$

$$V(S) = \{\ x \in \mathbf{R}^S\ :\ x_i \leq q_i \ \text{ for every } i \in S\ \}\ , \text{ for any } S \neq I\ ,$$

and $q \in int\ V(I)$.

Let us suppose that any $z \in \partial V(I) \cap (\mathbf{R}^I_+ + q)$ is strongly Pareto optimal. Then $N(V) = \lambda q$, where λ is such that $\lambda q \in \partial V(I)$.

Remark 5.2. It follows immediately from the example that for two-person TU games the nucleolus coincides with the Ortmann's proportional solution.

Corollary 5.1. If $V \in CG_+$ is a hyperplane game, then the (pre)nucleolus consist of precisely one point.

The following Proposition is a simple corollary of the definition of the proportional excess and the equality $\gamma(V, x) = \gamma(A{*}V, A{*}x)$ for any $A \in \mathbf{R}^S_{++}$.

Proposition 5.3. The nucleolus and the prenucleolus are scale covariant, i.e. if $A \in \mathbf{R}^I_{++}$, then for any $V \in CG_+$, $N(AV) = A*N(V)$, and $PN(AV) = A*PN(V)$, where game AV is defined by

$$AV(S) = A*V(S) \text{ for every } S.$$

Remark 5.3. As we mentioned in Section 3 the proportional (pre)nucleolus coincides with the (pre)nucleolus defined by means of gauge excess.

Appendix

Proposition 1A (Rubinov and Yagubov (1986)). Let h be a real-valued function on $\mathbf{R}^n$. The following properties are equivalent:

(a) the function h is positively homogeneous, nonnegative and continuous;

(b) h coincides with the gauge function of a star-shaped set

$$V = \{x : h(x) \leq 1 \}.$$

Proof. (a) Let h be a positively homogeneous, nonnegative and continuous function and $V = \{x : h(x) \leq 1 \}$. Then

$$\gamma(V, x) = \inf \{\lambda > 0 : x \in \lambda V \} = \inf \{\lambda > 0 : h(x) \leq \lambda \} = h(x).$$

It is obvious that V is star-shaped.

(b) Let h coincide with the gauge of a star-shaped set V. Since V is star-shaped it follows from the definition that $h(x) \leq 1$ if $x \in V$, and if $h(x) < 1$ then $x \in V$.

Since V is closed then $V = \{x : \gamma(V, x) \leq 1 \}$.

It is clear that the gauge is both positively homogeneous and nonnegative. Let us show that the gauge is continuous. Since it is positively homogeneous, it is sufficient to check that the set

$V_1 = \{x : \gamma(V, x) \leq 1 \}$ is closed, and the set

$V_2 = \{x : \gamma(V, x) < 1 \}$ is open.

V_1 , clearly is closed since it coincides with V. Suppose now that V_2 is not open, $x \in V_2$ and there exists a sequence $\{x_k\}$ such that $x_k \to x$, $\gamma(V, x_k) \geq 1$. Without loss of generality we can assume that $\lim \gamma(V, x_k) = \nu \geq 1$. Take $y_k = x_k / \gamma(V, x_k)$. Then $\gamma(V, y_k) = 1$, and therefore y_k is a boundary point of V. Since $y_k \to x/\nu$, then the point x/ν is also the boundary point of V . If $x \neq$

$\boldsymbol{O}$ then (since $\gamma(V,x) < 1$) the ray L_x intersects the boundary of V at least in two different points $x/\gamma(V, x)$ and x/ν, which contradicts the definition of star-shaped set.

If $x = \boldsymbol{O}$ then the ray L_x lies entirely in V and does not contain any boundary points of V. Thus the gauge of a star-shaped set must also be continuous. □

Remark 1A. Since the gauge is continuous and int V coincides with the set $\{x : \gamma(V, x) < 1\}$, the set V must be regular, i.e. it coincides with the closure of its interior.

Proposition 2A (Rubinov and Yagubov (1986)). Let A be a set of indices and U_a , $a\in A$ be a star-shaped set with gauge $\gamma(U_a, .)$. If the function $g(x) = \inf_{a\in A} \gamma(U_a, .)$ is continuous, then it is the gauge of the set $cl \bigcup_a U_a$. If the function $f(x) = \sup_{a\in A} \gamma(U_a, .)$ is finite and continuous, then it is the gauge of the set $\bigcap_a U_a$.

Proof. Since function $\inf_{a\in A} \gamma(U_a, .)$ is continuous, it is, by Proposition 1, the gauge of some star-shaped set V^1. Let us check that $V^1 = cl \bigcup_a U_a$. Indeed, the continuity of g and $\gamma(U_a, .)$ implies that

$$\text{int } V^1 = \{x : g(x) < 1\} = \left\{x : \inf_a \gamma(U_a, x) < 1\right\} = \bigcup_a \text{int}\, U_a .$$

By regularity

$$V^1 = cl\, \text{int}\, V^1 = cl \bigcup_a \text{int}\, U_a = cl \bigcup_a U_a .$$

The proof the second part of the Proposition is similar. □

Proof of Proposition 4.1. Let $V, V' \in CG^S_+$, and let

$$V \oplus V' = cl \bigcup_{\alpha\in[0,1]} (\alpha V \cap (1-\alpha)V') .$$

By Proposition 2A, the set $V\oplus V' \in CG^S_+$. Let us define a set W as follows:

$$W = \{x\in \mathbf{R}^S_+ : \gamma(V, x) + \gamma(V', x) \le 1 \}.$$

Since our game subsets are normally generated it is sufficient to prove that $(V\oplus V')_+ = W$.

a) Let us first show that $(V \oplus V')_+ \subset W$. Indeed, let $x \in (V \oplus V')_+$. Without loss of generality we can suppose $x \neq \boldsymbol{O}$, since $\boldsymbol{O}$, clearly, belongs to both sets under consideration. There is a sequence $\{x^m\}$, $m = 1,2, \dots$ such that $x^m \xrightarrow[m \to \infty]{} x$, and for any x^m there is $\alpha^m \in [0,1]$ such that

$x^m \in \alpha^m V$ and $x^m \in (1-\alpha^m)\, V'$. Hence, since V_+ and V'_+ are compact,

$x^m = \alpha^m y^m$ for some $y^m \in V$, and $x^m = (1-\alpha^m) z^m$ for some $z^m \in V'$.

Therefore $\gamma(V, y^m) \leq 1$, $\gamma(V', z^m) \leq 1$, and $\gamma(V, x^m/\alpha^m) \leq 1$, $\gamma(V', x^m/(1-\alpha^m)) \leq 1$. Then we have $\gamma(V, x^m) \leq \alpha^m$, $\gamma(V', x^m) \leq 1-\alpha^m$. Therefore $\gamma(V, x^m) + \gamma(V', x^m) \leq 1$ and $x^m \in W$. Since W is closed $x \in W$.

b) Let now $x \in W$, $x \neq \boldsymbol{O}$, then $\gamma(V, x) + \gamma(V', x) \leq 1$. Clearly $\gamma(V, x) > 0$ and $\gamma(V', x) > 0$. Let us take

$$\alpha = \frac{\gamma(V,x)}{\gamma(V,x)+\gamma(V',x)} > 0 \text{ , then } 1-\alpha = \frac{\gamma(V',x)}{\gamma(V,x)+\gamma(V',x)} > 0 \text{ .}$$

For this α we have

$$\gamma(\alpha V, x) = \inf \{\lambda : x \in \lambda \alpha V\} = \inf \{\lambda : x/\alpha \in \lambda \alpha V\} = \gamma(V, x/\alpha) =$$

$$= \frac{1}{\alpha}\gamma(V,x) = \gamma(V,x)\frac{\gamma(V,x)+\gamma(V',x)}{\gamma(V,x)} \leq 1.$$

Similarly, $\gamma((1-\alpha)\, V', x) \leq 1$. Therefore $x \in \alpha V \cap (1-\alpha)\, V'$ for α chosen, and $x \in V \oplus V'$.

(Another proof of the proposition can be found in Rubinov and Yagubov (1986)). □

Proposition 3A. Let a NTU game V be such that dom $v(S, .) = \{\alpha \in \mathbf{R}^I_+ : v(S, \alpha) < +\infty \}$ is nonempty and closed for every S. If $x,y \in \mathbf{R}^I$ such that $x_i < y_i$ for every $i \in S$, then $E_S(V,x) > E_S(V, y)$.

Proof. Let $x,y \in \mathbf{R}^I$, and $x_i < y_i$ for every $i \in S$. Then $y^S = x^S + z^S$, where $z^S > \boldsymbol{0}^S$. Thus

$E_s (V, x) = \min \{ v(S,\alpha) - (x,\alpha) : \alpha \in T^S \} =$

$$= \min \{ v(S,\alpha) - (x,\alpha) - (z,\alpha) + (z,\alpha) : \alpha \in T^S \} \geq$$

$$\geq \min \{ v(S,\alpha) - (x,\alpha) - (z,\alpha) : \alpha \in T^S \} + \min \{ (z,\alpha) : \alpha \in T^S \} >$$

$$= \min \{ v(S,\alpha) - (x,\alpha) - (z,\alpha) : \alpha \in T^S \} =$$

$$= \min \{ v(S,\alpha) - (y,\alpha): \alpha \in T^S \} = E_s (V, y) . \qquad \square$$

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