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The Rich and the Poor in a Simple Model of Growth and Distribution

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Богатые и бедные в простой модели роста и распределения
На английском языке

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Keywords: Growth, Inequality, Relative Consumption
JEL Classification: O41, E21, D31, D91

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The Rich and the Poor in a Simple Model of Growth and Distribution

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Abstract

We consider a model of economic growth with altruistic consumers who care both about their consumption relative to others and the disposable income of their offsprings. We show that if the parameter accounting for the importance of positional concerns is lower than a certain threshold, then the wealth of all agents converges irrespective of the initial distribution of wealth. If, however, it is higher than the threshold, then all the capital is eventually owned by the households which were the richest from the outset.

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1 Introduction

It is a widely accepted fact that the rich save more than the poor (see, e.g., Dynan et al, 2004). Does it necessarily imply that inequality necessarily continues to increase over time? "How important for deciding your current wealth position is the wealth status of an ancestor 500 years ago?" (Bliss, 2004, p. 124). Answering these questions is a complex task: on the one hand, inequality affects capital accumulation and, on the other hand, inequality changes through the accumulation process. The aim of this paper is to answer these questions

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within the context of a simple model of economic growth with successive generations of altruistic consumers.

Theoretical arguments that saving propensities increase with wealth date back at least to Fisher (1930) and Keynes (1936). However, if rich consumers save at higher rates, saving rates should rise over time as everyone becomes richer, which is not supported by empirical data: national saving rates remain roughly constant as income grows. A natural explanation of this discrepancy was proposed by Duesenberry (1949), who argued that poverty is relative and people make decisions based not on absolute income but on relative income inasmuch as the propensity to save of an individual is an increasing function of his or her percentile position in the income distribution.

A model of economic growth with increasing saving propensities was proposed by Schlicht (1975) and Bourguignon (1981). In that model, multiple two-class locally stable steady-state equilibria may emerge and the steady-state capital stock depends on the proportion in which the population is split into two classes[1].

A natural rationalization of the Duesenberry approach in terms of intertemporal utility maximization is proposed by Alvarez-Cuadrado and Long (2011), who present an overlapping generations model with heterogeneous agents caring about their consumption relative to others (in modern economic literature this type of interdependence is referred to as jealousy, envy or the desire to *keep up with the Joneses* and is considered to be a particular type of consumption externalities) and the bequests they leave to their offsprings[2].

In modern literature, the implications of consumption externalities for a range of important issues (asset pricing, stabilization policy, consumption, capital accumulation and growth) have been investigated. However, in most models with consumption externalities, a representative agent framework is adopted, where macroeconomic equilibrium is obtained by imposing symmetry and putting each individual's consumption equal to the average consumption level. However, "it would seem that heterogeneity across agents is a fundamental component of any analysis of consumption externalities, such as the effect of jealousy or keeping up with the Joneses, for the behavior of the aggregate magnitudes" (García-Peñalosa and Turnovsky, 2008, p. 440).

[1] Whereas Schlicht (1975) and Bourguignon (1981) offer a general treatment of convex saving functions without analyzing the reasons for convexity, Moav (2002) reaches analogous conclusions assuming intertemporally maximizing agents with convex bequest functions in a model of a small open economy.

[2] Borissov and Lambrecht (2009) and Borissov (2013) propose another rationalization of the Duesenberry approach; they consider an AK-model with endogenous time preferences and assume that the rich are more patient than the poor.

The role of consumption externalities in the presence of heterogeneous agents is studied in several recent papers. García-Peñalosa and Turnovsky (2008) examine a model where infinitely-lived agents differ in terms of the initial capital endowment and their reference group; they find that i) the aggregate dynamics are independent of the distribution of wealth across agents and are identical to those obtained in the model with homogeneous agents when a symmetric equilibrium is imposed and ii) in a growing economy, keeping up with the Joneses results in less inequality than would prevail in an economy with no consumption externalities. Fisher and Heijdra (2009) model agent heterogeneity in a framework of the Blanchard-Yaari model with individuals differing in age and, thus, in their consumption levels and asset holdings; they show that consumption externalities lower consumption and the capital stock in long-run equilibrium. Mino and Nakamoto (2012) consider a model with two types of agents and show that consumption externalities, together with heterogeneity of agents, can lead to a variety of dynamic behaviors.

Alvarez-Cuadrado and Long (2012) explore the impact of positional concerns on the intragenerational distribution of wealth and on the intergenerational transmission of inequality in a modification of the model introduced in Alvarez-Cuadrado and Long (2011). They provide some analytical results under a simple production technology linear in capital and labor and illustrate numerically the interaction between inequality and positional concerns in an environment with a Cobb-Douglas production function. In contrast to Schlicht (1975) and Bourguignon (1981), in that model, aggregate dynamics are independent of the distribution of income and the steady-state capital stock does not depend on the wealth distribution.

In this paper, we focus on heterogeneity across agents. We present an economy populated by altruistic agents heterogeneous in their initial wealth who care about their consumption relative to others. The main difference of our approach from the approach by Alvarez-Cuadrado and Long (2012) is that they assume the joy of giving form of altruism, whereas we assume family altruism, proposed by Lambrecht et al (2006). In the case of joy of giving altruism (Abel and Warshawsky, 1988; Andreoni, 1989), the amount and structure of bequests left by parents are not related to their children's preferences, but rather to the pleasure parents derive from giving; bequests enter in the parental utility function as a consumption good. As for family altruism, the utility of altruists depends on their children's disposable income.

The use of family altruism enables us to obtain clear-cut results concerning the dynamics of wealth and income distribution in a model with a Cobb-Douglas production function. Our results complement the literature that explores the relationship between consumption externalities and inequality. We show that if the parameter accounting for the importance of positional concerns (the degree of envy) is lower than a certain threshold, then there is a unique steady-state equilibrium, which is characterized by perfect income and wealth equality. Moreover, any equilibrium path converges to this steady-state equilibrium irrespective of the initial distribution of wealth. If the degree of envy is higher than the threshold, then in steady-state equilibria, the population splits into two classes, the rich and the poor. Under this scenario, the long-run stock of capital depends on the proportion between the rich and the poor, and, on any equilibrium path, all the capital is eventually owned by the households which were the wealthiest from the outset.

The rest of the paper is organized as follows. The next section presents the basic model. In Section 3, we define equilibria, describe their structure and prove their existence and uniqueness. Section 4 provides a description of steady-state equilibria. Section 5 then investigates equilibrium dynamics of wealth distribution. In Section 6, we shortly discuss our results. The final section concludes.

2 The Model

2.1 Population

We consider a closed economy populated by successive generations of consumers. Time is discrete and infinite with $t = -1, 0, 1, \ldots$. Population consists of N dynasties. Each individual is endowed with one unit of labor, lives for one period and gives birth to one offspring.

2.2 Production

Every period the economy produces a composite good that may be consumed or invested. At each time t, output, Y_t, is determined by the Cobb-Douglas production function

$$Y_t = K_t^{\alpha} N^{1-\alpha},\ 0 < \alpha < 1,$$

where K_t is the time t stock of physical capital, which fully depreciates during one time period. In *per capita* terms

$$y_t = k_t^{\alpha},$$

where $y_t := Y_t/N$ and $k_t := K_t/N$. Markets are assumed to be competitive and hence the interest rate r_t and the wage rate w_t are determined as follows:

$$1 + r_t = \alpha k_t^{\alpha-1},\ w_t = (1-\alpha)k_t^{\alpha}.$$

2.3 Utility maximization

Consider an agent who belongs to dynasty $j \in \{1, \ldots, N\}$ and lives in period t. The disposable income of this agent is $(1+r_t)s^j_{t-1} + w_t$, where $s^j_{t-1} \geq 0$ is the bequest left by her parent. She spends her disposable income on her personal consumption, $c \geq 0$, and the bequest she leaves to her offspring, $s \geq 0$. Thus, her budget constraint is as follows:

$$c + s = (1 + r_t)s^j_{t-1} + w_t.$$

Her preferences are given by the following utility function:

$$\ln(c - \gamma \bar{c}_t) + \delta \ln(w_{t+1} + (1 + r_{t+1})s),$$

where $\bar{c}_t$ is the average level of consumption of generation t, which is taken by all agents as given, $0 \leq \gamma < 1$ is the degree of envy and $\delta > 0$ is the degree of altruism.

Like Alvarez-Cuadrado and Long (2012), we follow Ljungqvist and Uhlig (2000) in adopting an additive specification for relative consumption: the satisfaction derived from consumption does not depend on the absolute level of consumption itself but rather on how it compares to the consumption of a reference group, namely, the average consumption of agents belonging to the same generation. As for altruism, we assume a form of altruism which is different from the form of altruism adopted by Alvarez-Cuadrado and Long (2012), who assume that agents derive pleasure from giving itself, which, in their model, implies that bequests enter in the parental utility function as a consumption good. By contrast, we follow Lambrecht et al (2006) in assuming that the utility of altruists depends on their children's disposable income.

Let

$$\xi := \frac{1-\alpha}{\alpha}.$$

For all $t = 0, 1, \ldots,$ we have

$$w_t = \xi(1 + r_t)k_t,$$

and hence

$$\ln(w_{t+1} + (1 + r_{t+1})s) = \ln(1 + r_{t+1}) + \ln(\xi k_{t+1} + s).$$

Therefore, the problem of the consumer who belongs to dynasty j and lives in period t can be written down as follows:

$$\left\{ \begin{array}{c} \max_{c \geq 0, s \geq 0} \{\ln(c - \gamma \bar{c}_t) + \delta \ln(\xi k_{t+1} + s)\} \\ c + s = (1 + r_t)(\xi k_t + s^j_{t-1}) \end{array} \right. . \quad (1)$$

Under the assumption that $\gamma\bar{c}_t < (1+r_t)(\xi k_t + s^j_{t-1})$, there is a unique solution to this problem, (c^j_t, s^j_t). If $\delta(1+r_t)(\xi k_t + s^j_{t-1}) \leq \delta\gamma\bar{c}_t + \xi k_{t+1}$, then it is determined by

$$s^j_t = 0,\ c^j_t = (1+r_t)(\xi k_t + s^j_{t-1}),$$

and if $\delta(1+r_t)(\xi k_t + s^j_{t-1}) > \delta\gamma\bar{c}_t + \xi k_{t+1}$, then it is determined by

$$s^j_t = \frac{\delta(1+r_t)(\xi k_t + s^j_{t-1}) - (\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta},$$

$$c^j_t = \frac{(1+r_t)(\xi k_t + s^j_{t-1}) + (\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta}.$$

In other terms,

$$s^j_t = \max\{0, \frac{\delta(1+r_t)(\xi k_t + s^j_{t-1}) - (\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta}\}, \tag{2}$$

$$c^j_t = (1+r_t)(\xi k_t + s^j_{t-1}) - s^j_t. \tag{3}$$

3 Equilibria

3.1 Time t equilibrium

Definition. *Let the bequests $s^j_{t-1} \geq 0$, $j = 1, \ldots, N$, left by the agents who live in period $t-1$ be given. Let further $k_t = \frac{\sum_{j=1}^N s^j_{t-1}}{N}$. A tuple $\{(c^j_t, s^j_t)^N_{j=1}, k_{t+1}\}$ constitutes a time t equilibrium if*

- *(c^j_t, s^j_t) is a solution to (1) at $\bar{c}_t = \frac{\sum_{i=1}^N c^i_t}{N}$, $j = 1, \ldots, N$;*
- *$k_{t+1} = \frac{\sum_{j=1}^N s^j_t}{N} > 0$.*

Note that unlike the OLG models with altruism, where the life-cycle motive for saving coexists with the bequest motive, in our model, there is no life-cycle motive and hence the total savings and the total bequests are equal; therefore, in equilibrium, the per capita stock of capital equals the average bequest.

Consider a time t equilibrium $\{(c^j_t, s^j_t)^N_{j=1}, k_{t+1}\}$ and describe its properties. Obviously,

$$\bar{c}_t + k_{t+1} = k^\alpha_t = (1+r_t)(\xi + 1)k_t$$

and hence

$$\bar{c}_t = (1 + r_t)(\xi + 1)k_t - k_{t+1}. \tag{4}$$

Therefore, taking account of (2), for $j = 1, \ldots, N$,

$$s_t^j = \max\{0, \frac{\delta(1 + r_t)(\xi - \gamma(\xi + 1))k_t + (\delta\gamma - \xi)k_{t+1} + \delta(1 + r_t)s_{t-1}^j}{1 + \delta}\}. \tag{5}$$

It follows that

$$\begin{aligned}&(1 + \delta)Nk_{t+1} \\ &\quad = \sum_{j=1}^{N} \max\{0, \delta(1 + r_t)(\xi - \gamma(\xi + 1))k_t + (\delta\gamma - \xi)k_{t+1} + \delta(1 + r_t)s_{t-1}^j\}.\end{aligned}$$

In other terms, k_{t+1} is a solution to the equation

$$g(k) = 0, \tag{6}$$

where

$$\begin{aligned}g(k) &:= (1 + \delta)Nk \\ &\quad - \sum_{j=1}^{N} \max\{0, \delta(1 + r_t)(\xi - \gamma(\xi + 1))k_t + (\delta\gamma - \xi)k + \delta(1 + r_t)s_{t-1}^j\}.\end{aligned}$$

Note that $\gamma\bar{c}_t < w_t + (1 + r_t)s_{t-1}^j = (1 + r_t)(\xi k_t + s_{t-1}^j)$ and hence

$$\gamma[(1 + r_t)(\xi + 1)k_t - k_{t+1}] < (1 + r_t)(\xi k_t + s_{t-1}^j). \tag{7}$$

Thus, if the tuple $\{(c_t^j, s_t^j)_{j=1}^N, k_{t+1}\}$ is a time t equilibrium, then *i)* $k_{t+1} > 0$ is a solution to (6); *ii)* for all $j = 1, \ldots, N$, (c_t^j, s_t^j) is determined by (5) and (3); *iii)* (7) is satisfied.

It is easy to prove that the converse is also true. Therefore, we can formulate the following lemma.

Lemma 1. *Suppose that i)* $k_{t+1} > 0$ *is a solution to (6); ii) for all* $j = 1, \ldots, N$, (c_t^j, s_t^j) *is determined by (3) and (5); iii) (7) is satisfied. Then the tuple* $\{(c_t^j, s_t^j)_{j=1}^N, k_{t+1}\}$ *is a time* t *equilibrium.*

Let us now prove the existence and uniqueness of a time t equilibrium under the assumption that $\gamma < \gamma_0$, where

$$\gamma_0 := \frac{\xi(\xi + 1 + \delta)}{\xi^2 + (2 + \delta)\xi + 1}.$$

Proposition 1. *Suppose that* $\gamma < \gamma_0$. *Then for any* $\{(s^j_{t-1})^N_{j=1}, k_t\}$ *such that* $s^j_{t-1} \geq 0,\ j = 1, \ldots, N,$ *and* $k_t = \frac{\sum_{j=1}^N s^j_{t-1}}{N} > 0$, *there exists a unique time* t *equilibrium* $\{(c^j_t, s^j_t)^N_{j=1}, k_{t+1}\}$.

Proof. By Lemma 1, it is sufficient to show that equation (6) has a unique positive solution and that if k_{t+1} is its solution and $s^j_t,\ j = 1, \ldots, N$, are determined by (5), then (7) is satisfied.

It is clear that $s^j_{t-1} \geq k_t$ for a least one j and hence

$$\begin{aligned}&\max\{0, \delta(1+r_t)(\xi - \gamma(\xi+1))k_t + \delta(1+r_t)s^j_{t-1}\}\\ &\qquad = \delta(1+r_t)(\xi - \gamma(\xi+1))k_t + \delta(1+r_t)s^j_{t-1}\\ &\qquad \geq \delta(1+r_t)(\xi + 1 - \gamma(\xi+1))k_t = \delta(1+r_t)(1-\gamma)(\xi+1)k_t > 0.\end{aligned}$$

Therefore, $g(0) < 0$. It is clear that $g(k)$ is an increasing function. Moreover,

$$\begin{aligned}&g[(1+r_t)(1+\xi)k_t] \geq (1+\delta)N(1+r_t)(1+\xi)k_t\\ &-\sum_{j=1}^N[\delta(1+r_t)(\xi-\gamma(\xi+1))k_t + (\delta\gamma-\xi)(1+r_t)(1+\xi)k_t + \delta(1+r_t)s^j_{t-1}]\\ &\qquad = (1+\delta)N(1+r_t)(1+\xi)k_t\\ &-\sum_{j=1}^N[\delta(1+r_t)(\xi-\gamma(\xi+1))k_t + (\delta\gamma-\xi)(1+r_t)(1+\xi)k_t + \delta(1+r_t)k_t]\\ &\qquad = N(1+r_t)(1+\xi)^2k_t > 0.\end{aligned}$$

Therefore, there is a unique solution to the equation $g(k) = 0$. It is obvious that this solution is positive.

Let us now show that if k_{t+1} is a solution to (6), then

$$\gamma[(1+r_t)(\xi+1)k_t - k_{t+1}] < (1+r_t)\xi k_t \tag{8}$$

and hence (7) is satisfied. Indeed, let k_{t+1} be a solution to (6) and s^j_t be determined for all $j = 1, \ldots, N$, by (5). We have

$$\begin{aligned}(1+\delta)Nk_{t+1} &\geq \sum_{j=1}^N[\delta(1+r_t)(\xi-\gamma(\xi+1))k_t + (\delta\gamma-\xi)k_{t+1} + \delta(1+r_t)s^j_{t-1}]\\ &= \delta(1+r_t)(\xi-\gamma(\xi+1))Nk_t + (\delta\gamma-\xi)Nk_{t+1} + \delta(1+r_t)Nk_t\end{aligned}$$

and hence

$$k_{t+1} \geq (1+r_t)\frac{\delta(1-\gamma)(\xi+1)}{1+\xi+\delta(1-\gamma)}k_t.$$

It is not difficult to check that $\gamma < \gamma_0 \Leftrightarrow \gamma[(\xi+1)-\frac{\delta(1-\gamma)(\xi+1)}{1+\xi+\delta(1-\gamma)}] < \xi$. Therefore, taking account of the inequality $\gamma < \gamma_0$, we get

$$\gamma[(1+r_t)(\xi+1)k_t - k_{t+1}] \\ \leq (1+r_t)\gamma[(\xi+1) - \frac{\delta(1-\gamma)(\xi+1)}{1+\xi+\delta(1-\gamma)}]k_t < (1+r_t)\xi k_t$$

and hence (8) is satisfied. □

Remark. *It is possible to show that if $\gamma > \gamma_0$ and N is sufficiently large, then there exist bequests $s^j_{t-1} > 0,\ j = 1, \ldots, N$, such that no time t equilibrium exists.*

3.2 Equilibrium paths

Definition. *Let the bequests $s^j_{-1} \geq 0,\ j = 1, \ldots, N$, left by the agents who live in period $t = -1$ be given. Let further $k_0 = \sum_{j=1}^N s^j_{-1}$. A sequence $\{(c^j_t, s^j_t)_{j=1}^N, k_{t+1}\}_{t=0}^\infty$ constitutes an equilibrium path starting from $(s^j_{-1})_{j=1}^N$ if for each $t = 0, 1, \ldots,$ $\{(c^j_t, s^j_t)_{j=1}^N, k_{t+1}\}$ is a time t equilibrium.*

The following existence and uniqueness theorem follows directly from Proposition 1.

Theorem 1. *If $\gamma < \gamma_0$, then for any $(s^j_{-1})_{j=1}^N$ such that $\sum_{j=1}^N s^j_{t-1} > 0$, there exists a unique equilibrium path $\{(c^j_t, s^j_t)_{j=1}^N, k_{t+1}\}_{t=0}^\infty$ starting from $(s^j_{-1})_{j=1}^N$.*

In what follows we assume that $\gamma < \gamma_0$.

4 Steady-State Equilibria

Definition. *A tuple $\{(c^j, s^j)_{j=1}^N, k\}$ is called a steady-state equilibrium if $k > 0$ and the sequence $\{(c^j_t, s^j_t)_{j=1}^N, k_{t+1}\}_{t=0}^\infty$ given for all $t = 0, 1, \ldots,$ by*

$$k_{t+1} = k;\ (c^j_t, s^j_t) = (c^j, s^j),\ j = 1, \ldots, N,$$

is an equilibrium path starting from $(s^j)_{j=1}^N$.

Let the function $h : [0, \gamma_0) \times [0, 1] \to \mathbb{R}_+$ be defined by

$$h(\gamma, x) = \frac{\delta(1 + x\xi - x\gamma - x\gamma\xi)}{1 + \delta + x\xi - x\delta\gamma},$$

and let $k(\gamma, x)$ denote the positive solution to the following equation in k:

$$k = h(\gamma, x)\alpha k^{\alpha}.$$

Let further

$$\gamma_1 := \frac{\delta\xi}{\xi + \delta\xi + 1}.$$

It is easily checked that $\gamma_1 < \gamma_0$ and that

$$\gamma \gtrless \gamma_1 \ \Leftrightarrow \ \frac{\delta}{(1 + \delta)h(\gamma, x)} \gtrless 1$$
$$\Leftrightarrow \ \delta\xi - \delta\gamma\xi - \delta\gamma + \delta\gamma h(\gamma, x) - \xi h(\gamma, x) \gtrless 0 \quad (9)$$

for any $x \in (0, 1]$.

The following proposition describes the structure of steady-state equilibria.

Proposition 2. *1) If $\gamma < \gamma_1$, then there is a unique steady-state equilibrium $\{(c^j, s^j)_{j=1}^N, k\}$, which is determined as follows:*

$$k = k(\gamma, 1); \ s^j = k \text{ and } c^j = k^{\alpha} - k, \ j = 1, \dots, N.$$

2) If $\gamma > \gamma_1$, then for any non-empty subset J of the set of dynasties, $\{1, \dots, N\}$, there exists a unique steady-state equilibrium $\{(c^j, s^j)_{j=1}^N, k\}$ such that

$$s^j > 0, \ j \in J; \ s^j = 0 \ j \notin J.$$

It is determined by

$$k = k(\gamma, \frac{|J|}{N});$$

$$s^j = \frac{N}{|J|}k \text{ and } c^j = \frac{N}{|J|}(k^{\alpha} - k) - (\frac{N - |J|}{|J|})(1 - \alpha)k^{\alpha}, \ j \in J;$$

$$s^j = 0 \text{ and } c^j = (1 - \alpha)k^{\alpha}, \ j \notin J,$$

where $|J|$ is the cardinality of J.

Proof. The proposition follows from Theorem 2, which will be proved in the next section. □

Proposition 2 first maintains that if $\gamma < \gamma_1$, then there is a unique steady-state equilibrium, which is characterized by absolute income and wealth equality. Secondly, if $\gamma > \gamma_1$, then steady-state equilibria are multiple. In a steady-state equilibria the population splits into two classes, the rich and the poor, and all the capital is owned by the rich, whereas the poor leave no bequests and spend all their incomes on consumption. Any division of the population into the rich and the poor is possible except the degenerate division where all agents are poor (the case were all agents are rich is however possible). It is important to note that the steady-state capital stock depends on the proportion between the rich and the poor.

5 Equilibrium Dynamics of Wealth Distribution

In this section we prove that if $\gamma < \gamma_1$, then any equilibrium path converges to the unique steady-state equilibrium, which is characterized by perfect wealth equality, whereas if $\gamma > \gamma_1$, then, in the long run, all dynasties become poor, except for those that were the wealthiest initially.

Let $\{(c_t^j, s_t^j)_{j=1}^N, k_{t+1}\}_{t=0}^\infty$ be an equilibrium path starting from $(s_{-1}^j)_{j=1}^N$ such that $\sum_{j=1}^N s_{-1}^j > 0$. With no loss of generality we assume

$$s_{-1}^1 \geq s_{-1}^2 \geq \ldots \geq s_{-1}^N.$$

By L we denote the number of agents j such that $s_{-1}^j = s_{-1}^1$. In other terms,

$$s_{-1}^1 = s_{-1}^2 = \ldots = s_{-1}^L > s_{-1}^{L+1} \geq \ldots \geq s_{-1}^N.$$

It is not difficult to check that if $s_{t-1}^j \geq s_{t-1}^i$ then $s_t^j \geq s_t^i$. Moreover, if $s_{t-1}^j > s_{t-1}^i$ and $s_t^j > 0$, then $s_t^j > s_t^i$. Therefore,

$$s_t^1 = s_t^2 = \ldots = s_t^L > s_t^{L+1} \geq \ldots \geq s_t^N, \ t = 0, 1, \ldots. \tag{10}$$

By $M(t)$ we denote the number of agents which leave positive bequests in period t:

$$s_t^j > 0, \ j = 1, \ldots, M(t); \ s_t^j = 0, \ j = M(t) + 1, \ldots, N.$$

Theorem 2. 1) *If $\gamma < \gamma_1$, then:*

$$M(t) = N, \ t = 0, 1, \ldots, \tag{11}$$

$$k_{t+1} = h(\gamma, 1)(1 + r_t)k_t,\ t = 0, 1, \ldots, \tag{12}$$

$$\lim_{t\to\infty} \frac{s^j_{t-1}}{k_t} = 1,\ j = 1, \ldots, N, \tag{13}$$

$$\lim_{t\to\infty} k_t = k(\gamma, 1). \tag{14}$$

2) If $\gamma > \gamma_1$, then: the sequence $\{M(t)\}_{t=0}^{\infty}$ is non-increasing; there exists T such that for $t = T+1, T+2, \ldots,$

$$M(t) = L, \tag{15}$$

$$k_{t+1} = h(\gamma, L/N)(1 + r_t)k_t, \tag{16}$$

$$\frac{s^j_{t-1}}{k_t} = \frac{N}{L},\ j = 1, \ldots, L;\ s^j_{t-1} = 0,\ j = L+1, \ldots, N; \tag{17}$$

and

$$\lim_{t\to\infty} k_t = k(\gamma, L/N). \tag{18}$$

To prove Theorem 2, we need the following lemma.

Lemma 2. *If $M(t) \geq M(t-1)$ for some t, then*

$$k_{t+1} = h(\gamma, m(t))(1 + r_t)k_t, \tag{19}$$

$$\delta\gamma\bar{c}_t + \xi k_{t+1} = (1 + r_t)[\delta\gamma\xi + \delta\gamma - \delta\gamma h(\gamma, m(t)) + \xi h(\gamma, m(t))]k_t, \tag{20}$$

and

$$\frac{s^j_t}{k_{t+1}} = \frac{\delta\xi - h(\gamma, m(t))\xi - \delta\gamma - \delta\gamma\xi + \delta\gamma h(\gamma, m(t))}{(1+\delta)h(\gamma, m(t))} + \frac{\delta}{(1+\delta)h(\gamma, m(t))}\frac{s^j_{t-1}}{k_t},\ j = 1, \ldots, M(t), \tag{21}$$

where $m(t) = M(t)/N$.

Proof. Suppose that $M(t) \geq M(t-1)$. We have

$$s^j_t = \frac{\delta(1 + r_t)(\xi k_t + s^j_{t-1}) - (\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta},\ j = 1, \ldots, M(t).$$

Since $\sum_{j=1}^{M(t)} s_{t-1}^j \geq \sum_{j=1}^{M(t-1)} s_{t-1}^j = Nk_t$, we get

$$Nk_{t+1} = \sum_{j=1}^{M(t)} \frac{\delta(1+r_t)(\xi k_t + s_{t-1}^j) - (\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta}$$
$$= \frac{\delta(1+r_t)(M(t)\xi k_t + Nk_t) - M(t)(\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta}.$$

Applying (4) gives

$$(1+\delta)k_{t+1} = \delta(1+r_t)(m(t)\xi+1)k_t - m(t)(\delta\gamma\bar{c}_t + \xi k_{t+1})$$
$$= \delta(1+r_t)(m(t)\xi+1)k_t - m(t)\delta\gamma(1+r_t)(\xi+1)k_t + m(t)\delta\gamma k_{t+1} - m(t)\xi k_{t+1}$$

and hence

$$(1+\delta+m(t)\xi - m(t)\delta\gamma)k_{t+1} = \delta(1+r_t)(1+m(t)\xi - m(t)\gamma - m(t)\gamma\xi)k_t,$$

which proves (19). A rather easy calculation using (4) shows that (20) is also satisfied.

Therefore, for $j = 1, \ldots, M(t)$,

$$s_t^j = \frac{\delta(1+r_t)(\xi k_t + s_{t-1}^j) - (\delta\gamma\bar{c}_t + \xi k_{t+1})}{1+\delta}$$
$$= (1+r_t)\frac{[\delta\xi - \delta\gamma\xi - \delta\gamma + \delta\gamma h(\gamma, m(t)) - \xi h(\gamma, m(t))]k_t + \delta s_{t-1}^j}{1+\delta},$$

and hence

$$\frac{s_t^j}{k_{t+1}} = \frac{s_t^j}{(1+r_t)h(\gamma, m(t))k_t}$$
$$= \frac{\delta\xi - \delta\gamma\xi - \delta\gamma + \delta\gamma h(\gamma, m(t)) - \xi h(\gamma, m(t))}{(1+\delta)h(\gamma, m(t))} + \frac{\delta}{(1+\delta)h(\gamma, m(t))}\frac{s_{t-1}^j}{k_t},$$

which proves (21). □

Proof of Theorem 2. Case 1: $\gamma < \gamma_1$**.** A routine calculation shows that for any $t = 0, 1, \ldots,$ $h(\gamma, 1)(1+r_t)k_t$ is a solution to the following equation in k:

$$(1+\delta)Nk = \sum_{j=1}^{N}[\delta(1+r_t)(\xi - \gamma(\xi+1))k_t + (\delta\gamma - \xi)k + \delta(1+r_t)s_{t-1}^j].$$

Moreover, it is easy to check that $\delta(\xi-\gamma(\xi+1))+(\delta\gamma-\xi)h(\gamma,1)>0$ and, hence,

$$\begin{aligned}\delta(1+r_t)(\xi-\gamma(\xi+1))k_t+(\delta\gamma-\xi)h(\gamma,1)(1+r_t)k_t+\delta(1+r_t)s^j_{t-1}\\ \geq\delta(1+r_t)(\xi-\gamma(\xi+1))k_t+(\delta\gamma-\xi)h(\gamma,1)(1+r_t)k_t\\ =[\delta(\xi-\gamma(\xi+1))+(\delta\gamma-\xi)h(\gamma,1)](1+r_t)k_t>0.\end{aligned}$$

It follows that for all $j=1,\dots,N$,

$$\begin{aligned}&\max\{0,\delta(1+r_t)(\xi-\gamma(\xi+1))k_t+(\delta\gamma-\xi)h(\gamma,1)(1+r_t)k_t+\delta(1+r_t)s^j_{t-1}\}\\ &=\delta(1+r_t)(\xi-\gamma(\xi+1))k_t+(\delta\gamma-\xi)h(\gamma,1)(1+r_t)k_t+\delta(1+r_t)s^j_{t-1}>0.\end{aligned}$$

Therefore, $h(\gamma,1)(1+r_t)k_t$ is a solution to (6). Hence, by Lemma 1, (12) is satisfied, which implies (14).

Since, by (2) and (12),

$$\begin{aligned}s^j_t=\delta(1+r_t)(\xi-\gamma(\xi+1))k_t\\ +(\delta\gamma-\xi)h(\gamma,1)(1+r_t)k_t+\delta(1+r_t)s^j_{t-1}>0,\ j=1,\dots,N,\end{aligned}$$

(11) is satisfied as well. To prove (13), it is sufficient to apply Lemma 2, namely, to note that, by (11) and (21),

$$\frac{s^j_t}{k_{t+1}}=\frac{\delta\xi-\delta\gamma\xi-\delta\gamma+\delta\gamma h(\gamma,1)-\xi h(\gamma,1)}{(1+\delta)h(\gamma,1)}+\frac{\delta}{(1+\delta)h(\gamma,1)}\frac{s^j_{t-1}}{k_t},\quad j=1,\dots,N.$$

and, by (9), $\delta\xi-\delta\gamma\xi-\delta\gamma+\delta\gamma h(\gamma,1)-\xi h(\gamma,1)>0$ and $\frac{\delta}{(1+\delta)h(\gamma,1)}<1$. □

Proof of Theorem 2. Case 1: $\gamma>\gamma_1$**.** Let us show that $s^j_{t-1}=0\ \Rightarrow\ s^j_t=0$ and hence $M(t)\leq M(t-1),\ t=0,1,\dots$. Indeed, assume the converse. Then $s^i_{t-1}=0$ and $s^i_t>0$ for some t and i. Taking account of (2), we get $\delta(1+r_t)\xi k_t>\delta\gamma\bar{c}_t+\xi k_{t+1}$ and, hence, $s^j_t>0,\ j=1,\dots,N$. Therefore, $M(t)=N>M(t-1)$. By Lemma 2, (20) is satisfied and, hence,

$$\begin{aligned}\delta(1+r_t)\xi k_t>\delta\gamma\bar{c}_t+\xi k_{t+1}=(1+r_t)[\delta\gamma(\xi+1)-\delta\gamma h(\gamma,1)+\xi h(\gamma,1)]k_t\\ =(1+r_t)\frac{\delta(\xi+\gamma)(\xi+1)}{1+\xi+\delta(1-\gamma)}k_t.\end{aligned}$$

Thus, $\xi>\frac{(\xi+1)(\gamma+\xi)}{1+\xi+\delta(1-\gamma)}$, or, equivalently, $\gamma<\gamma_1$, which is a contradiction.

It follows that there are $M > 0$ and T such that $M(t) = M$ for $t \geq T$. By (21),

$$\frac{s_t^j}{k_{t+1}} = \frac{\delta\xi - \delta\gamma\xi - \delta\gamma + \delta\gamma h(\gamma,m) - \xi h(\gamma,m)}{(1+\delta)h(\gamma,m)} + \frac{\delta}{(1+\delta)h(\gamma,m)}\frac{s_{t-1}^j}{k_t}, \; j = 1,\ldots,M, \; t = T+1, T+2, \ldots,$$

where $m = M/N$. At the same time, by (9), $\delta\xi - \delta\gamma\xi - \delta\gamma + \delta\gamma h(\gamma,m) - \xi h(\gamma,m) < 0$ and $\frac{\delta}{(1+\delta)h(\gamma,m)} > 1$. Therefore,

$$\frac{s_t^1}{k_{t+1}} = \frac{s_t^2}{k_{t+1}} = \ldots = \frac{s_t^M}{k_{t+1}} = \frac{N}{M}, \; t = T+1, T+2, \ldots,$$

because otherwise we would have $\lim_{t\to\infty} \frac{s_{t-1}^j}{k_t} = \pm\infty$, $j = 1,\ldots,M$. At the same time, by (10), for each t, if $j \leq L$ and $i > L$, then $s_t^j > s_t^i$. Therefore, $M = L$. Hence, (15) and (17) are satisfied. Finally, taking account of (19), we deduce that (16) is also satisfied. Finally, (18) follows from (16). □

6 Discussion

It is interesting to compare our results with the results by García-Peñalosa and Turnovsky (2008) and Alvarez-Cuadrado and Long (2012). In both papers, the steady-state equilibrium is unique and the aggregate dynamics are independent of the distribution of wealth across agents. However, these two papers arrive at different conclusions with regards to the dynamics of wealth inequality. While García-Peñalosa and Turnovsky (2008) argue that *keeping up with the Joneses* reduces wealth inequality following an expansion in the aggregate capital stock, Alvarez-Cuadrado and Long (2012) show that envy increases the degree of inequality in consumption, capital ownership, and bequests.

In this paper, Proposition 2 and Theorem 2 show that, depending on the degree of envy, two qualitatively different cases are possible.

1. If the degree of envy is low, $\gamma < \gamma_1$, then: *i*) the steady-state equilibrium is unique and the aggregate dynamics are independent of the distribution of wealth across households; *ii*) wealth and incomes of all dynasties converge irrespective of the initial distribution of wealth. The uniqueness

of the steady-state equilibrium and the independence of the aggregate dynamics of the distribution of wealth across agents is in full accord with García-Peñalosa and Turnovsky (2008) and Alvarez-Cuadrado and Long (2012). As for the convergence of wealth and income, it is consistent with García-Peñalosa and Turnovsky (2008), but conflicts with Alvarez-Cuadrado and Long (2012).

2. If the degree of envy is higher, $\gamma > \gamma_1$, then: *i*) in steady-state equilibria, the population is split into two classes, the rich and the poor, and the steady-state of capital depends on the proportion in which the split occurs; *ii*) eventually, all capital is owned by the dynasties that were the wealthiest at the outset whereas all other dynasties become poor. Qualitatively, the dynamics obtained in this case are in sharp contrast with the results by García-Peñalosa and Turnovsky (2008) and complements the results by Alvarez-Cuadrado and Long (2012). These dynamics are essentially the same as in Borissov and Lambrecht (2009) and Borissov (2013), who consider a model where infinitely-lived consumers have endogenous time preferences.

We should also comment the dependence of the steady-state equilibrium capital stock, $k(\gamma, \frac{|J|}{N})$, on the degree of envy, γ, and the share of the rich in the population, $\frac{|J|}{N}$. To describe this dependence, it is sufficient to calculate the derivatives $\frac{\partial h(\gamma,x)}{\partial \gamma}$ and $\frac{\partial h(\gamma,x)}{\partial x}$. It is easy to check that $\frac{\partial h(\gamma,x)}{\partial \gamma} < 0$ for any $x \in (0,1]$ and that $\frac{\partial h(\gamma,x)}{\partial x} > 0$ if $\gamma < \gamma_1$ and $\frac{\partial h(\gamma,x)}{\partial x} < 0$ if $\gamma > \gamma_1$. It follows that for any split of the population into the rich and the poor, a higher degree of envy leads to a lower steady state capital stock, because $\frac{\partial k(\gamma,|J|/N)}{\partial \gamma} < 0$ for any $|J|/N$. If the degree of envy is high, $\gamma > \gamma_1$, then the smaller the share of the rich in the population and hence the higher the level of income and wealth inequality, the higher the steady-state capital stock.

Given that bequests constitute the only channel of inter-generational transmission of inequality in our model, it is interesting to specify the impact of the degree of altruism on wealth inequality in steady-state equilibria. It is easy to check that γ_1 is an increasing function of δ. Therefore, a higher degree of altruism results in a larger range of γ over which two-class steady-state equilibria are impossible. At the same time, for a given γ, a change in δ will have no impact on wealth inequality in the steady-state equilibrium as long as γ remains inside the range $\gamma_1 < \gamma < \gamma_0$, where wealth distribution is determined by the proportion between the rich and the poor, or inside the range $0 < \gamma < \gamma_1$, where perfect equality is ensured.

7 Conclusion

In this paper, we have introduced a simple model of economic growth and distribution with altruistic consumers who care about their consumption relative to others and the disposable income of their offsprings. Our main findings are as follows:

- If the parameter accounting for the importance of positional concerns is lower than a certain threshold, then there is a unique steady-state equilibrium, which is characterized by absolute income and wealth equality, and all equilibrium paths converge to this steady-state equilibrium irrespective of the initial state.

- If the parameter accounting for the importance of positional concerns is higher than the threshold, steady-state equilibria are multiple. In this case, in steady-state equilibria, the population splits into two classes, the rich and the poor, and all the capital is owned by the rich, whereas the poor leave no bequests and spend all their incomes on consumption. As for the dynamics of inequality, in the long run, all dynasties become poor except for those who were the wealthiest from the outset.

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