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Показатели темпа роста
На английском языке

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Abstract: This paper develops an axiomatic theory of an economic variable average growth rate (average rate of change) measurement. The structures that we obtain generalize the conventional measures for average rate of growth (such as the difference quotient, the continuously compounded growth rate, etc.) to an arbitrary domain of the underlying variable and comprise various models of growth. These structures can be described with the help of intertemporal choice theory by means of parametric families of time preference relations on the "prize-time" space with a parameter representing the subjective discount rate.

Keywords: average growth rate; average rate of change; time preferences; discounting
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A THEORY OF AVERAGE GROWTH RATE INDICES

Alexander Alexeev, Mikhail Sokolov

1. Introduction

A large number of implications in economic statistics, macroeconomics, and finance are based on comparing the average growth rates (average rates of change) of a variable over a specified period of time: the compound annual growth rate (the average increase in the value of an investment) is used to compare gains from risk-free investment alternatives; economic growth rate (growth rate of real GDP per capita), inflation rate (the average rate of increase of a price index), and population growth rate are basic statistical information for the international comparison of economic and demographic performance, etc. For values x and x' of the variable at time t and t' ($> t$), a widely used measure of the average growth rate is given by

$$I_0(x,t;x',t') := \frac{\ln x' - \ln x}{t' - t} \tag{1}$$

(the exponential growth rate) or by an order-preserving transformation of I_0.

Every time index I_0 is used as a measure of average rate of growth, the following assumptions are implicitly made:

1. The domain of the variable under consideration is the set of positive real numbers: $x, x' \in \mathrm{R}_{++}$.
2. The average rate of growth is strictly decreasing in the initial value x of the variable and strictly increasing in the final value x'.
3. Scale invariance: $I_0(\lambda x, t; \lambda x', t') = I_0(x,t;x',t') \ \forall \lambda > 0$.
4. Invariance with respect to time shift: $I_0(x, t+\tau; x', t'+\tau) = I_0(x,t;x',t') \ \forall \tau$.
5. The average rate of growth over a consolidated period should fall between growth rates over its subperiods: for any x, x', x'', and $t < t' < t''$

$$\begin{aligned}\min\{I_0(x,t;x',t'), I_0(x',t';x'',t'')\} &\le \\ \le I_0(x,t;x'',t'') &\le \\ \le \max\{I_0(x,t;x',t'), I_0(x',t';x'',t'')\}&.\end{aligned}$$

It can easily be checked that assumptions 1–5 are independent: any four of them do not imply the fifth. Moreover, they characterize I_0 up to an order-preserving transformation among continuous real-valued functions on the set $\{(x,t;x',t') \in (\mathrm{R}_{++} \times \mathrm{R})^2 : t < t'\}$ (see Theorem 10 and Example 4 below for details).

However, the applicability of index I_0 is quite limited: each of assumptions 1–5 seems to be unnecessarily restrictive for a number of applications. Assumptions 1–3 are too restrictive in the case when the economic variable has a more general domain than R_{++}. This is the case when the variable is vector-, function-, or set-valued (e.g., assumption 3 is ill-defined unless the operation of scalar multiplication is defined on the domain). For instance, the average rate of change in prices, inflation rate, implies the vector-valued economic variable, the vector of prices. For another example, assume that the average rate of growth considered is the rate of return for a stock portfolio measured under risk, then the economic variable, the value of the portfolio, is random and is therefore function-valued rather than scalar-valued. One can also expect a violation of assumption 4, stationarity, if the real rate of return of a portfolio is evaluated in the presence of time-varying inflation. Finally, if x and x' in (1) are treated as money, then $I_0(x,t;x',t')$ is the internal rate of return, the solution d of the equation $e^{-dt}x = e^{-dt'}x'$. In this case the functional form of I_0 is a consequence of the exponential discounting model, so assumptions 4 and 5 may be violated for time-inconsistent models of discounting. In particular, assumption 5 does not hold if one defines the average rate of growth using hyperbolic discounting, as the solution d of the equation $x = x'/(1+d(t'-t))$.

In the present paper, we try to develop a unified (i.e., independent of the nature of the economic variable considered) theory of average growth rates that is free of the mentioned limitations. In particular, we characterize all possible functional forms for average rates of growth that correspond to relaxing one or more of assumptions 1–5.

More formally, the problem we deal with can be formulated as follows. Let X be an arbitrary nonempty set representing the domain of an economic variable. A total preorder $\trianglerighteq$ is assumed to be defined on X. The relation $\trianglerighteq$ formalizes the notion of growth of the variable over time: given observations $x \in \mathrm{X}$ and $x' \in \mathrm{X}$ of the variable at time t and t' ($> t$), we say that there is *growth* over the period from t to t' if $x' \vartriangleright x$, where $\vartriangleright$ is the asymmetric part of $\trianglerighteq$. The task is to define a real-valued function I on the set of pairs of dated observations $\{(x,t;x',t') \in (\mathrm{X}\times\mathrm{R})^2 : t < t'\}$ such that (A) I is consistent with $\trianglerighteq$ and (B) the value $I(x,t;x',t')$ can be interpreted as the variable's average growth rate per time unit over the period from t to t'. By consistency in (A) we mean that I is "strictly decreasing" in x and "strictly increasing" in x' (compare with assumption 2): $I(y,t;x',t') \le I(x,t;x',t') \;\Leftrightarrow\; y \trianglerighteq x$; $I(x,t;y',t') \ge I(x,t;x',t')$ $\Leftrightarrow\; y' \trianglerighteq x'$. The possibility of the interpretation in (B) is delineated by various axioms that seem to be reasonable when considering "average rate of growth". Most of the axioms are along the lines of assumptions 3–5.

Here are a few examples that can be reduced to the stated problem setting:

– the economic variable: the value of a portfolio of assets; the domain: $\mathrm{X} = \mathrm{R}_{++}$; $\trianglerighteq$ is the natural order on R_{++}; the intuitive meaning for the function I: the portfolio (average) rate of return;

– the economic variable: welfare of an economic agent; value of the variable: the agent's income and the n-dimensional price vector he/she faces (there are n goods in the economy); the domain: $\mathrm{X} = \mathrm{R}^{n+1}_{++}$; $\trianglerighteq$ is the indirect preference relation induced by the agent's indirect utility function;

the intuition behind I : the agent's welfare growth rate (see Example 2 below for details);

– the economic variable: productive potential of the economy; value of the variable: the production possibilities set, a subset of R^n, where n is the number of produced or consumed goods and services in the economy; X is a collection of production possibilities sets; the relation $\trianglerighteq$ is consistent with the set inclusion relation ($x \supseteq x' \;\Rightarrow\; x \trianglerighteq x'$) and orders production possibilities sets – production capacities – from less to more productive; we think of I as the rate of potential economic growth (here we adopt the "naive" definition of potential economic growth as a long-term expansion of the productive potential of the economy) (see Example 1 below for details).

Problems very similar to those described above are studied in axiomatic index number theory with the variable being the price/quantity vector of goods and services in the economy and time of the observations being fixed (e.g., see Balk, 2008). Investment valuation is another closely related field in the case of monetary values of the variable: several generalizations of I_0 are axiomatized by Promislow and Spring (1996) and Vilenskii and Smolyak (1998). However, in this paper we will try to show that the problem is along the lines of intertemporal choice. More precisely, there is a dual representation of the average growth rate as a parametric family of time preferences on the "prize-time" space (as defined by Ok and Masatlioglu, 2007, §2.1) with the parameter representing subjective discount rate. The axioms and assumptions on I impose restrictions on all the elements of the family; these restrictions are found to be well-known axioms that are usually used to characterize time preferences: transitivity, stationarity, homotheticity, separability, etc.

The paper is organized as follows. In section 2, in order to formalize the concept of average growth rate we introduce the main object of our analysis, called ARG. An ARG (an abbreviation for "average rate of growth") is a total preorder on the set of pairs of the variable dated observations with the help of which average growth rates can be compared;

the function I above is treated as a numerical representation of an ARG. A one-to-one correspondence between ARGs and particular indexed families of time preferences on the “prize-time” space is established in section 3. In section 4, the correspondence is used to derive representations for ARGs from the families of time preferences that are basic in intertemporal choice theory. In section 5, our main result is presented: all of the ARGs introduced in section 4 can be characterized by a combination of a few easily interpretable axioms that are similar to assumptions 3–5 above. Two important special cases of an ARG are studied in section 6: the case when the economic variable under consideration takes monetary values and the case of probabilistic uncertainty with respect to values of the variable. All proofs are given in the Appendix.

2. Primitives

Let $(\mathrm{X}, d_{\mathrm{X}})$ be a separable metric space. The set X represents the *domain* of an economic variable under consideration. Elements of X are treated as *values* of the variable. We define $\mathrm{P} := \mathrm{X} \times \mathrm{R}$ and $\mathrm{V} := \{(x, t; x', t') \in \mathrm{P}^2 : t < t'\}$. Elements (x, t) of P are referred to as *dated values,* dated observations of the variable. We metrize R by the Euclidean metric; P, P^2, V, and V^2 are metrized by the product metric (note that they are separable as the direct products of separable spaces).

In order to formalize the concept of average growth rate of the variable the following two major simplifications are made: path independence (growth over the specified period is fully characterized by the initial and final values of the variable) and ordinality (comparability of average growth rates is the only feature required). Path independence allows us to use a pair of dated observations of the variable, an element of V, as a primitive. Ordinality means that the average rate of growth is measured on an ordinal scale and induces a total preorder on V. So average growth rates in this paper are compared by means of a binary relation $\succeq$ on the set V of ordered pairs of dated values. The statement $(x, t; x', t') \succeq (y, \tau; y', \tau')$ means that the average rate of growth over the period from t to t' with the initial value x and the final value x' is no less than the average rate of growth over the period from τ to τ' with y and y' as the initial and final values respectively.

The following definition introduces the main object of our analysis.

Definition 1.

A binary relation $\succeq$ on V (with $\sim$ and $\succ$ being the symmetric and asymmetric parts of $\succeq$) is said to be an *average rate of growth* (*ARG*) for X if the following conditions are satisfied:

A1. *Total preorder*: $\succeq$ is complete and transitive.

A2. *Continuity*: $\succ$ is an open subset of V^2.

A3. *Time consistency*:

$$(x,t;x',t') \succeq (x,t;y',t') \Rightarrow (y,\tau;x',\tau') \succeq (y,\tau;y',\tau') \ \forall\, \tau < \tau', y \in \mathrm{X};$$

$$(x,t;x',t') \succeq (x,t;y',t') \Leftrightarrow (y',t;x,t') \succeq (x',t;x,t').$$

A1 and A2 are standard assumptions. By the theorem of Debreu (1954), they guarantee that an ARG has a continuous numerical representation $I : \mathrm{V} \to \mathrm{R}$:

$$v \succeq v' \Leftrightarrow I(v) \geq I(v'). \tag{2}$$

In what follows, the function I will be called an *index* of the ARG.

Under A1 and A2, assumption A3 is equivalent to the existence of a continuous total preorder $\trianglerighteq$ on X (with $\approx$ and $\triangleright$ being the symmetric and asymmetric parts of $\trianglerighteq$) such that

$$\begin{aligned} (x,t;x',t') \succeq (x,t;y',t') &\Leftrightarrow x' \trianglerighteq y'; \\ (x,t;x',t') \succeq (y,t;x',t') &\Leftrightarrow y \trianglerighteq x \end{aligned} \tag{3}$$

(compare with the consistency assumption (A) in the Introduction).

The intuition behind the relation $\trianglerighteq$ is as follows. An economic agent is assigned to the problem of average rate of growth measurement. The relation $\trianglerighteq$ describes the agent's preferences on X so that given $(x,t;x',t')$

if $x' \vartriangleright x$ then the agent detects growth of the variable over the period from t to t' (compare with the definition of growth in the Introduction). By (3), the preferences are assumed to be stable over time. In what follows, a continuous numerical representations of $\trianglerighteq$ is called a *utility function* of the agent.

We denote $\mathrm{V}_+ := \{(x,t;x',t') \in \mathrm{V} : x' \vartriangleright x\}$ (the growth set), $\mathrm{V}_0 := \{(x,t;x',t') \in \mathrm{V} : x \approx x'\}$ (the permanence set), and $\mathrm{V}_- := \{(x,t;x',t') \in \mathrm{V} : x \vartriangleright x'\}$ (the decrease set). An ARG is *nondegenerate* if $\mathrm{V}_0 \neq \mathrm{V}$. An ARG is said to be *regular* if for any $v \in \mathrm{V}$ the sets

$$\overline{\mathrm{U}}(v) := \{v' \in \mathrm{V} : v' \succeq v\} \cup \{(x,t;x',t') \in \mathrm{P}^2 : x' \trianglerighteq x,\ t = t'\},$$

$$\overline{\mathrm{L}}(v) := \{v' \in \mathrm{V} : v \succeq v'\} \cup \{(x,t;x',t') \in \mathrm{P}^2 : x \trianglerighteq x',\ t = t'\}$$

are closed in P^2. Regularity requires ARG to be time sensitive and means that the average rate of growth for $v \in \mathrm{V}_+$ ($v \in \mathrm{V}_-$) can be set arbitrarily high (resp. low) by reducing the delay. Indeed, under regularity, given $x' \vartriangleright x$ ($x \vartriangleright x'$) and t, for any $v \in \mathrm{V}$ there exists $\varepsilon > 0$ such that $(x,t;x',t') \succ v$ (resp. $v \succ (x,t;x',t')$) for all $t' \in (t, t+\varepsilon)$.

Clearly, an index of an ARG is defined up to an order-preserving transformation. For instance, a wide range of rate of return measures used in finance are order-preserving transformations of I_0 and, thus, define the same ARG: $I_0 - r$, where r is the inflation rate, is the real effective continuously compounded rate of return, e^{I_0} is the nominal compound index of growth, $e^{I_0} - 1$ is the nominal compound growth rate, etc.

The next proposition states that an index of an ARG can be decomposed into two components (the proposition follows immediately from Definition 1 and the proof is omitted).

Proposition 1.

A binary relation $\succeq$ on V is an ARG if and only if there exist continuous functions $u:\mathrm{X}\to\mathrm{R}$ and $q:\{(y,t;y',t')\in(u(\mathrm{X})\times\mathrm{R})^2 : t<t'\}\to\mathrm{R}$ such that q is strictly decreasing in its first argument, strictly increasing in its third argument, and

$$I(x,t;x',t') := q\big(u(x),t;u(x'),t'\big) \tag{4}$$

is a numerical representation (2) of $\succeq$.

According to Proposition 1, an index of an ARG is completely determined by the functions u and q, where u is a utility function that describes the agent's preferences over X and q is itself an index of an ARG for the set $u(\mathrm{X})\subseteq\mathrm{R}$ with the natural order on it. Despite being trivial, Proposition 1 captures the point of an ARG: ARG actually measures the average growth rate of the agent's utility, regardless of the nature of the variable considered. Thus, without loss of generality, we may study ARGs for subsets of R with the natural order on them.

Example 1. Measuring the rate of potential economic growth

In economic theory, potential economic growth is sometimes defined as an increase in the capacity of an economy to produce goods and services. Describing the production capacity by means of the production set, the set of all feasible production plans, one can measure the rate of potential economic growth through the rate of expansion of the production set. We try to formalize this highly stylized observation by virtue of an ARG.

Let $(\mathrm{Y},d_{\mathrm{Y}})$ be a separable metric space. An element of Y is referred to as production plan. The set of all feasible (at a given state of the economy) production plans, the production set, is a nonempty compact (resources are assumed to be limited) subset of Y. A collection X of production sets endowed with the Hausdorff metric $d_{\mathrm{X}}(x,x') := \max\left\{\sup_{y\in x}\inf_{y'\in x'} d_{\mathrm{Y}}(y,y'),\ \sup_{y'\in x'}\inf_{y\in x} d_{\mathrm{Y}}(y,y')\right\}$ is a separable metric space (e.g., see Aliprantis and Border, 2006, corollary 3.90 and theorem 3.91). Thus one can define an ARG $\succeq$ for the domain X with the economic variable representing production capacity of the economy. A reasonable restriction on the corresponding relation $\trianglerighteq$ is the consistency with the set inclusion relation:

$$x \subseteq x' \;\Rightarrow\; x' \trianglerighteq x\,. \tag{5}$$

To determine $\trianglerighteq$ let us assume that the associated economic agent is a social planner whose preferences over production plans Y are represented by a continuous utility function w. w induces a natural ordering on X:

$$x \trianglerighteq x' \;\Leftrightarrow\; \max_{y \in x} w(y) \geq \max_{y' \in x'} w(y')\,.$$

Then (5) holds and $\trianglerighteq$ is continuous by Berge's maximum theorem (Aliprantis and Border, 2006, theorems 17.15 and 17.31) as needed. Identifying an index of potential economic growth with a numerical representation I of $\succeq$, we get (4) with $u(x) = \max_{y \in x} w(y)$.

3. A representation of ARG by a collection of time preference relations

The introduced notion of ARG can be described by means of the agent's time preferences relations over the set of dated values P, the "prize-time" space, as defined by Ok and Masatlioglu (2007).

Definition 2.

A binary relation $\underline{\blacktriangleright}$ on P (with $\sim$ and $\blacktriangleright$ being the symmetric and asymmetric parts of $\underline{\blacktriangleright}$) is said to be a *time preference* if the following conditions are satisfied:

B1. *Completeness*: $\underline{\blacktriangleright}$ is complete.

B2. *Continuity*: $\blacktriangleright$ is an open subset of P^2.

B3. *Weak transitivity*: there exists a total preorder $\trianglerighteq$ on X such that if $(x,t) \,\underline{\blacktriangleright}\, (x',t')$ and $x' \trianglerighteq x''$, then $(x,t) \,\underline{\blacktriangleright}\, (x'',t')$. Moreover, if either antecedent inequality is strict, so is the conclusion.

Definition 2 permits time preference to be intransitive. This is consistent with some empirical evidence against transitivity of time preference (e.g., see Roelofsma and Read, 2000; Read, 2001). However, according to precondition B3, the only possible source of intransitivity is the passage of time. Intransitivity in prizes (values) is not allowed. This type of intransitivity is unrealistic, at least in the context of monetary prizes, as would conflict with the monotonicity of preferences for money (e.g., see Ok and Masatlioglu, 2007, p. 218).

Definition 2 is stronger than the one given by Ok and Masatlioglu (2007, §2.1).[1] Indeed, B3 with $(x,t)=(x',t')$ implies

$$(x,t)\ \underline{\blacktriangleright}\ (x'',t) \ \Leftrightarrow\ x\ \trianglerighteq\ x''$$

as is assumed instead of B3 in their definition. However, the definition covers most of the time preference models considered in the experimental and theoretical literature on intertemporal choice.

A time preference $\underline{\blacktriangleright}$ is said to be *well-behaved* if there exist x, x', and $t \neq t'$ such that $(x,t) \sim (x',t')$. Put differently, a well-behaved time preference has a nontrivial indifference relation. Under rather general assumptions, a time preference is well-behaved (e.g. if X is connected and $\triangleright$ is nonempty). Time preferences that are not well-behaved turn out to be insensitive to time delay in the sense that $(x,t) \blacktriangleright (x',t')$ whenever $x \triangleright x'$ and are excluded from consideration.[2]

[1] We depart slightly from the original formulation of Ok and Masatlioglu (2007), who, in fact, consider time preferences over the space $\mathrm{X}\times[0,\infty)$ rather than $\mathrm{X}\times\mathrm{R}$.

[2] Let a time preference $\underline{\blacktriangleright}$ be not well-behaved. Fix $x \triangleright x'$ and define the disjoint sets $\mathrm{T}_1 := \{(t,t') : (x,t)\ \underline{\blacktriangleright}\ (x',t')\}$, $\mathrm{T}_2 := \{(t,t') : (x',t')\ \underline{\blacktriangleright}\ (x,t)\}$. Since $\underline{\blacktriangleright}$ is continuous and complete, T_1 and T_2 are closed in R^2 and $\mathrm{T}_1 \cup \mathrm{T}_2 = \mathrm{R}^2$. By connectedness of R^2, $\mathrm{T}_1 \in \{\varnothing, \mathrm{R}^2\}$ and $\mathrm{T}_2 = \mathrm{R}^2 \setminus \mathrm{T}_1$. Since $(x,t)\ \underline{\blacktriangleright}\ (x',t)$ for all t, we conclude that $\mathrm{T}_1 = \mathrm{R}^2$. Thus, $(x,t) \blacktriangleright (x',t')$ whenever $x \triangleright x'$.

Throughout this section the associated economic agent is assumed to have state-dependent intertemporal preferences. That is he/she has no unique time preference, but rather by a collection of time preferences depending on the subjective discount rate.

Definition 3.

A collection of well-behaved time preferences $\{\underline{\blacktriangleright}_d,\ d \in \mathrm{D} \subseteq \mathrm{R}\}$ on P is called a *TP family* if the following conditions are satisfied:

C1. *Consistency*: for all $\underline{\blacktriangleright}_d$ B3 holds with the same total preorder $\unrhd$.

C2. *Monotonicity*: $(x,t)\ \underline{\blacktriangleright}_d\ (x',t'),\quad t<t' \quad \Rightarrow \quad (x,t)\ \blacktriangleright_{d'}\ (x',t')$ $\forall\, d' > d$.

C3. *Solvability*: $\forall\,(x,t),(x',t') \in \mathrm{P},\ t<t'\ \ \exists d \in \mathrm{D}:\ (x,t)\ \sim_d\ (x',t')$.

Conditions C1–C3 have obvious interpretations. According to C1, $\underline{\blacktriangleright}_d$ is complied (in the sense of B3) with the same total preorder $\unrhd$ that describes the agent's preferences over undated values. As in the previous section, a continuous numerical representations of $\unrhd$ is called a utility function of the agent. By C2, the increase in d makes the later value of x and x' less preferable. This allows us to interpret d as the subjective discount rate. Finally, according to C3, for any dated values (x,t) and (x',t') with $t \neq t'$ there exists a discount rate d with respect to which (x,t) and (x',t') are indifferent. Condition C3 implies that a TP family is rich enough to include "positive" time preferences as well as "zero" and "negative" ones. Indeed, given $t<t'$ C3 holds regardless of the fact whether $(x,t;x',t')$ is in V_+, V_0, or V_-.

Each TP family $\{\underline{\blacktriangleright}_d,\ d \in \mathrm{D}\}$ induces a regular ARG $\succeq$ by

$$(x,t;x',t') \succeq (y,\tau;y',\tau') \iff \forall d \in \mathrm{D} \text{ if } (x,t)\ \underline{\blacktriangleright}_d\ (x',t'), \text{ then } (y,\tau)\ \underline{\blacktriangleright}_d\ (y',\tau'). \tag{6}$$

One can prove (see Proposition 2 below) that the constructed relation $\succeq$ is well defined and is indeed a regular ARG with the numerical representation I (2) of the form:

$$I(x,t;x',t') = d \;\Leftrightarrow\; (x,t) \sim_d (x',t'). \tag{7}$$

Conversely, a regular ARG with an index I generates a TP family $\{\underline{\blacktriangleright}_d,\ d \in \mathrm{D} := I(\mathrm{V})\}$ by the rule

$$(x,t)\ \underline{\blacktriangleright}_d\ (x',t') \;\Leftrightarrow\; \begin{cases} I(x,t;x',t') \le d & \text{if } t < t' \\ x \trianglerighteq x' & \text{if } t = t' \\ I(x',t';x,t) \ge d & \text{if } t > t' \end{cases}. \tag{8}$$

One can check that $\{\underline{\blacktriangleright}_d,\ d \in \mathrm{D}\}$ is indeed a TP family (Proposition 2). For instance, the ARG for $\mathrm{X} = \mathrm{R}_{++}$ with the index I_0 (1) generates the TP family with the present value representation: for any $d \in \mathrm{D} := \mathrm{R}$

$$(x,t)\ \underline{\blacktriangleright}_d\ (x',t') \;\Leftrightarrow\; e^{-dt}x \ge e^{-dt'}x'. \tag{9}$$

Conversely, the TP family (9) induces I_0 by (7).

Now we present the main result of this section.

Proposition 2.

(6) and (8) define a one-to-one correspondence between the set of regular ARGs and the set of TP families.[3]

The established representation of a regular ARG by a TP family allows us to introduce the following basic procedure for the construction of an ARG. The agent's intertemporal preferences are represented by a TP family $\{\underline{\blacktriangleright}_d,\ d \in \mathrm{D}\}$ with the parameter d interpreted as a subjective discount rate. The family induces an ARG $\succeq$ with the index that takes each $(x,t;x',t') \in \mathrm{V}$ to a unique discount rate d, the calibration parameter, that equalizes (x,t) and (x',t') (7). The ARG defined this way represents the

[3] We identify TP families that are order preserving reparameterizations of each other.

"internal rate" of utility growth (by analogy with the internal rate of return in capital budgeting). This procedure is used in the next section to construct ARGs related to the TP families that are important in intertemporal choice theory.

4. Examples

In this section we introduce six representations for ARGs by means of the TP families that are standard in intertemporal choice theory. All the representations will be axiomatized in the next section with the aid of the properties similar to 3–5 in the Introduction.

Index I_1: ARGs induced by the exponential discounting model

Let elements of a TP family $\{\succsim_d,\ d \in \mathrm{D}\}$ be represented in the form of the basic exponential discounting model

$$(x,t) \succsim_d (x',t') \Leftrightarrow e^{-dt}U(x) \geq e^{-dt'}U(x'), \tag{10}$$

where $U : \mathrm{X} \to \mathrm{R}_{++}$ is a utility function and d is the discount rate. Then the procedure of section 3 induces the ARG with the representation

$$I_1(v) = \frac{u(x') - u(x)}{t' - t}, \tag{11}$$

where $u := \ln U$. The functional form (11), the difference quotient, emphasizes that the ARG measures the average growth rate of the agent's utility, regardless of the nature of the variable considered. Comparing (9) and (10), one may observe that I_1 generalizes I_0 to an arbitrary domain of the underlying economic variable.

Index I_2: ARGs induced by the multiplicative discounting model

Let elements of a TP family be represented in the form of the multiplicative discounting model (e.g., see Fishburn and Rubinstein, 1982, §4)

$$(x,t) \succsim_d (x',t') \Leftrightarrow U(x) f_d(t) \geq U(x') f_d(t'), \tag{12}$$

where $U : \mathrm{X} \to \mathrm{R}_{++}$ is a utility function and $\{ f_d : \mathrm{R} \to \mathrm{R}_{++},\ d \in \mathrm{D} \}$ is a family of continuous discount functions: $f_d(t)$ is the discount factor at time t and the discount rate d. For any $t < t'$ the function $d \mapsto f_d(t')/f_d(t)$, the "relative" discount factor, is strictly decreasing. The TP family induces the ARG with the index I_2 that takes each $(x,t;x',t') \in \mathrm{V}$ to a unique solution d of the equation

$$U(x) f_d(t) = U(x') f_d(t'), \tag{13}$$

the "internal rate" of utility growth. Comparing (10) and (12), we conclude that I_2 extends I_1 to an arbitrary multiplicative discounting scheme in time preferences.

Index I_3: ARGs induced by transitive time preferences of a general form

A further generalization of I_0 is obtained from the transitive TP family of a general form

$$(x,t) \succeq_d (x',t') \iff h_d\big(u(x),t\big) \geq h_d\big(u(x'),t'\big), \tag{14}$$

where u is a utility function, h_d is continuous and strictly increasing in its first argument. The family induces the ARG with the index I_3 that takes each $(x,t;x',t') \in \mathrm{V}$ to a unique solution d of the equation

$$h_d\big(u(x),t\big) = h_d\big(u(x'),t'\big). \tag{15}$$

To give the intuitive meaning of h_d we assume that there exists time t_0, a reference point, such that

$$h_d\big(u(x),t_0\big) = u(x) \ \forall\, x, d\,. \tag{16}$$

Then $h_d\big(u(x),t\big)$ is the discounted (to time t_0 at the rate d) utility of x that is to be obtained at time t. The representation (14) is the most general form of absolute discounting in the sense that preferences between dated values are evaluated through a present value computation. To demonstrate the validity of this interpretation and motivate Eq. (15) we use (slightly modified) reasoning of Manca (1969). For a fixed utility function u denote $\mathrm{Y} := u(\mathrm{X})$. Let $w : \mathrm{Y} \times \mathrm{R}^2 \to \mathrm{Y}$ be the "present value operation" that maps a dated value (x, t) to its worth $w(u(x), t, t')$ at time t'. A reasonable condition on w is given by

$$w(y, t, t'') = w(w(y, t, t'), t', t''). \tag{17}$$

Loosely speaking, Eq. (17) states that discounting is path independent. In addition, we assume that for each t and t' the function $w(\cdot, t, t')$ is a bijection of Y onto itself, that is delay can always be compensated by utility change. As a corollary, the worth of (x, t) at time t is given by $u(x)$: $w(u(x), t, t) = u(x) \;\; \forall\, x$. Under the aforementioned assumption, the general solution of the functional equation (17) is given by (Ng, 1978, corollary 3.7)

$$h(w(y, t, t'), t') = h(y, t), \tag{18}$$

where $h : \mathrm{Y} \times \mathrm{R} \to \mathrm{Y}$ is an arbitrary function such that for each t $h(\cdot, t)$ is a bijection of Y onto itself. The result remains valid if Eq. (17) holds for $t \le t' \le t''$ or $t \ge t' \ge t''$. Moreover, for any fixed time t_0, the reference point, h can be chosen such that $h(y, t_0) = y \;\; \forall\, y$. Thus, $h(u(x), t)$ represents the discounted (to time t_0) utility of dated value (x, t): $h(u(x), t) = h(w(u(x), t, t_0), t_0) = w(u(x), t, t_0)$. These arguments justify the functional form of the index I_3 and validate the intuitive meaning of $h_d(u(x), t)$. The time preference model with contemplation costs (Ok and Masatlioglu, 2007, p. 228) is an example of a time preference relation represented by (14) that is not reduced to multiplicative discounting (12).

Index I_4: ARGs induced by stationary transitive time preferences

An important special case of Eq. (15) with the functions h_d represented in the form

$$h_d(y,t) = g_d(y) + \delta_d t, \tag{19}$$

where $\delta_d \in \{-1,0,1\}$, g_d is continuous and strictly increasing, is studied by Eichhorn (1978) (see also Castillo et al., 2005, example 5.3). This functional form is consistent with the assumption of stationarity of the "present value operation" w:

$$w(y,t,t') = w(y,t+\tau,t'+\tau) \ \ \forall\tau . \tag{20}$$

Indeed, under weak regularity conditions (see Moszner, 1989, theorem 3 for details), the general continuous solution of the system of functional equations (17) and (20) is given by (18) with $h(y,t) = g(y) + \delta t$, $\delta \in \{-1,0,1\}$, where the function g is strictly increasing and onto R. In what follows, we denote by I_4 the index of an ARG that takes each $(x,t;x',t') \in \mathrm{V}$ to a unique solution d of Eq. (15) with the functions h_d of the form (19). Note: if $g_d(y) = \begin{cases} y/|d| & \text{if } d \neq 0 \\ y & \text{otherwise} \end{cases}$ and $\delta_d = -\operatorname{sgn} d$, then I_4 reduces to I_1.

Now let us turn to intransitive time preference models.

Index I_5: ARGs induced by stationary intransitive time preferences

Let elements of a TP family be represented in the form

$$(x,t) \succsim_d (x',t') \ \Leftrightarrow \ u(x) \geq h\big(u(x'), d(t'-t)\big), \tag{21}$$

where h is continuous, strictly increasing in its first argument, strictly decreasing in its second argument, and $h(y,t) = y' \ \Leftrightarrow \ h(y',-t) = y$ (so that $h(y,0) = y$). Here $h\big(u(x'), d(t'-t)\big)$ is interpreted as the worth at time t of the dated value (x',t') under the discount rate d. The functional form (21) incorporates a number of (not necessarily transitive) time preference models that assume dependence on dates through the delay $t'-t$ (e.g., see Read, 2001; Scholten and Read, 2006; Ok and Masatlioglu, 2007, §3.3). The TP family induces the ARG with the representation

$$I_5(v) := \frac{g\big(u(x), u(x')\big)}{t' - t}, \quad (22)$$

where $g(\cdot, y')$ is the inverse to $h(y', \cdot)$ (by the definition of a TP family, Definition 3, the inverse is well defined). One may think of $g(y, \cdot)$ as a gain-loss (or relative) utility given the reference utility level y. The properties of h imply that g is continuous, skew-symmetric ($g(y', y) = -g(y, y')$), and $g(y, \cdot)$ is strictly increasing. The functional form (22) formalizes the intuition that an index of an ARG is the total change divided by the time taken for that change.

Index I_6: ARGs induced by path-dependent intransitive time preferences

Let elements of a TP family be represented in the form

$$(x, t) \trianglerighteq_d (x', t') \Leftrightarrow g\big(u(x), u(x')\big) \leq f_d(t, t'), \quad (23)$$

where the functions f_d and g are continuous and skew-symmetric, for any $t < t'$ and y the functions $f_\bullet(t, t')$ and $g(y, \cdot)$ are strictly increasing. An element of this family is referred to as path-dependent time preference of Ok and Masatlioglu (2007, §4.2). A similar representation is used in the tradeoff model of Scholten and Read (2010). Here g is a gain-loss utility function and $\exp\big(-f_d(t, t')\big)$ is interpreted as a "relative" (in terms of Ok and Masatlioglu) discount factor between dates t and t' at the discount rate d. The TP family induces the ARG with the index

$$I_6(v) := h\big(g\big(u(x), u(x')\big), t, t'\big), \quad (24)$$

where $h(\cdot, t, t')$ is the inverse to $f_\bullet(t, t')$ (by condition C3, the function $h(\cdot, t, t')$ is well defined provided that $t < t'$).

5. Axiomatizations

In this section axiomatizations of the introduced representations I_k, $k = 1,...,6$ are given by means of the hypotheses (axioms) that seem to be reasonable when considering "average rate of growth" and are related to assumptions 3–5 in the Introduction.

5.1. Hypotheses

Throughout this subsection $\succeq$ is an ARG; all hypotheses are imposed on $\succeq$. We do not try to keep the number of hypotheses used to a minimum. Instead, we try to state the hypotheses so that they are conceptually distinct and apparently simple, even at the cost of increase of their number.

(i) *Time sensitivity*: A) if $x' \rhd x$ ($x \rhd x'$), then $(x,\tau;x',\tau') \succ (x,t;x',t')$ whenever $[\tau,\tau'] \subset [t,t']$ (resp. $[t,t'] \subset [\tau,\tau']$); $v \sim v'$ whenever $v, v' \in \mathrm{V}_0$;

B) given $v = (x,t;x',t')$ and v' in V, if either $v, v' \in \mathrm{V}_+$ or $v, v' \in \mathrm{V}_-$, then there exist t_1, t_2, t_1', t_2' such that $(x,t_1;x',t') \succeq v' \succeq (x,t_2;x',t')$ and $(x,t;x',t_1') \succeq v' \succeq (x,t;x',t_2')$.

(ii) *Value sensitivity*: for every $(x,t;x',t'), v' \in \mathrm{V}$ there exist y and y' in X such that $(y,t;x',t') \sim v' \sim (x,t;y',t')$.

(iii) *Internality*: if $t < t' < t''$, then $(x,t;x',t') \succeq (x',t';x'',t'') \Leftrightarrow (x,t;x',t') \succeq (x,t;x'',t'') \Leftrightarrow (x,t;x'',t'') \succeq (x',t';x'',t'')$.

(iv) *Skew-symmetry*: $(x,t;x',t') \succeq (y,\tau;y',\tau')$ implies $(y',\tau;y,\tau') \succeq (x',t;x,t')$.

(v) *Separability*: $(x,t;x',t') \succeq (y,t;y',t')$ implies $(x,\tau;x',\tau') \succeq (y,\tau;y',\tau')$ for any $\tau < \tau'$.

(vi) *Time-shift invariance*: there exists a strictly increasing function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ such that $(x,t;x',t') \sim (x,\varphi^{-1}(\varphi(t)+\tau);x',\varphi^{-1}(\varphi(t')+\tau))$ for any $\tau \in \mathrm{R}$.

(vii) *Interval scalability*: there exists a strictly increasing function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ such that $(x,t;x',t') \succeq (x,\tau;x',\tau')$ implies $(x,\varphi^{-1}(a\varphi(t)+b);x',\varphi^{-1}(a\varphi(t')+b)) \succeq$
$(x,\varphi^{-1}(a\varphi(\tau)+b);x',\varphi^{-1}(a\varphi(\tau')+b))\ \forall\, a>0\,, b$.

Note: hypotheses (i)–(vii) are not independent (for instance, (i) and (vii) imply (vi) (see Proposition 3 in the Appendix); (vii) implies (v)). Before interpreting them, we observe that the hypotheses impose restrictions on all the elements of the induced TP family. Some of the restrictions are found to be well-known axioms that are usually used in intertemporal choice theory to characterize time preference relations. So a part of the interpretations are given in the context of the TP family.

Hypothesis (i) (part A) is closely related to axiom A3 of Fishburn and Rubinstein (1982) and monotonicity axiom of Ok and Masatlioglu (2007) on time preference. This states that an ARG is strictly decreasing (increasing) with respect to delay if $v \in \mathrm{V}_+$ (resp. $v \in \mathrm{V}_-$) (an analogue of the impatience condition of Fishburn and Rubinstein) and is independent of the delay if $v \in \mathrm{V}_0$ (an analogue of the procrastination condition). Thus, delay is undesirable in the case of growth of the variable and desirable in the case of decrease. One can also interpret the procrastination condition as a generalization of the identity test in index number theory (e.g., see Balk, 2008, §3.2). In the context of the induced TP family $\{\underline{\blacktriangleright}_d,\ d \in \mathrm{D}\}$, (i) (part A) says that the family can be decomposed into 3 components $\mathrm{D} = \mathrm{D}_0 \cup \mathrm{D}_+ \cup \mathrm{D}_-$, the set of "zero" ($\mathrm{D}_0$), "positive" ($\mathrm{D}_+$), and "negative" ($\mathrm{D}_-$) time preferences, where

$$\mathrm{D}_0 := \{d_0\},\ \text{where}\ (x,t)\ \underline{\blacktriangleright}_{d_0}\ (x',t') \iff x \trianglerighteq x',$$

$$\mathrm{D}_+ := \{d \in \mathrm{D} : d > d_0\} = \{d \in \mathrm{D} : (x,t)\ \underline{\blacktriangleright}_d\ (x',t'),\ t<t' \Rightarrow$$
$$(x,\tau)\ \blacktriangleright_d\ (x',\tau')\ \ [t,t'] \subset [\tau,\tau']\},$$

$$\mathrm{D}_- := \{d \in \mathrm{D} : d < d_0\} = \{d \in \mathrm{D} : (x,t)\ \underline{\blacktriangleright}_d\ (x',t'),\ t<t' \Rightarrow$$
$$(x,\tau)\ \blacktriangleright_d\ (x',\tau')\ \ [\tau,\tau'] \subset [t,t']\}.$$

(the interpretation is valid, since hypothesis (i) implies an ARG to be regular). Since a TP family is defined up to an order preserving reparameterization, without loss of generality, we may assume that $d_0 = 0$. Hypothesis (i) (part B) is an analogue of time sensitivity axiom of Ok and Masatlioglu (2007). This means that average growth rate can be set arbitrarily low/high by changing delay. For example, if a price index doubled between t and t', where $t' - t$ is a quarter of a century, then inflation is in the usual target range, but if $t' - t$ is a year, then there is hyperinflation.

Hypothesis (ii) parallels closely to outcome sensitivity axiom of Ok and Masatlioglu (2007). According to (ii), every possible magnitude of the average growth rate can be obtained by changing values. Loosely speaking, both (i) (part B) and (ii) state that time and value are equivalent in some sense: delay can always be compensated by changing in values and vice verse.

Hypothesis (iii) formalizes the term “average” in the abbreviation “ARG”. This property is an ordinal formulation of the idea that the average growth rate over the consolidated period should fall between average growth rates over its subperiods (compare with assumption 5 in the Introduction). In other words, the consolidated index of an ARG is a Cauchy mean of the partial indices. The hypothesis validates the following “obvious” guidance: to guarantee the target level of an average growth rate over the given period it suffices to keep the target on each subperiod. A closely related axiom is used by Vilenskii and Smolyak (1998) to characterize internal rate of return. A weaker form of this hypothesis, the weak monotonicity axiom, is also used to characterize difference measurement structures (Krantz et al., 1971, chapter 4). For a regular ARG, (iii) holds if and only if all elements of the induced TP family are transitive. Rather natural in the rate of growth context, this assumption, however, does not allow to describe some “anomaly” effects of intertemporal choice, in particular, subadditivity (Read, 2001). In the language of the present model, subadditivity holds if

$$(x,t;x',t') \sim (x',t';x'',t'') \Rightarrow (x,t;x'',t'') \succ (x,t;x',t').$$

Hypothesis (iv) is a generalization of the time reversal test in index number theory (see, for instance, Balk, 2008, §3.4.1). This states that an ARG evaluates growth and decrease in a similar way. Indeed, given a restriction of an ARG to the growth set $\mathrm{V}_+ \times \mathrm{V}_+$, (iv) describes how it could be extended to the decrease set $\mathrm{V}_- \times \mathrm{V}_-$. A similar sign-reversal type axiom is also used in measurement theory to characterize algebraic-difference structures (Krantz et al., 1971, §4.4). Though most of the introduced examples of ARGs (namely, the representations I_0, I_1, I_2, I_5, and I_6) satisfy (iv), in the TP family context, the hypothesis contradicts some empirical studies (Thaler, 1981; Benzion et al., 1989) that discover the sign effect, an asymmetry in evaluating gains (the case of growth) and losses (the case of decrease).

Hypothesis (v) ensures the separation of the effects of time and values in an ARG. This states that the restriction of the ordering $\succeq$ to simultaneous entities is independent of dates. This is the classical separability condition (also known as the independence condition), which is used to characterize decomposable representations (e.g., see Krantz et al., 1971, definition 11, p. 301; Fishburn and Rubinstein, 1982, axiom B2).

Hypothesis (vi), a stationarity-like assumption, is a generalization of assumption 4 in the Introduction. To motivate (vi) let us assume first that φ is the identity transformation. Then (vi) holds if and only if elements of the induced TP family are stationary in the sense of axiom A5 of Fishburn and Rubinstein (1982):

$$(x,t) \trianglerighteq_d (x',t') \Rightarrow (x,t+\tau) \trianglerighteq_d (x',t'+\tau) \ \forall \tau . \tag{25}$$

Stationarity (25) is a theoretically appealing assumption. This states that the time preference between two dated values preserves under time shift so that preference between (x,t) and (x',t') depends on t and t' through their difference $t'-t$. However, the empirical evidence points to violation of this assumption (e.g., see Thaler, 1981; Benzion et al., 1989; Loewenstein and Prelec, 1992). For instance, (25) does not hold if the agent uses nonconstant discount rate. The introduced weaker form (vi) of the usual stationarity assumption preserves invariance with respect to time shift interpretation and overcomes some of the "anomalies" in intertemporal choice, in particular, the delay effect (common difference effect, decreasing impatience). The delay effect predicts (e.g., see Thaler, 1981; Benzion et al., 1989) the decrease of discount rate as waiting time increases. More precisely, a time preference $\trianglerighteq$ satisfies *decreasing* (*constant*, *increasing*) *impatience* if for any $t < t'$

$$(x,t) \sim (x',t') \Rightarrow (x',t'+\tau) \trianglerighteq \text{(resp. } \sim, \text{not } \triangleright) \ (x,t+\tau) \ \forall \tau > 0$$

$$\text{provided that } x' \trianglerighteq x;$$

$$(x,t) \sim (x',t') \Rightarrow (x,t+\tau) \trianglerighteq \text{(resp. } \sim, \text{not } \triangleright) \ (x',t'+\tau) \ \forall \tau > 0$$

$$\text{provided that } x \trianglerighteq x'.$$

Now let (i) and (vi) hold (so that $\succeq$ is regular), then elements of the induced TP family satisfy decreasing (constant, increasing) impatience if and only if the transformation φ in (vi) is concave (resp. linear, convex).

We think of the transformation φ as a time change. φ reflects the varying "speed" of operational time and is chosen such that in the "new" time scale the subject discount rate of the agent is constant and, therefore, in the "new" time scale elements of the induced TP family are stationary. Hypothesis (vi) is quite weak. E.g., this does not contradict the exponential discounting TP family with time-varying discount rate

$$(x,t) \trianglerighteq_d (x',t') \Leftrightarrow e^{-d\varphi(t)}U(x) \geq e^{-d\varphi(t')}U(x').$$

In a rather similar context φ is interpreted as time-weighing (Scholten and Read, 2010) or the time perception function which indicates how fast time is perceived to pass in the agent's mind (Ahlbrecht and Weber, 1995).

Hypothesis (vii) is "covariance with respect to an affine transformation"-like assumption and is similar to (vi). Here φ has the same interpretation as above (time change). According to (vii), the "new" time is measured on an interval scale and $\succeq$ is covariant with respect to a change of its measurement scale. Under (i) hypothesis (vii) is a combination of (vi) and the property

$$(x,t;x',t') \succeq (x,\tau;x',\tau') \Rightarrow$$
$$(x,\varphi^{-1}(a\varphi(t));x',\varphi^{-1}(a\varphi(t'))) \succeq \qquad (26)$$
$$(x,\varphi^{-1}(a\varphi(\tau));x',\varphi^{-1}(a\varphi(\tau'))) \ \forall\, a>0$$

(see Proposition 3 in the Appendix). Covariance with respect to a similarity transformation of the "new" time (26) is an obligatory feature of an ARG, since it measures change per unit of time.

5.2. Main results

In this subsection axiomatizations of the representations I_k, $k=1,...,6$ are given by means of hypothesis (i)–(vii).

Our first result states that the ARG induced by the path-dependent intransitive TP family (23) is related to the separability hypothesis (v).

Theorem 1.

Let $\succeq$ be an ARG. The following two statements are equivalent:

(a) (v) holds;

(b) there exist a utility function u and continuous real-valued functions g and h such that I_6 (24) represents $\succeq$, $g(\cdot,y')$ is strictly decreasing, $g(y,\cdot)$ and $h(\cdot,t,t')$ are strictly increasing.

If, in addition, (i) (part A) and (iv) hold in (a), then g and h in (b) can be chosen such that g is skew-symmetric, $h(\cdot,t,t')$ is odd, $h(y,\tau,\tau')>h(y,t,t')$ whenever $[\tau,\tau']\subset[t,t']$ and $y>0$.

The next theorem axiomatizes I_5 (22) by virtue of hypotheses (i) and (vii) with the identity transformation φ.

Theorem 2.

Let $\succeq$ be an ARG. The following two statements are equivalent:

(a) (i) and (vii) hold;

(b) there exist a utility function u, a strictly increasing function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$, and a continuous function $g : u(\mathrm{X})^2 \to \mathrm{R}$ such that

$$I_5'(v) := \frac{g\big(u(x), u(x')\big)}{\varphi(t') - \varphi(t)} \tag{27}$$

represents $\succeq$. $g(y, \cdot)$ is strictly increasing, $g(\cdot, y')$ is strictly decreasing, and $g(y, y) = 0$.

If, in addition, (iv) holds in (a), then g in (27) can be chosen skew-symmetric. If, further, (vii) in (a) holds with the identity transformation φ, then I_5 (22) represents $\succeq$.

According to the next result, internality hypothesis (iii) is a necessary and sufficient condition for a regular ARG to be represented by I_3 (15).

Theorem 3.

Let $\succeq$ be a regular ARG. The following two statements are equivalent:

(a) (iii) holds;

(b) $\succeq$ is induced by the TP family (14), where u is a utility function, h_d, $d \in \mathrm{D}$ are continuous and strictly increasing with respect to the first argument.

Furthermore, if (ii) also holds in (a), then for every fixed time t_0, a reference point, the representations h_d for elements of the TP family can be chosen such that the normalization condition (16) holds.

The next theorem axiomatizes I_4 (19) with particular δ_d.

Theorem 4.

Let $\succeq$ be an ARG. The following two statements are equivalent:

(a) (i), (iii), and (vi) hold;

(b) $\succeq$ is induced by the TP family $\{\succeq_d,\ d \in \mathrm{D} \supseteq \{0\}\}$ of the form

$$(x,t) \succeq_d (x',t') \Leftrightarrow$$
$$g_d\big(u(x)\big) - \varphi(t)\,\mathrm{sgn}\, d \ge g_d\big(u(x')\big) - \varphi(t')\,\mathrm{sgn}\, d\,, \tag{28}$$

where u is a utility function, the function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ is strictly increasing, the functions $g_d : u(\mathrm{X}) \to \mathrm{R}$, $d \in \mathrm{D}$ are continuous and strictly increasing, for any $y < y'$ the function $d \mapsto g_d(y') - g_d(y)$ is strictly increasing (decreasing) on the set $\mathrm{D} \cap (-\infty, 0)$ (resp. $\mathrm{D} \cap \mathrm{R}_{++}$).

In particular, if (vi) in (a) holds with the identity transformation φ, then (28) induces I_4 with $\delta_d = -\mathrm{sgn}\, d$.

A sufficient condition for an ARG to be representable through I_2 (13) is given by Theorem 5.

Theorem 5.

Let $\succeq$ be a nondegenerate ARG and let the domain X be connected. The following two statements are equivalent:

(a) (i), (ii), (iii), (iv), and (v) hold;

(b) $\succeq$ is induced by the TP family (12) with a utility function $U : \mathrm{X} \xrightarrow{\text{onto}} \mathrm{R}_{++}$ and a family $\{ f_d : \mathrm{R} \to \mathrm{R}_{++},\ d \in \mathrm{D} \supset \{0\}\}$ of continuous discount functions. If $d > 0$, f_d is strictly decreasing and onto R_{++}. $f_d = 1/f_{-d}$ for any $d \in \mathrm{D}$. For every $t < t'$ the function $d \mapsto f_d(t')/f_d(t)$ is strictly decreasing.

For any time t_0, a reference point, the family $\{ f_d,\ d \in \mathrm{D}\}$ in (b) can be chosen such that $f_\bullet(t_0) \equiv 1$.

Our next result axiomatizes I_1 (11).

Theorem 6.

Let $\succeq$ be a nondegenerate ARG and let the domain X be connected. The following two statements are equivalent:

(a) (i), (ii), (iii), (iv), (v), and (vi) hold;

(b) there exist a utility function $u : \mathrm{X} \xrightarrow{\text{onto}} \mathrm{R}$ and a strictly increasing function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ such that

$$I_1'(v) := \frac{u(x') - u(x)}{\varphi(t') - \varphi(t)} \tag{29}$$

represents $\succeq$.

If (vi) in (a) holds with the identity transformation φ, then I_1 (11) represents $\succeq$.

Finally, we axiomatize a state-dependent generalization of I_1' (29) by virtue of hypotheses (i), (iii), and (vii).

Theorem 7.

Let $\succeq$ be an ARG. The following two statements are equivalent:

(a) (i), (iii), and (vii) hold;

(b) there exist utility functions u_-, u_+ and a strictly increasing function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ such that

$$I_1''(v) := \begin{cases} \dfrac{u_+(x') - u_+(x)}{\varphi(t') - \varphi(t)} & \text{if } v \in \mathrm{V}_+ \\[2ex] \dfrac{u_-(x') - u_-(x)}{\varphi(t') - \varphi(t)} & \text{otherwise} \end{cases} \tag{30}$$

represents $\succeq$.

If, in addition, (iv) holds in (a), then u_- and u_+ in (b) can be chosen to be identical so that I_1' (29) represents $\succeq$.

Distinct utility functions u_- and u_+ for the decrease (V_-) and growth (V_+) sets in (30) are tools to model the sign effect, an asymmetry in evaluating losses and gains. Note that the functions u_- and u_+ are related by an order-preserving transformation, since they are numerical representations of the same relation $\trianglerighteq$.

Example 2. An economic agent's welfare growth rate

In this example, we introduce a measure of an economic agent's welfare growth rate by virtue of an ARG. In microeconomic theory, welfare of an agent, the economic variable considered, is usually measured on the basis of his/her income and the vector of prices he/she faces. Thus, we can define an ARG $\succeq$ for the domain $\mathrm{X} = \mathrm{R}_{++}^{n+1}$ (metrized by the Euclidean metric) and an element $x = (M, p)$ of X is treated as the agent's income M and the n-dimensional price vector p he/she faces (there are n goods in the economy). Then $\trianglerighteq$ is the indirect preference relation induced by the agent's indirect utility function U. If $\succeq$ satisfies (i), (iii), (iv), and (vii), we get (Theorem 7) that

$$I(v) = \frac{h \circ U(x') - h \circ U(x)}{\varphi(t') - \varphi(t)} \tag{31}$$

represents $\succeq$, where h is a strictly increasing and continuous function. (31) represents welfare change per transformed (by φ) time unit. For instance, if $h(\cdot)$ is chosen to be the agent's expenditure function $e(\cdot, p_0)$ with a fixed reference price vector p_0, then the numerator in (31) is the general variation (Hammond, 1994), a money metric measure of welfare change that includes the compensating and equivalent variations as particular cases. With $h(\cdot) = \ln e(\cdot, p_0)$ (31) reduces to a generalization of the Allen welfare index (e.g., see Ebert, 1984, §4; Diewert and Nakamura, 1993, §7.3) to varying delay.

Example 3. Average velocity along a reference direction

In this example, we apply ARG to characterize the formula for average velocity along a reference direction. Let X be a separable real Hilbert space. An element $x \in \mathrm{X}$ is treated as a position vector and $(x,t;x',t') \in \mathrm{V}$ expresses the change in position during the time interval from t to t'. In order to compare average changes in position (average velocities) a nondegenerate ARG $\succeq$ is defined for the domain X. Let $\succeq$ satisfy (i), (iii), (vii) with the identity transformation φ. We also assume that the average velocity of $(x,t;x',t')$ depends on the positions x and x' through the displacement $\Delta x := x' - x$:

$$(x,t;x',t') \sim (x+y,t;x'+y,t') \ \ \forall\, y \in \mathrm{X}. \tag{32}$$

Then $\trianglerighteq$ is translation invariant: $x' \trianglerighteq x \ \Rightarrow \ x'+y \trianglerighteq x+y$. Indeed, if $x' \trianglerighteq x$, then $(x+y,t;x'+y,t') \sim (x,t;x',t') \succeq (x,t;x,t') \sim (x+y,t;x+y,t')$ and, therefore, $x'+y \trianglerighteq x+y$. Thus, (30) with the identity transformation φ represents $\succeq$ (Theorem 7) and

$$u_+(x') - u_+(x) = u_+(x'+y) - u_+(x+y) \tag{33}$$

holds for any $x, x', y \in \mathrm{X}$. The general non-constant continuous solution of the Cauchy type functional equation (33) is given by $u_+(x) = \langle x, x_+ \rangle + c_+$, where $\langle \cdot,\cdot \rangle$ is the inner product, $x_+ \in \mathrm{X} \setminus \{0\}$, and c_+ is a constant. A similar argument shows that $u_-(x) = \langle x, x_- \rangle + c_-$. Since u_- and u_+ are defined up to a positive affine transformation and represent the same total preorder, c_-, c_+, x_-, x_+ can be chosen such that $c_+ = c_- = 0$, $x_+ = x_- = x_0$, and $\|x_0\| = 1$. Thus,

$$I(v) = \frac{\langle x' - x, x_0 \rangle}{t' - t} = \frac{\langle \Delta x, x_0 \rangle}{\Delta t}, \tag{34}$$

the scalar projection of Δx on x_0 over time $\Delta t := t' - t$, represents $\succeq$. In n-dimensional Euclidean space ($n \le 3$) Eq. (34) is known as the average velocity along the reference direction x_0. For instance, in the one-dimensional case $X = R$ we have that $I(v) = \pm \Delta x / \Delta t$ represents $\succeq$.

Alternatively, to get representation (34), we may replace assumption (32) with invariance with respect to a change of scale (scale invariance) and reference point (translation invariance):

$$(x,t;x',t') \succeq (y,\tau;y',\tau') \;\Rightarrow$$
$$(\lambda x + z,t;\lambda x' + z,t') \succeq (\lambda y + z,\tau;\lambda y' + z,\tau') \;\; \forall\, \lambda > 0\,,\; z \in X\,. \tag{35}$$

Indeed, from (35) it follows that $x' \trianglerighteq x \;\Rightarrow\; \lambda x' + z \trianglerighteq \lambda x + z$. Setting $t = \tau$ and $t' = \tau'$ in (35) and using representation (30), we get

$$u_+(x') - u_+(x) \geq u_+(y') - u_+(y) \;\Rightarrow$$
$$u_+(\lambda x' + z) - u_+(\lambda x + z) \geq u_+(\lambda y' + z) - u_+(\lambda y + z)\,.$$

Thus, $x \mapsto u_+(x)$ and $x \mapsto u_+(\lambda x + z)$ are related by a positive affine transformation (Basu, 1983):

$$u_+(\lambda x + z) = a(\lambda, z)u_+(x) + b(\lambda, z),\; a(\lambda, z) > 0\,. \tag{36}$$

The general non-constant continuous solution of the functional equation (36) is given by $u_+(x) = \langle x, x_+ \rangle + c_+$, where $x_+ \in X \setminus \{0\}$ and c_+ is a constant (the proof is a minor modification of that given by Aczél et al. (1986)). Hence, (34) holds.

6. Two special cases

In the present section two special cases of an ARG are considered: the case when the variable under consideration takes monetary values and the case of probabilistic uncertainty with respect to the values of the variable.

6.1. An application to investment performance

We now discuss a widespread application of average growth rates to the measurement of financial operations profitability. Our main objective here is to characterize the compound growth rate formula I_0 (1) and generalize I_0 to non-exponential modes of time discounting and vector monetary rewards. Our consideration is restricted to the simplest investment projects with only two transactions, initial investment and final gain. Several generalizations of (1) to an arbitrary set of transactions (cash flows) are axiomatized by Promislow and Spring (1996), Promislow (1997), Vilenskii and Smolyak (1998).

Throughout this subsection the economic variable under consideration is assumed to take monetary values. An element $v=(x,t;x',t')$ of the set V is interpreted as a simple investment project with the amount x of money invested at time t and the amount x' of money gained at time t' (one can think of x \$ (x' \$) as a portfolio position at time t (t')). Thus, by $v \succeq v'$ we mean that the project v is (weakly) more profitable than v'. The agent associated to $\succeq$ is now treated as an investor.

The interpretation requires additional structures on X and V. Two operations are assumed to be defined on V: the operation of changing volume of investment (e.g., by means of leverage) and the operation of concatenation of simultaneous investment projects. To formalize the operations throughout this subsection the set X is assumed to be an abstract convex cone over R_{++}. By an *abstract convex cone* over R_{++} we mean the quadruple $(\mathrm{X}, d_{\mathrm{X}}, +, \cdot)$ such that $(\mathrm{X}, d_{\mathrm{X}})$ is a metric space, $(\mathrm{X},+)$ is a commutative semigroup, and $\cdot : \mathrm{R}_{++} \times \mathrm{X} \to \mathrm{X}$ is the continuous operation of multiplication of elements of X by positive real numbers $(\lambda, x) \mapsto \lambda x$ satisfying the following properties: $\lambda(x+y)=\lambda x+\lambda y$, $(\lambda+\mu)x=\lambda x+\mu x$, $\lambda(\mu x)=(\lambda\mu)x$, $1x=x$, $d_{\mathrm{X}}(x+z, y+z)=d_{\mathrm{X}}(x,y)$, and $d_{\mathrm{X}}(\lambda x, \lambda y)=\lambda d_{\mathrm{X}}(x,y)$.[4] As a consequence of the axioms from the definition, we have that the addition

[4] We depart slightly from the conventional definition of an abstract convex cone (Chistyakov, 2010), which requires X to contain an identity element and $\cdot$ be the operation of multiplication of elements by non-negative real numbers.

operation $+$ is a continuous mapping from $\mathrm{X}\times\mathrm{X}$ into X (Chistyakov, 2010). A convex cone in a real normed vector space is an obvious example of an abstract convex cone.

Definition 4.

An ARG $\succeq$ for X is said to be an *average rate of return* (*ARR*) if the domain is an abstract convex cone $(\mathrm{X}, d_{\mathrm{X}}, +, \cdot)$ and the following condition is satisfied:

A4. *Nonsatiety*: $x+y \vartriangleright x$.

Definition 4 has a straightforward interpretation. The imposed additional structure on X is minimally reasonable to define the operation of changing volume of investment $(x,t;x',t') \mapsto (\lambda x,t;\lambda x',t')$, $\lambda > 0$ and the operation of concatenation $(x+y,t;x'+y',t')$ of independent simultaneous investment projects $(x,t;x',t')$ and $(y,t;y',t')$. Finally, according to axiom A4, the investor prefers more money to less.

The following hypotheses with respect to an ARR seem to be reasonable.

(viii) *Scale invariance*: $(x,t;x',t') \sim (\lambda x,t;\lambda x',t')\ \forall\, \lambda \in \mathrm{R}_{++}$.

(ix) *Invariance with respect to simultaneous realizations*: $(x,t;x',t') \sim (y,t;y',t') \Rightarrow (x,t;x',t') \sim (x+y,t;x'+y',t')$.

Interpretations of the hypotheses are as follows. Scale invariance (viii) states that profitability takes no account of the investment size and hence is a relative measure. Most of the known measures of profitability satisfy this property. Another possible interpretation of (viii) is invariance with respect to a change of monetary units (compare with assumption 3 in the Introduction). Scale invariance implies homotheticity of the investor's preferences:

$$x' \trianglerighteq x \Rightarrow \lambda x' \trianglerighteq \lambda x\ \forall\, \lambda > 0. \tag{37}$$

Indeed, by (viii) and A3, if $x' \trianglerighteq x$, then $(\lambda x, t; \lambda x', t') \sim (x, t; x', t') \succeq (x, t; x, t') \sim (\lambda x, t; \lambda x, t')$. Hence, $\lambda x' \trianglerighteq \lambda x$. Similar arguments show that (viii) holds for a regular ARR if and only if the induced TP family consists of time preferences that are homothetic with respect to money:

$$(x,t) \blacktriangleright_d (x',t') \Rightarrow (\lambda x, t) \blacktriangleright_d (\lambda x', t') \ \forall \lambda > 0 .$$

By hypothesis (ix), if two simultaneous projects are equally profitable, profitability of the consolidated project will be the same. Hypothesises (iii) and (ix) are along the lines of the internality axiom of Vilenskii and Smolyak (1998). Given two projects, the axiom states that profitability of the consolidated project falls between profitabilities of the individual projects. This property is extremely appealing in practice, since this allows investors to decompose investment decisions into separate evaluation of individual investment projects. Hypothesis (ix) implies translation invariance of the investor's preferences: $x' \trianglerighteq x \Rightarrow x' + y \trianglerighteq x + y \ \forall y \in \mathrm{X}$. For a regular ARR hypothesis (ix) is equivalent to translation invariance with respect to money for all elements of the induced TP family.

Both properties (viii) and (ix) impose restrictions on the investor's preferences $\trianglerighteq$ (the function u) rather than on the functional form of the average growth rate q of the investor's utility in the representation (4).

Let $\succeq$ be an ARR. A utility function u on X is said to be *homogeneous* if $u(\mathrm{X}) = \mathrm{R}_{++}$ and $u(\lambda x) = \lambda u(x) \ \forall \lambda > 0$ and *linear* if u is homogeneous and additive: $u(x+y) = u(x) + u(y)$.

Theorem 8.

Let $\succeq$ be an ARR. The following statements are equivalent:

(a) (viii) (resp. (ix)) holds;

(b) there exist a homogeneous (resp. linear) utility function u and a continuous and strictly increasing in the first argument function $h:\{(y,t,t')\in \mathrm{R}_{++}\times \mathrm{R}^2 : t<t'\}\to \mathrm{R}$ such that

$$I_7(v) := h\left(\frac{u(x')}{u(x)}, t, t'\right) \tag{38}$$

represents $\succeq$.

The obtained representation I_7 (38) complies with the special case of the relative discounting model of Ok and Masatlioglu (2007) and Dubra (2009). This special case corresponds to a homogeneous utility function u. Indeed, let an ARR with the index I_7 be regular and satisfy (ii), then for any $t<t'$ $h(\mathrm{R}_{++},t,t') = I_7(\mathrm{V})$ and the induced TP family has the relative discounting representation

$$(x,t) \trianglerighteq_d (x',t') \Leftrightarrow u(x) f_d(t,t') \geq u(x'), \tag{39}$$

where $f_\bullet(t,t')$ is the inverse to $h(\cdot,t,t')$ for $t<t'$, $f_d(t,t)=1$, and $f_d(t,t') := 1/f_d(t',t)$ for $t>t'$.

The next axiomatization gives sufficient conditions for an ARG to be induced by the special case of multiplicative transitive TP family (12) with homogeneous utility.

Theorem 9.

Let $\succeq$ be a regular ARG. The following two statements are equivalent:

(a) (ii), (iii), and (viii) (resp. (ix)) hold;

(b) $\succeq$ is induced by the TP family (12) with a homogeneous (resp. linear) utility function U and a family $\{ f_d,\ d\in \mathrm{D} \}$ of positive and continuous discount functions such that for every $t<t'$ the function $d \mapsto f_d(t')/f_d(t)$ is strictly decreasing.

For any time t_0, a reference point, the family $\{ f_d ,\ d \in \mathrm{D} \}$ in (b) can be chosen such that $f_{\bullet}(t_0) \equiv 1$.

As a direct consequence of Theorem 2 and Theorem 8, we obtain the following result.

Corollary 1.

Let $\succeq$ be an ARR. The following statements are equivalent:

(a) (i), (vii), and (viii) (resp. (ix)) hold;

(b) there exist a homogeneous (resp. linear) utility function u, strictly increasing and continuous functions $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ and $h : \mathrm{R}_{++} \to \mathrm{R}$ with $h(1) = 0$ such that

$$I_8(v) := \frac{h\left(\dfrac{u(x')}{u(x)}\right)}{\varphi(t') - \varphi(t)} \tag{40}$$

represents $\succeq$.

To simplify interpretation of I_8 (40) we assume that the ARR also satisfies (ii) and (iv), then h can be chosen such that $h(1/y) = -h(y)$, h is onto R, and elements of the induced TP family are represented by

$$(x,t) \trianglerighteq_d (x',t') \iff u(x) f\big(d(\varphi(t') - \varphi(t))\big) \geq u(x'), \tag{41}$$

where f is the inverse to h; f is strictly increasing and satisfies $f(-\tau) = 1/f(\tau)$. Representation (41) is the special case of the discounting by intervals model of Scholten and Read (2006). This special case corresponds to homogeneous utility function u. Note that in accordance with the interpretation of (vii) the "new" time $\varphi(\cdot)$ is measured on an interval scale, thus d in (41) is just a scale parameter. In the "new" time scale, $\trianglerighteq_d$ is also coincident with the stationary relative discounting model of Ok and Masatlioglu (2007, §3.3). Several proposals for the function f and the time-weighing function φ in (41) are suggested in the literature (Loewenstein and Prelec, 1992; Rachlin, 2006; Scholten and Read, 2006, 2010; Bleichrodt et al., 2009).

Theorem 10.

Let $\succeq$ be an ARR. The following statements are equivalent:

(a) (i), (iii), (vii), and (viii) (resp. (ix)) hold;

(b) (iii), (vi), and (viii) (resp. (ix)) hold;

(c) there exist a homogeneous (resp. linear) utility function u and a strictly increasing function $\varphi : \mathrm{R} \xrightarrow{\text{onto}} \mathrm{R}$ such that

$$I_9(v) := \frac{\ln\left(\dfrac{u(x')}{u(x)}\right)}{\varphi(t') - \varphi(t)} \tag{42}$$

represents $\succeq$.

Note: if $\trianglerighteq$ is fixed, Theorem 10 characterizes a unique ARG. Indeed, given $\trianglerighteq$ a homogeneous utility function u is defined up to a multiplicative constant and the ratio $u(x')/u(x)$ is uniquely determined. In contrast, Theorems 1–9 and Corollary 1 characterize rich families of ARGs (these families have cardinality of the continuum).

Since hypothesises (i)–(ix) still have economic interpretations if elements of the set V are treated as liabilities (getting an amount x of money at time t and returning an amount x' of money at time $t' > t$) and $\succeq$ is a relation of the cost of borrowing, the results of this subsection also axiomatize the corresponding indices of the cost of borrowing.

Example 4. Index I_0

A characterization of the compound annual growth rate I_0 (1) is the most straightforward application of Theorem 10. Let $\succeq$ be an ARR for the convex cone $\mathrm{X} = \mathrm{R}_{++}$ in the one-dimensional Euclidean space so that $\trianglerighteq$ is the natural order on X. If $\succeq$ satisfies (iii), (viii) (or (ix)), and (vi) with the identity transformation φ, then I_0 represents $\succeq$.

Example 5. Index I_0 under interval uncertainty

How should one compare the compound growth rates I_0 of investment alternatives under interval uncertainty with respect to amounts being invested and gained? The standard approach, the interval analysis, treats growth rates as intervals as well (Moore, 1979, chapter 9). Thus, to compare growth rates one should construct a preference relation on the space of real intervals. A somewhat more straightforward approach is presented below.

To describe investment alternatives under interval uncertainty with respect to amounts being invested and gained consider the space $\mathrm{X} := \{x = [x_1, x_2] : x_1, x_2 \in \mathrm{R}_{++},\ x_1 \le x_2\}$ of compact real intervals with the usual operations of scalar multiplication $\cdot$ and interval addition (the Minkowski addition) $+$ in interval arithmetic. An element $x \in \mathrm{X}$ represents the interval to which amounts being invested/gained belong. The usual metrization of X is given by the Hausdorff metric $d_{\mathrm{X}}(x, x') := \max\{|x_1' - x_1|, |x_2' - x_2|\}$ (Moore, 1979, §4.1). Then the space $(\mathrm{X}, d_{\mathrm{X}})$ is separable and $(\mathrm{X}, d_{\mathrm{X}}, +, \cdot)$ is an abstract convex cone.

Let $\succeq$ be an ARR for X. If $\succeq$ satisfy (iii), (viii), and (vi) with the identity transformation φ, then there exists a homogeneous utility function u such that

$$I(v) = \frac{\ln\left(\dfrac{u(x')}{u(x)}\right)}{t' - t}$$

represents $\succeq$ (Theorem 10). By A4, u is nondecreasing: $u(x) \le u(x')$ whenever $x_i \le x_i'$, $i = 1, 2$. Hence,

$\min_{i=1,2} x'_i/x_i \le u(x')/u(x) \le \max_{i=1,2} x'_i/x_i$ for any x and x'. Consequently, $I(v)$ is a Cauchy mean of elementary indices $I_0(x_i, t; x'_i, t')$, $i = 1, 2$:

$$\min_{i=1,2} I_0(x_i, t; x'_i, t') \le I(v) \le \max_{i=1,2} I_0(x_i, t; x'_i, t') .$$

In particular, if (ix) holds instead of (viii), then there exists a constant $w \in [0,1]$ such that

$$I(v) = \frac{\ln\left(\dfrac{wx'_1 + (1-w)x'_2}{wx_1 + (1-w)x_2}\right)}{t' - t} . \tag{43}$$

The representation (43) coincides with I_0 for the amounts invested and gained equal to the weighted arithmetic mean of endpoints of the corresponding intervals.

Example 6. The Allen and Lowe quantity indices

Consider an economy with n goods. Let $\mathrm{X} := \{x \in \mathrm{R}^n : x \ge 0, x \ne 0\}$ be a convex cone in the n-dimensional Euclidean space. An element of X is treated as the goods quantities transacted at a given state of the economy. Our purpose is to construct an elementary quantity index, an ARR $\succeq$ for X, which summarizes the change in the quantities transacted from one time period to another. In the classical setting, bilateral index number theory deals with price/quantity changes over comparable periods. In terms of the present model, this means that we should work with the restriction $\succeq_\tau$ of $\succeq$ to the set $\{(v_1, v_2) = ((x_1, t_1; x'_1, t'_1), (x_2, t_2; x'_2, t'_2)) \in \mathrm{V}^2 : t'_1 - t_1 = t'_2 - t_2 = \tau\}$. Let $\succeq$ satisfy (viii) and (vi) with the identity transformation φ, then (Theorem 8) there exists a homogeneous utility function $u : \mathrm{X} \to \mathrm{R}_{++}$ such that $v_1 \succeq_\tau v_2 \Leftrightarrow Q_A(x_1, x'_1) \ge Q_A(x_2, x'_2)$, where $Q_A(x, x') := u(x')/u(x)$ is known as the Allen or Malmquist quantity index (in the case of homothetic preferences the two indices coincide), a measure of change in quantities transacted from a microeconomic point of view (see, for instance, Diewert and Nakamura, 1993, §7.3). In particular, if (viii) is replaced by (ix), then $v_1 \succeq_\tau v_2 \Leftrightarrow Q_L(x_1, x_1') \geq Q_L(x_2, x_2')$, where $Q_L(x, x') := \langle x', p \rangle / \langle x, p \rangle$, $p \in \mathrm{R}^n_{++}$, is the Lowe quantity index (Diewert and Nakamura, 1993, §2.2). If further, (iii) holds, then $I(x,t;x',t') = Q_A(x,x')^{\frac{1}{t'-t}}$ (resp. $I(x,t;x',t') = Q_L(x,x')^{\frac{1}{t'-t}}$), an obvious generalization of Q_A (resp. Q_L) to varying delay, represents $\succeq$ (Theorem 10).

6.2.A remark on average rates of growth in risky environments

Strictly speaking, the theory of average growth rates presented here is incompatible with probabilistic uncertainty. For instance, it is inapplicable in the case when growth rates are defined over lotteries on the entire space V. However, the theory incorporates the cases when probabilistic uncertainty concerns with values of the variable rather than time provided that the considered space of probability distributions on the set of values is separable. Such a case is considered in this subsection.

Let $(\mathrm{Y}, d_\mathrm{Y})$ be a separable metric space, the sample space of values of the economic variable; let $\mathrm{B}(\mathrm{Y})$ be the Borel σ-algebra on $(\mathrm{Y}, d_\mathrm{Y})$ and let X be a collection of probability measures on the measurable space $(\mathrm{Y}, \mathrm{B}(\mathrm{Y}))$. X endowed with the Prokhorov metric d_X is a separable metric space; moreover, d_X metrizes weak convergence of probability measures (Billingsley, 1999, section 1.6). Thus, one can define an ARG for X if by continuity of $\trianglerighteq$ one means continuity with respect to weak convergence of probability measures.

As an example, we consider a generalization of the compound growth rate I_0 (1) to the case of probabilistic uncertainty with respect to values.

Example 7. Index I_0 under probabilistic uncertainty: the case of risk-neutral investor

Let $(\mathrm{X}, d_{\mathrm{X}})$ be the space of all probability measures on $(\mathrm{R}_{++}, \mathrm{B}(\mathrm{R}_{++}))$ with finite first moment endowed with the Prokhorov metric d_{X} and such that

$$\mathbf{E}e^{-sX} \le (1 + s\mathbf{E}X)^{-1} \;\forall\, s \ge 0 \tag{44}$$

(the following notational convention is used throughout this example: for a probability measure $x \in \mathrm{X}$, X is a random variable on the probability space $\langle \mathrm{R}_{++}, \mathrm{B}(\mathrm{R}_{++}), \mathrm{P} \rangle$ that induces x). The assumption (44) is technical, this states that the set X consists exactly of those probability distributions that infinite-degree weakly stochastically dominate the exponential distribution with the same mean (Thistle, 1993, proposition 4). An element of X is treated as an investment under probabilistic uncertainty with respect to amounts being invested and gained. Buying a stock at a (random) market price tomorrow and selling it the day after tomorrow is an example of such an investment.

We are going to introduce a measure of profitability of (risky) investments for a risk-neutral investor by virtue of an ARG $\succeq$ for X. Let the ARG satisfy axiom A4, (viii), and (ix), where «+» denotes the operation of convolution of probability measures and λx ($\lambda > 0$) is a probability measure induced by λX. The set X is closed under convolution of probability measures and scalar multiplication (Basu and Mitra, 1998, §2.1), thus, A4, (viii), and (ix) are well defined.

To derive a numerical representation for $\succeq$ we adopt the idea of Vilenskii and Smolyak (1998). To $v = (x, t; x', t') \in \mathrm{V}$ with x' being a degenerate probability measure assign the sequences

$$X_k := \frac{1}{k}\sum_{i=1}^{k} Y_i\,,\; v_k := (x_k, t; x', t')\,,\; k = 1, 2, \ldots,$$

where Y_i, $i=1,2,...$ are independent copies of X. By the law of large numbers, the sequence $\{X_k\}_{k=1}^{\infty}$ converges in probability (and, therefore, in distribution) to the mathematical expectation $\mathbf{E}X$. By (viii) and (ix), $v \sim v_k$ for every k. Applying A4, A2, and continuity of the mathematical expectation operator $\mathbf{E}$ in X (Basu and Mitra, 1998, theorem 2.1), we get that for any $x, x' \in \mathrm{X}$ $x \trianglerighteq x' \Leftrightarrow \mathbf{E}X \geq \mathbf{E}X'$ as desired for a risk-neutral investor. By arguments similar to those used in the proof of Theorem 8 it follows that there exists a continuous and strictly increasing in the first argument function $h:\{(y,t,t') \in \mathrm{R}_{++} \times \mathrm{R}^2 : t < t'\} \to \mathrm{R}$ such that

$$I(v) = h\left(\frac{\mathbf{E}X'}{\mathbf{E}X}, t, t'\right) \tag{45}$$

represents $\succeq$. In view of (38) and (39), under (ii) elements of the TP family induced by a regular ARG with the index (45) are the special cases of time preferences over risky outcomes introduced by Dubra (2009, theorem 2).

Thus, to get a reasonable generalization of an ARR with $\mathrm{X} = \mathrm{R}_{++}$ to the case of probabilistic uncertainty one may use the expected values of the corresponding random variables as proxies for amounts being invested and gained.

7. Conclusion

In this paper we deal with the problem of average rate of growth (change, increase) measurement in a unified manner regardless of the nature of the underlying variable. Our findings may be summarized as follows:

– The functional form of average rate of growth depends, among other things, on preferences of an economic agent associated with the problem; various measures of average rate of growth can be specified for the same underlying variable depending on the agent's preferences.

– Average growth rates actually measure rates of growth of the agent's utility.

– The average rate of growth admits a dual representation by means of a parametric family of time preferences with the parameter being treated as a state-dependent subjective discount rate of the agent.

– Average rates of growth related to the families of time preferences that are important in intertemporal choice theory can be characterized by several easily interpretable axioms. The axioms impose restrictions on all the elements of the families; these restrictions parallel closely to axioms that are widely used in intertemporal choice theory to characterize time preferences.

We hope that the results obtained can serve as an additional argument to justify the characterized measures of average rate of growth and determine the limits of their applicability.

8. Appendix: Proofs

Proof of Proposition 2.

First, we check that given a TP family on P, (6) induces a regular ARG $\succeq$ for X. By construction, $\succeq$ is well defined and satisfies A1 and A3. For any $v \in \mathrm{V}$, the strict lower contour set $\mathrm{L}(v) := \{v' \in \mathrm{V} : v \succ v'\} = \{(x,t;x',t') \in \mathrm{V} : (x,t) \blacktriangleright_{I(v)} (x',t')\}$ of v is open in P^2 (and therefore in V) as the intersection of open sets $\blacktriangleright_{I(v)}$ and V. Similar arguments show that the strict upper contour set $\mathrm{U}(v) := \{v' \in \mathrm{V} : v' \succ v\} = \{(x,t;x',t') \in \mathrm{V} : (x',t') \blacktriangleright_{I(v)} (x,t)\}$ is open. In the context of total preorders, openness of $\mathrm{L}(v)$ and $\mathrm{U}(v)$ for all $v \in \mathrm{V}$ is equivalent to A2. Finally, the ARG $\succeq$ is regular: $\overline{\mathrm{L}}(v)$ ($\overline{\mathrm{U}}(v)$) is closed as the intersection of closed sets $\{(x,t;x',t') \in \mathrm{P}^2 : t \le t'\}$ and $\underline{\blacktriangleright}_{I(v)}$ (the complement to $\blacktriangleright_{I(v)}$).

Now we prove that given a regular ARG $\succeq$, (8) generates a TP family. By construction, $\{\underline{\blacktriangleright}_d,\ d \in I(\mathrm{V})\}$ satisfies C1–C3, and B1, B3 hold for each element of the family. $\underline{\blacktriangleright}_d$ are continuous. Indeed, given $d \in I(\mathrm{V})$ take $v \in \mathrm{V}$ such that $I(v) = d$. From regularity of $\succeq$ it follows that the complement of $\blacktriangleright_d$ is a closed subset of P^2 as the union of closed sets $\{(x,t;x',t')\in \mathrm{V}:(x',t')\ \underline{\blacktriangleright}_d\ (x,t)\}\cup\{(x,t;x',t')\in \mathrm{P}^2 : x' \trianglerighteq x,\ t=t'\} =$ $=\overline{\mathrm{U}}(v)$ and $h(\overline{\mathrm{L}}(v))$, where h is a homeomorphism of P^2 onto itself given by $(x,t;x',t')\mapsto(x',t';x,t)$.

Let A be the set of regular ARGs for X and let B be the set of TP families on P such that TP families that are order preserving reparameterizations of each other are identified. Let $f:\mathrm{B}\to\mathrm{A}$ and $g:\mathrm{A}\to\mathrm{B}$ be the maps defined by (6) and (8), respectively. Then $f\circ g$ and $g\circ f$ are the identity maps. ■

Proof of Theorem 1.

(a) $\Rightarrow$ (b). Let q be defined as in Proposition 1. By (v)

$$q(y,t;y',t')\ge q(z,t;z',t') \ \Leftrightarrow\ g(y,y')\ge g(z,z'),$$

where $g(y,y'):=q(y,0;y',1)$. Thus, $q(y,t;y',t')$ is a function of only $g(y,y')$, t, and t' so that (24) represents $\succeq$.

If, in addition, (iv) holds, then

$$h\big(g(y,y'),t,t'\big)\ge h\big(g(z,z'),t,t'\big) \ \Leftrightarrow$$
$$h\big(g(y',y),t,t'\big)\le h\big(g(z',z),t,t'\big).$$

Therefore, $g(y',y)$ is a strictly decreasing function of $g(y,y')$. If, further, (i) (part A) holds, then the functions h and g in the representation (24) can be chosen such that g is skew-symmetric and h is odd.

(b) $\Rightarrow$ (a). Trivial. ■

In what follows, we need the following proposition.

Proposition 3.

Let $\succeq$ be an ARG, then (i) and (vii) imply (vi) (with the same transformation φ).

Proof.

Given φ, define a binary relation $\succeq_\varphi$ on V by

$$\begin{gathered}(x,t;x',t') \succeq_\varphi (y,\tau;y',\tau') \Leftrightarrow \\ (x,\varphi^{-1}(t);x',\varphi^{-1}(t')) \succeq (y,\varphi^{-1}(\tau);y',\varphi^{-1}(\tau'))\end{gathered} \tag{46}$$

with $\sim_\varphi$ and $\succ_\varphi$ defined as usual. Since φ is strictly increasing, onto R, and, therefore, continuous, $\succeq_\varphi$ is an ARG satisfying (i) (with $\succeq$ replaced by $\succeq_\varphi$). In this notation, (vi) and (vii) take the form:

(vi)$_\varphi$ $\quad (x,t;x',t') \sim_\varphi (x,t+\tau;x',t'+\tau)\ \forall \tau \in \mathrm{R}$;

(vii)$_\varphi$ $\quad (x,t;x',t') \succeq_\varphi (x,\tau;x',\tau') \Rightarrow$

$(x,at+b;x',at'+b) \succeq_\varphi (x,a\tau+b;x',a\tau'+b)\ \forall a>0,\ b$.

Fix $v_0 = (x,t_0;x',t_0') \in \mathrm{V}$, if $v_0 \in \mathrm{V}_0$, then (vi)$_\varphi$ follows from (i). In what follows, we assume that $v_0 \in \mathrm{V}_+$ (the arguments in case of $v_0 \in \mathrm{V}_-$ are similar). Assume the contrary: there exists $\tau > 0$ such that

$$v_0 \succ_\varphi (x,t_0+\tau;x',t_0'+\tau) \tag{47}$$

(if $\tau < 0$, then the arguments are the same). Denote $t_1 := t_0 + \tau$ and let t_1' be a unique solution of the equation $v_0 \sim_\varphi (x,t_1;x',t_1')$. By (i), A2, and (47) t_1' is well defined and $0 < t_1' - t_1 < t_0' - t_0$.

Let $a \in (0,1)$, $b \in \mathrm{R}$ be a solution of the system of linear equations $at_0 + b = t_1$, $at_0' + b = t_1'$. Construct two sequences $\{t_k\}_{k=0}^\infty$, $\{t_k'\}_{k=0}^\infty$ by $t_{k+1} := at_k + b$, $t_{k+1}' := at_k' + b$, $k = 0,1,...$ Thus defined sequences are increasing, $t_k < t_k'$, $k = 0,1,...$, and converge to $t^* := b/(1-a)$. We define $v_k := (x,t_k;x',t_k')$, $k = 0,1,...$

Let t $(< t^*)$ be a unique (by (i) and A2) solution of the equation $v_0 \sim_\varphi (x,t;x',t^*)$. Then, by (vii)$_\varphi$ with the constants a and b, $v_0 \sim_\varphi v_k$, $k = 1,2,...$ Therefore,

$$(x,t;x',t^*) \sim_\varphi v_k,\ k = 0,1,... \tag{48}$$

Since $\{t_k\}_{k=0}^{\infty}$ and $\{t'_k\}_{k=0}^{\infty}$ are increasing and convergent, there exists k such that $t \le t_k < t'_k < t^*$. Combining this and (48) we obtain a contradiction with part A of (i). ■

Proof of Theorem 2.

(a) $\Rightarrow$ (b). Let $\succeq_\varphi$ be defined by (46). Propositions 1 and 3 ensure that there exist a utility function u and a continuous function $g_1 : \mathrm{Y}^2 \times \mathrm{R}_{++} \to \mathrm{R}$ with $\mathrm{Y} := u(\mathrm{X})$ such that $(x,t;x',t') \mapsto g_1\big(u(x),u(x'),t'-t\big)$ is a numerical representations of $\succeq_\varphi$. By (i) (part A), $g_1(y,y',\cdot)$ is strictly decreasing if $y < y'$, strictly increasing if $y > y'$, and identically equal to a constant if $y = y'$.

If Y is a singleton, then (22) with $g \equiv 0$ represents $\succeq_\varphi$. Otherwise, put

$$g_2(t) := \begin{cases} g_1(y_0, y'_0, 1/t) & \text{if } t > 0 \\ g_1(y_0, y_0, 1) & \text{if } t = 0 \\ g_1(y'_0, y_0, -1/t) & \text{if } t < 0 \end{cases},$$

where $y_0 < y'_0$ are arbitrary but fixed elements of Y. By (i), thus defined function g_2 is strictly increasing and onto $g_1(\mathrm{Y}^2 \times \mathrm{R}_{++})$. Therefore, the function $(x,t;x',t') \mapsto g_3\big(u(x),u(x'),t'-t\big)$ with $g_3 := g_2^{-1} \circ g_1$ is well defined and represents $\succeq_\varphi$.

Given $(y,y',\tau) \in \mathrm{Y}^2 \times \mathrm{R}_{++}$, put $\tau' := g_3(y,y',\tau)$. Then from the definition of g_3 and (vii), we get $g_3(y,y',a\tau) = \tau'/a = g_3(y,y',\tau)/a$ $\forall\, a > 0$. With the function $g(\cdot) := g_3(\cdot,1)$ and $a = t'-t$ it takes the form (22). From (i) it follows that $g(y,\cdot)$ is strictly increasing, $g(\cdot,y')$ is strictly decreasing, and $g(y,y) = 0$.

If, in addition, (iv) holds, then $g(y,y')/g(y',y) = g(z,z')/g(z',z)$ for every $y < y'$ and $z < z'$ in Y. This is possible if and only if the quotient $g(y,y')/g(y',y)$ is identically equal to a constant $c < 0$ on the set $\{(y,y') \in \mathrm{Y}^2 : y < y'\}$. Thus, I_5 (22) with the skew-symmetric function $\tilde{g}(y,y') := \begin{cases} g(y,y') & \text{if } y < y' \\ -cg(y,y') & \text{otherwise} \end{cases}$ instead of g represents $\succeq_\varphi$.

(b) $\Rightarrow$ (a). Straightforward. ■

Proof of Theorem 3.

For a regular ARG $\succeq$, (iii) holds if and only if all elements of the induced TP family $\{\trianglerighteq_d,\ d \in \mathrm{D}\}$ are transitive (i.e. continuous total preorders). Let u be a utility function and let $h_d(u(\cdot),\cdot)$ be a continuous numerical representation of $\trianglerighteq_d$, then (14) holds.

If (ii) holds, then $\tilde{h}_d(y,t) := h_d^{-1}(h_d(y,t),t_0)$, where $h_d^{-1}(\cdot,t_0)$ is the inverse to $h_d(\cdot,t_0)$ and t_0 is fixed, is well defined and also represents $\trianglerighteq_d$ so that the normalization condition (16) holds. ■

Proof of Theorem 4.

(a) $\Rightarrow$ (b). Let $\succeq_\varphi$ be defined by (46). Since $\succeq_\varphi$ satisfies (i) and (iii), it is regular and is induced by the transitive TP family $\{\trianglerighteq_d,\ d \in \mathrm{D}\}$ (14) by means of (15) (Theorem 3). By (i) (part A), the parameterization of the family can be chosen such that $0 \in \mathrm{D}$, $h_0(y,t) = y$ for every $y \in \mathrm{Y} := u(\mathrm{X})$, and h_d is strictly increasing (decreasing) with respect to the second argument for $d < 0$ (resp. $d > 0$). We define $\tilde{h}_d(t) := h_d(y_0,t)$, $d \in \mathrm{D}$, where y_0 is an arbitrary but fixed element of Y. Condition (i) (part B) guarantees that given $(d,y,t) \in \{\mathrm{D} \setminus \{0\}\} \times \mathrm{Y} \times \mathrm{R}$ there exists a unique t' such that

$h_d(y,t) = \tilde{h}_d(t')$. Put $f_0(y,t) := y$ and $f_d(y,t) := -\tilde{h}_d^{-1}(h_d(y,t))\operatorname{sgn} d$, $d \neq 0$. f_d is a continuous numerical representation of $\underline{\blacktriangleright}_d$ and, by (vi), satisfies the equation $f_d(y,t+\tau) = f_d(y,t) - \tau \operatorname{sgn} d \ \forall \tau$. We define $g_d(y) := f_d(y,0)$, then $f_d(y,t) = f_d(y,0) - t\operatorname{sgn} d = g_d(y) - t\operatorname{sgn} d$. By (i) (part A), for any $y < y'$ the function $d \mapsto g_d(y') - g_d(y)$ is strictly increasing (decreasing) on the set $\mathrm{D} \cap (-\infty, 0)$ (resp. $\mathrm{D} \cap \mathrm{R}_{++}$).

(b) $\Rightarrow$ (a). Trivial. ■

Proof of Theorem 5.

(a) $\Rightarrow$ (b). Let u and I be defined as in Proposition 1. By Theorem 1 I can be represented in the form (24) with g being skew-symmetric and $h(\cdot, t, t')$ is odd (so that $I(v) = 0 \ \forall v \in \mathrm{V}_0$). On the other hand, I is an implicit function that takes each $(x,t;x',t') \in \mathrm{V}$ to a unique solution d of Eq. (15) such that the normalization condition (16) holds (Theorem 3). Since X is connected, $\mathrm{Y} := u(\mathrm{X})$ $\mathrm{G} := g(\mathrm{Y}^2)$, and $\mathrm{D} := I(\mathrm{V}) \supset \{0\}$ are non-degenerate intervals. Fix $d > 0$ in D. For $z \in \mathrm{G}$ and t, let $k_d(z,t)$ be a unique solution t' of the equation $h(z,t,t') = d$ if $z > 0$, of the equation $h(-z,t',t) = d$ if $z < 0$, and t otherwise. The function k_d is well defined, $k_d(\cdot, t)$ is strictly increasing ((i), part A), onto R ((ii)), and hence, continuous. Similar arguments show that $k_d(z,\cdot)$ is continuous and strictly increasing. Thus, the functions k_d, h_d, and h_d^{-1}, where $h_d^{-1}(y,\cdot)$ is the inverse to $h_d(y,\cdot)$, are continuous and strictly monotone in all arguments. Then

$$k_d\big(g(y,y'),t\big) = t' = h_d^{-1}\big(y', h_d(y,t)\big) \tag{49}$$

holds for all y, y', and t. The general continuous and strictly monotone solution of the equation of generalized associativity (49) is given by Maksa

(2005): there exist continuous and strictly increasing functions α, β_d, and γ such that

$$k_d(z,t) = \beta_d^{-1}\left(\gamma(z) + \beta_d(t)\right),$$
$$g(y,y') = \gamma^{-1}\left(\alpha(y') - \alpha(y)\right),$$
$$h_d(y,t) = \alpha^{-1}\left(\alpha(y) - \beta_d(t)\right),$$

α and β_d are onto R, γ is odd. Thus, the restriction of the relation $\succeq$ to the growth set $\mathrm{V}_+ \times \mathrm{V}_+$ is represented by means of (13) with $U := e^{\alpha \circ u}$ and $f_d := e^{-\beta_d}$. From (iv) it follows that the restriction can be extended to $\succeq$ by letting $f_0 :\equiv 1$ and $f_d := 1/f_{-d}$, $d < 0$. By A3, for any $t < t'$ the function $d \mapsto f_d(t')/f_d(t)$ is strictly increasing and (by (16)) $f_\bullet(t_0) \equiv 1$.

(b) $\Rightarrow$ (a). Straightforward. ■

Proof of Theorem 6.

(a) $\Rightarrow$ (b). Let $\succeq_\varphi$ be defined by (46). Since $\succeq_\varphi$ is an ARG that satisfies (i), (ii), (iii), (iv), and (v), it can be represented through Eq. (13) (Theorem 5). Fix $d \in \mathrm{D}$. By (ii) and (vi), for any t there exist x and x' such that the equation

$$\frac{f_d(t)}{f_d(0)} = \frac{U(x')}{U(x)} = \frac{f_d(t+\tau)}{f_d(\tau)} \tag{50}$$

holds for any τ. Eq. (50) is the exponential Cauchy functional equation with respect to f_d, its continuous solution is given by $f_d(t) = \alpha(d)e^{-\beta(d)t}$ for some functions $\alpha > 0$ and β. β is strictly increasing, since the function $d \mapsto f_d(t')/f_d(t)$ is strictly decreasing for every $t < t'$. The images of $u := \ln U$ and β are R.

(b) $\Rightarrow$ (a). Trivial. ■

Proof of Theorem 7.

(a) $\Rightarrow$ (b). Let $\succeq_\varphi$ be defined by (46). By Theorem 2, there exist u and g such that I_5 (22) represents $\succeq_\varphi$. If $\mathrm{Y} := u(\mathrm{X})$ has either 1 or 2 elements, then I_1 (11) represents $\succeq_\varphi$. Otherwise, for $(y, y', y'') \in \mathrm{Y}^3$ such that $(y' - y)(y'' - y') > 0$, from (iii) with $t = -1$, $t' = 0$, and $t'' = g(y', y'')/g(y, y')$ we obtain

$$g(y, y') + g(y', y'') = g(y, y'') . \tag{51}$$

(51) remains valid for $(y' - y)(y'' - y') \geq 0$. We define

$$\tilde{u}_-(y) := \begin{cases} g(y_0, y) & \text{if } y < y_0 \\ -g(y, y_0) & \text{if } y \geq y_0 \end{cases}, \quad \tilde{u}_+(y) := \begin{cases} -g(y, y_0) & \text{if } y < y_0 \\ g(y_0, y) & \text{if } y \geq y_0 \end{cases},$$

where y_0 is an arbitrary but fixed element of Y. Since u_- and u_+ are continuous and strictly increasing, $u_- := \tilde{u}_- \circ u$ and $u_+ := \tilde{u}_+ \circ u$ are utility functions. To complete the proof, we show that the general solution of the Sincov functional equation (51) on the domain $\{(y, y', y'') \in \mathrm{Y}^3 : (y' - y)(y'' - y') \geq 0\}$ is given by

$$g(y_1, y_2) = \begin{cases} \tilde{u}_+(y_2) - \tilde{u}_+(y_1) & \text{if } y_1 < y_2 \\ \tilde{u}_-(y_2) - \tilde{u}_-(y_1) & \text{if } y_1 \geq y_2 \end{cases}. \tag{52}$$

Indeed, take any $y_1, y_2 \in \mathrm{Y}$. Assume that $y_1 < y_2$ (the arguments in the case of $y_1 \geq y_2$ are similar). If $y_0 \leq y_1$, then (52) follows from (51) with $y = y_0$, $y' = y_1$, and $y'' = y_2$. If $y_1 < y_0 \leq y_2$, then (51) with $y = y_1$, $y' = y_0$, $y'' = y_2$ implies (52). If $y_2 < y_0$, then (52) follows from (51) with $y = y_1$, $y' = y_2$, and $y'' = y_0$.

If, in addition, (iv) holds, then g is skew-symmetric and the functions $\tilde{u}_-$ and $\tilde{u}_+$ are identical.

(b) $\Rightarrow$ (a). Straightforward. ■

Proof of Theorem 8.

(a) $\Rightarrow$ (b). Let u and q be defined as in Proposition 1. First, let us prove that under (viii) u can be chosen such that

$$u(\mathrm{X}) = \mathrm{R}_{++} \text{ and } u(\lambda x_0) = \lambda \quad \forall \lambda > 0, \tag{53}$$

where x_0 is an arbitrary but fixed element of X. Fix $x_0 \in \mathrm{X}$. Since the map $\lambda \mapsto u(\lambda x_0)$ is strictly increasing (axiom A4) and continuous, it suffices to show that for every $x \in \mathrm{X}$ there exists $\lambda \in \mathrm{R}_{++}$ such that $u(\lambda x_0) = u(x)$. Assume the converse: there exists x such that $u(x) < u(\lambda x_0)$ $\forall \lambda > 0$ (the arguments in the case of $u(x) > u(\lambda x_0)$ are similar). Then $u(2x) < u(\lambda x_0) < u(\lambda x_0 + (1-\lambda)x)$, $\lambda \in (0,1)$, where the first inequality follows from (37), while the second one follows from axiom A4. Tending λ to $+0$ and using continuity of u, we obtain a contradiction with axiom A4: $u(2x) \le u(x)$.

In what follows we assume that (53) holds. Then, by (viii),

$$\begin{aligned} & q(u(x), t; u(x'), t') = q(u(u(x)x_0), t; u(u(x')x_0), t') = \\ & = q(u(\lambda u(x)x_0), t; u(\lambda u(x')x_0), t') = q(\lambda u(x), t; \lambda u(x'), t') \end{aligned} \tag{54}$$

for any $\lambda > 0$. An application of (54) with $\lambda = 1/u(x)$ yields (38) with $h(y;t,t') := q(1,t;y,t')$. It follows from (viii) and (38) that

$$u(x')/u(x) = u(\lambda x')/u(\lambda x), \; x, x' \in \mathrm{X}, \; \lambda \in \mathrm{R}_{++}. \tag{55}$$

Setting in (55) $x = x_0$, we obtain that u is homogeneous.

Now let (ix) hold. With $y = x$ and $y' = x'$ (ix) reduces to $(x,t;x',t') \sim (2x,t;2x',t')$. By induction,

$$(x,t;x',t') \sim (\lambda x, t; \lambda x', t') \tag{56}$$

for any positive rational number λ. Now (56) holds for any $\lambda > 0$ since the set of positive rational numbers is everywhere dense in R_{++} and A2 holds.

Therefore, (viii) is valid and (38) represents the ARR. By (ix),

$$\frac{u(x')}{u(x)} = \frac{u(y')}{u(y)} \;\Rightarrow\; \frac{u(x')}{u(x)} = \frac{u(x'+y')}{u(x+y)} \,. \qquad (57)$$

From (57) with $x' = u(x)x_0$, $y' = u(y)x_0$, we obtain

$$u(x+y) = u(x) + u(y)\,.$$

(b) $\Rightarrow$ (a). Trivial. ■

Proof of Theorem 9.

(a) $\Rightarrow$ (b). Let u and $\{h_d,\; d \in \mathrm{D}\}$ be defined as in Theorem 3 such that the normalization condition (16) holds. By Theorem 8, u can be chosen to be homogeneous (resp. linear). Then h_d is homogeneous with respect to the first argument. We define $f_d(t) := h_d(1,t)$, then $h_d(u(x),t) = u(x)h_d(1,t) = u(x) f_d(t)$ so that (12) holds with $U := u$. ■

(b) $\Rightarrow$ (a). Straightforward. ■

Proof of Theorem 10.

(a) $\Rightarrow$ (b). This follows from Proposition 3.

(b) $\Rightarrow$ (c). Let $\succeq_\varphi$ be defined by (46). By Theorem 8 and (vi), there exist a homogeneous (resp. linear) utility function $u : \mathrm{X} \to \mathrm{R}_{++}$ and a continuous and strictly increasing in the first argument function $h : \mathrm{R}^2_{++} \to \mathrm{R}$ such that

$$I(x,t;x',t') = h\left(\frac{u(x')}{u(x)}, t'-t\right)$$

represents $\succeq_\varphi$.

Applying (iii) with $t'-t = t''-t' = \tau$ and $\dfrac{u(x')}{u(x)} = \dfrac{u(x'')}{u(x')} = \lambda$, we get $h(\lambda,\tau) = h(\lambda^2, 2\tau)$. By induction,

$$h(\lambda,\tau) = h(\lambda^r, r\tau) \qquad (58)$$

for any positive rational number r. Since h is continuous, (58) holds for any $r \in \mathrm{R}_{++}$. Applying (58) with $r = 1/\tau$, we obtain

$$I(x,t;x',t') = h\left(\frac{u(x')}{u(x)}, t'-t\right) = h\left(\left(\frac{u(x')}{u(x)}\right)^{\frac{1}{t'-t}}, 1\right).$$

Since h is strictly increasing in its first argument, I_9 represents $\succeq$.

(c) $\Rightarrow$ (a). Trivial. ■

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10. References

Aczél, J., Roberts, F.S., Rosenbaum, Z. (1986). On scientific laws without dimensional constants. *Journal of Mathematical Analysis and Applications*, 119, 389–416.

Ahlbrecht, A., Weber, M. (1995). Hyperbolic discounting models in prescriptive theory of intertemporal choice. *Zeitschrift für Wirtschafts- und Sozialwissenschaften*, 115, S535–S568.

Aliprantis, C.D., Border, K.C. (2006). *Infinite dimensional analysis: A hitchhiker's guide*. Berlin: Springer.

Balk, B.M. (2008). *Price and quantity index numbers: models for measuring aggregate change and difference*. New York: Cambridge University Press.

Basu, K. (1983). Cardinal utility, utilitarianism, and a class of invariance axioms in welfare analysis. *Journal of Mathematical Economics*, 12, 193–206.

Basu, S.K., Mitra, M. (1998). The class of life distributions that are Laplace order dominated by the exponential law and its ramifications. In A.P.

Basu, S.K. Basu, S. Mukhopadhyay (eds.). *Frontiers in reliability* (pp. 37–45). World Scientific.

Benzion, U., Rapoport, A., Yagil, J. (1989). Discount rates inferred from decisions – an experimental study. *Management Science*, 35(3), 270–284.

Billingsley, P. (1999). *Convergence of probability measures*. New York: John Wiley & Sons.

Bleichrodt, H., Rohde, K.I.M., Wakker, P.P. (2009). Non-hyperbolic time-inconsistency. *Games and Economic Behavior*, 66(1), 27–38.

Castillo, E., Iglesias, A., Ruiz-Cobo, R. (2005). *Functional equations in applied sciences*. Amsterdam: Elsevier.

Chistyakov, V.V. (2010). Modular metric spaces, I: Basic concepts. *Nonlinear Analysis*, 72(1), 1–14.

Debreu, G. (1954). Representation of a preference ordering by a numerical function. In R.M. Thrall, C.H. Coombs, R.L. Davis (eds.). *Decision processes* (pp. 159–165). New York: Wiley.

Diewert, W.E., Nakamura, A.O. (eds.). (1993). *Essays in index number theory*. Amsterdam: North-Holland.

Dubra, J. (2009). A theory of time preferences over risky outcomes. *Journal of Mathematical Economics*, 45, 576–588.

Ebert, U. (1984). Exact welfare measures and economic index numbers. *Journal of Economics*, 44(1), 27–38.

Eichhorn, W. (1978). *Functional equations in economics*. London: Addison-Wesley.

Fishburn, P.C., Rubinstein, A. (1982). Time preference. *International Economic Review*, 23(3), 677–694.

Hammond, P.J. (1994). Money metric measures of individual and social welfare allowing for environmental externalities. In W. Eichhorn (ed.). *Models and measurement of welfare and inequality* (pp. 694–724). Springer Verlag.

Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A. (1971). *Foundations of measurement*. Vol. 1. New York: Academic Press.

Loewenstein, G., Prelec, D. (1992). Anomalies in intertemporal choice: evidence and an interpretation. *The Quarterly Journal of Economics*, 107, 573–597.

Luce, R.D. (2000). *Utility of gains and losses: measurement-theoretical and experimental approaches*. Mahwah: Erlbaum.

Maksa, G. (2005). Quasisums and generalized associativity. *Aequationes Mathematicae*. Vol. 69. P. 6–27.

Manca, P. (1969). Equazioni funzionali e leggi di interesse finanziario. *Giornale dell'Istituto Italiano degli Attuari*, 32(2), 196–205.

Moore, R.E. (1979). *Methods and applications of interval analysis*. SIAM.

Moszner, Z. (1989). Une généralisation d'un résultat de J. Aczél et M. Hosszú sur l'équation de translation. *Aequationes Mathematicae*, 37, 267–278.

Ng, C.T. (1978). Inverse systems and the translation equation on topological spaces. *Acta Mathematica Academiae Scientiarum Hungaricae*, 31(3–4), 227–232.

Ok, E.A., Masatlioglu, Y. (2007). A theory of (relative) discounting. *Journal of Economic Theory*, 137(1), 214–245.

Promislow, S.D. (1997). Classification of usurious loans. *Proceedings of International AFIR Colloquium*, 739–759.

Promislow, S.D., Spring, D. (1996). Postulates for the internal rate of return of an investment project. *Journal of Mathematical Economics*, 26, 325–361.

Rachlin, H. (2006). Notes on discounting. *Journal of the Experimental Analysis of Behavior*, 85(3), 425–435.

Read, D. (2001). Is time-discounting hyperbolic or subadditive? *Journal of Risk and Uncertainty*, 23(1), 5–32.

Roelofsma, P., Read, D. (2000). Intransitive intertemporal choice. *Journal of Behavioral Decision Making*, 13, 161–177.

Scholten, M., Read, D. (2006). Discounting by intervals: a generalized model of intertemporal choice. *Management Science*, 52(9), 1424–1436.

Scholten, M., Read, D. (2010). The psychology of intertemporal tradeoffs. *Psychological Review*, 117(3), 925–944.

Thaler, R. (1981). Some empirical evidence on dynamic inconsistency. *Economics Letters*, 8, 201–207.

Thistle, P.D. (1993). Negative moments, risk aversion, and stochastic dominance. *The Journal of Financial and Quantitative Analysis*, 28(2), 301–311.

Vilenskii, P.L., Smolyak, S.A. (1998). Internal rate of return and its modifications. *Preprint # WP/98/060 CEMI RAS*. (in Russian).