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Portfolio Return Relative to a Benchmark

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Доходность портфеля по отношению к бенчмарку
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Keywords: portfolio performance; compound annual growth rate; benchmarking; index number theory; portfolio choice under uncertainty
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PORTFOLIO RETURN RELATIVE TO A BENCHMARK

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Abstract

Benchmarking is a universal practice in portfolio management and is well-studied in the optimal portfolio selection literature. This paper derives axiomatic foundations for a benchmark-based evaluation, which is generally grounded on the relative return. We show that the existence of a benchmark naturally arises from a few basic axioms and is tightly linked to the economic theory. Our method relies on the use of both axiomatic and economic approaches to index number theory. We also analyze the problem of optimal portfolio selection under complete uncertainty about a future price system, where the objective function is the relative return.

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1. Introduction

The use of benchmarks is a universal practice in both active and passive asset management.[1] A typical measure of performance under the active management is how much better a fund manager did relative to a particular benchmark. Under the passive management, performance is usually measured by a tracking error, i.e., by how much a fund manager deviated from a benchmark.[2] The most popular benchmarks include stock indices (e.g., S&P500), bond indices (e.g., Lehman Brothers Bond Index), or commodity indices (e.g., Goldman Sachs Commodity Index), though they can also be represented by macroeconomic indicators, such as inflation. The present paper considers the axiomatic foundations of the benchmark-based performance measure under the active management.

The practice of benchmarking spanned a vast literature in optimal portfolio selection framework.[3] In this paper, we tackle a different question: whether benchmarking can be justified by

[1] See Siegel (2003) for an extensive summary of the history and practice of benchmarking in the US.

[2] This measure gives rise to the tracking problem: replicating the benchmark as closely as possible using a small number of assets (Barro and Canestrelli, 2009; Beasley et al., 2003; Gilli and Këllezi, 2002).

[3] Earlier papers on benchmarks are concerned with the problem of maximizing expected utility subject to the constraint that wealth level does not fall below a deterministic threshold (Basak, 1995; Grossman and Zhou, 1996; Cox and Huang, 1989). Later papers consider the case with a stochastic benchmark. Browne (1999) was probably one of the first to look at various performance goals associated with a stochastic benchmark. Teplá (2001) studies a more standard problem of maximizing expected utility subject to the constraint that a wealth level does not fall below a stochastic

some basic axioms. Benchmarking is predominantly taken as a starting point in the current literature, which is not surprising since it is universally accepted as a standard practice and has intuitively appealing features. This does not necessarily imply that benchmarking should be used from a theoretical perspective. The question of why benchmarks are used in managers' evaluation remains open. There are some attempts to give it an economic interpretation or rationalize it by its positive impact on fund managers' incentives. This argument, however, is ad hoc and follows from a sheer, though insightful, intuition. The question of whether benchmarking serves to align managers' incentives with investors' incentives is debatable, and this explanation seems to work only in certain cases.

A typical economic interpretation of the benchmark-based evaluation is captured by the “keeping up with the Joneses” (KJ, hereafter) literature. It considers preferences of an agent, who is concerned with his/her consumption relative to the consumption of some reference group, “the Joneses.” This reference consumption can be, for instance, a country's average per capita consumption, or in the case of financial markets, a broad stock market index. KJ-assumption stems from the social and psychological aspects of the market participants stressed by Shiller (1993). Abel (1990) and Gali (1994) were the first to formalize KJ-preferences by specifying the appropriate utility functions and study their implications for the asset pricing. An appealing feature of this model is that it can explain the equity premium puzzle (this, however, comes at the expense of generating new puzzles (Abel, 1990)). Gómez et al. (2009) investigate the implications of the KJ-preferences for the international asset pricing. They argue that this model better captures data patterns than other international pricing models. Teplá (2001) establishes the equivalence between the problem of an agent with KJ-preferences and that of maximizing the expected utility of terminal wealth subject to a stochastic benchmark. The formal behavioral underpinnings of the KJ-model, however, are unclear. Abel (1990) and Gali (1994) do not discuss the axioms that lead to the KJ-utility specification.

Another common justification for benchmarking is that it can be a useful tool for reducing the tension between the incentives of fund managers and investors. Basak et al. (2007) focus on a particular source of this tension: the desire of managers to maximize the inflow of money in the fund they manage. It leads to an excessive risk exposure since positive performance is usually associated with the inflow of money in a fund. They argue that the costs of this behavior can be substantial, but the use of benchmarks can alleviate this issue almost entirely. Some authors, however, argue against using benchmarks in managers' evaluation. Roll (1992) finds that the fund

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benchmark. Basak et al. (2006) relax the benchmark constraint by allowing for a given probability of a shortfall. Davis and Lleo (2008) consider a risk-sensitive control problem, in which the investor's risk aversion enters the objective function directly. The benchmark portfolio approach to stochastic finance (Platen and Heath, 2006) argues to use the “best” performing (growth optimal) portfolio as a benchmark.

managers who maximize relative performance subject to a tracking-error-variance constraint select inefficient mean-variance portfolios. Similarly, Admati and Pfleiderer (1997) consider benchmark-based compensation of fund managers and show that it can have severe adverse effects. Their results imply that the use of benchmarks weakens managers' incentives and leads to selecting sub-optimal portfolios and is not consistent with optimal risk-sharing, among other things.

Both of these lines of reasoning that try to motivate the use of a benchmark in managers' evaluation are mostly ad hoc, even though they are based on a sound psychological intuition or a desire to better align managers' incentives. Our paper is an attempt to derive the existence of a benchmark from the basic axiomatic principles. We focus on a conventional measure of relative performance, the index of a portfolio return relative to a benchmark (the relative return index, for short),[4] derive an axiomatic characterization of this measure, analyze its properties and relate it to the usual microeconomic approach to the measurement of individual welfare change. Then we show how this measure can be applied to the problem of optimal portfolio selection under complete uncertainty.

We start by looking at what can constitute a benchmark. Following the quantitative finance literature, we argue that a reasonable requirement is that a benchmark must be a self-financing portfolio and proceed by characterizing the properties of the self-financing benchmarks. We then employ the axiomatic approach to index number theory (Balk, 2008) to provide cardinal and ordinal axiomatizations of the family of indices of relative portfolio returns, when the benchmark satisfies the self-financing condition. We prove that this family is indeed a natural generalization of the absolute ("non-benchmarked") return index, i.e., when the benchmark value is identically equal to a constant, to the case when there are several possibilities for numéraire. Our ordinal axiomatization is along the lines of Alexeev and Sokolov (2014). Though the characterizations do not explicitly assume benchmarking, the existence of a self-financing benchmark arises from a few general axioms that seem to be desirable when measuring portfolio return. Next, the relative return index is considered through the lens of the economic approach to index number theory (see Diewert (1993) for a review of the approach). We show that if a benchmark value is a concave function of asset prices, then the index coincides with the Allen welfare index (Diewert, 1993, section 3; Ebert, 1984, section 4) for an economic agent with neoclassical (strongly monotonic, upper semicontinuous, convex, and homothetic) preferences over the set of assets, which is a well-known measure of welfare change that has some appealing properties. In this case, the benchmark value as a function of asset prices has a genuine interpretation as the minimum capital needed to receive a utility unit under that prices. The index can also be considered as the absolute return index deflated by the Konüs cost-of-living index (Diewert, 1993, section 3). We conclude the analysis by considering the

[4] As opposed to the absolute return that does not take benchmarks into account.

problem of optimal portfolio selection under complete uncertainty about a future price system, where the objective function is the relative return index. We study the so-called conservative portfolios that have non-negative relative log-returns under any price change. We prove that they correspond to Wald's maximin criterion, which is one of the standard approaches to individual decision making under complete uncertainty (e.g., see Barberà et al. (2004)) and ambiguity, and in portfolio choice theory (Guidolin and Rinaldi, 2013), in particular. A game-theoretic characterization of such portfolios is provided.

The rest of the paper is organized as follows. Section 2 looks at benchmarks restricted to be self-financing portfolios and the implications of this restriction. Section 3 derives the axiomatizations for a family of indices of relative portfolio returns using axiomatic and economic approaches to index number theory. According to a classical result in index number theory, the Laspeyres and Paasche price/quantity indices are the only consistent pair of basket-type indices. In section 4, a similar result is established for the introduced family of indices. Section 5 applies the indices to the problem of optimal portfolio selection under complete uncertainty. Section 6 concludes. All proofs are given in the Appendix.

2. The relative return index

The following notational conventions are used throughout the paper. R_{++}, R_{+} and R are the sets of positive, non-negative and all real numbers, respectively. All operations with vectors are performed component-wise (e.g., $\boldsymbol{x}/\boldsymbol{y} = (x_1/y_1,...,x_n/y_n)$, where $\boldsymbol{x} = (x_1,...,x_n)$, $\boldsymbol{y} = (y_1,...,y_n)$). $\mathbf{0}_n$ and $\mathbf{1}_n$ are the n-dimensional null and the unit vectors. $\cdot$ is the dot product. Given a vector $\boldsymbol{x} = (x_1,...,x_n)$, by $\max(\boldsymbol{x})$ we mean $\max\limits_{i=1,...,n} x_i$; $\min(\boldsymbol{x})$ is defined in a similar manner. The gradient and the superdifferential of a function f at a point $\boldsymbol{x}$ is denoted by $\nabla f(\boldsymbol{x})$ and $\partial f(\boldsymbol{x})$.

We now introduce the main object of our study, relative (to a benchmark) portfolio return. The idea behind benchmarking is to compare the results of an asset management to the results of a particular alternative strategy, a benchmark, rather than to the results keeping assets in cash. Measuring the performance of an asset management against an alternative strategy is equivalent to measuring the performance of a portfolio valued in benchmark units. In other words, the benchmark can be thought of as a numéraire. Benchmarking, as applied to portfolio return, leads to the following index

$$I_f(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}', t') := \left(\frac{\overline{\boldsymbol{p}}' \cdot \boldsymbol{x}'}{f(\boldsymbol{p}')} \bigg/ \frac{\overline{\boldsymbol{p}} \cdot \boldsymbol{x}}{f(\boldsymbol{p})} \right)^{\frac{1}{t'-t}}, \tag{1}$$

which we call the relative return index. It is assumed that there are $n+1$ infinitely divisible assets in the market, $\boldsymbol{x} \in \mathrm{X} := \mathrm{R}_{+}^{n+1} \setminus \{\mathbf{0}_{n+1}\}$ are the quantities of assets in a portfolio at time t, $\boldsymbol{p} \in \mathrm{P} := \mathrm{R}_{++}^{n}$ are the prices of the first n assets measured in the units of the asset $n+1$ (the numéraire) at time t, $\overline{\boldsymbol{p}} := (\boldsymbol{p},1)$, $f(\boldsymbol{p})$ is the market value of a benchmark in the units of asset $n+1$ at time t (assumed to be a function of the price vector $\boldsymbol{p}$), $\boldsymbol{x}'$, $\boldsymbol{p}'$, and $f(\boldsymbol{p}')$ are the values of those variables at time t' ($>t$). The ratios $(\overline{\boldsymbol{p}} \cdot \boldsymbol{x})/f(\boldsymbol{p})$ and $(\overline{\boldsymbol{p}}' \cdot \boldsymbol{x}')/f(\boldsymbol{p}')$ in (1) are the market values of the portfolio at times t and t' measured in the benchmark units, so that index (1) is the usual index of return for a portfolio, whose value is measured in the benchmark units.

A reasonable limitation on a possible alternative strategy (benchmark) is to invest in a self-financing portfolio. Self-financing means that there are no exogenous inflows and outflows of capital in the portfolio. In other words, the purchase of a new asset must be financed by the sale of an old one, and vice versa. As we will see, the self-financing condition imposes certain restrictions on the value function of a benchmark.

Define a *portfolio* of $n+1$ assets as a function $\boldsymbol{x} : \mathrm{P} \to \mathrm{X}$, where the vector $\boldsymbol{x}(\boldsymbol{p}) = (x_1(\boldsymbol{p}),...,x_{n+1}(\boldsymbol{p}))$ comprises the quantities of assets in the portfolio at price system $\boldsymbol{p}$. A portfolio $\boldsymbol{x}$ is said to be *self-financing* if the function $\boldsymbol{p} \mapsto \overline{\boldsymbol{p}} \cdot \boldsymbol{x}(\boldsymbol{p})$, called the portfolio *value function* (measured in the units of the $(n+1)$th asset), is locally Lipschitz and

$$d\big(\overline{\boldsymbol{p}} \cdot \boldsymbol{x}(\boldsymbol{p})\big) = \sum_{i=1}^{n} x_i(\boldsymbol{p}) dp_i \tag{2}$$

almost everywhere (a.e.) with respect to the Lebesgue measure on P.[5]

The definition of a self-financing portfolio assumes no transaction costs to rebalance the portfolio and is an autonomous (meaning that the portfolio structure is a time-independent function of prices) and deterministic counterpart of the equivalent notion used in stochastic finance (e.g., see Björk, 2009, sections 6.1, 6.2). Informally, this definition states that local changes in the value of a self-financing portfolio ($d\big(\overline{\boldsymbol{p}} \cdot \boldsymbol{x}(\boldsymbol{p})\big)$) are caused by changes in prices ($\sum_{i=1}^{n} x_i(\boldsymbol{p}) dp_i$), but not changes in the structure of the portfolio ($\sum_{i=1}^{n+1} \overline{p}_i dx_i(\boldsymbol{p}) = 0$). An obvious example of a self-financing portfolio is a *constant* portfolio, for which $\boldsymbol{x}$ is a constant function. The definition implies that the behavior of a self-financing portfolio is locally equivalent to the behavior of a constant portfolio.

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[5] Recall that, according to Rademacher's theorem (Niculescu and Persson, 2006, theorem 3.11.1, p. 151), the differential of a locally Lipschitz function is defined a.e.

A self-financing portfolio defined this way is a.e. determined by its value function. Indeed, if f is the value function of a self-financing portfolio $\boldsymbol{x}$, then $\boldsymbol{x}(\boldsymbol{p}) = \nabla \bar{f}(\bar{\boldsymbol{p}})$ a.e. with respect to the Lebesgue measure on P, where

$$\bar{f}(\boldsymbol{p}, p_{n+1}) := p_{n+1} f(\boldsymbol{p}/p_{n+1}) \tag{3}$$

is the positively linearly homogeneous (homogeneous, for short) extension of f to R^{n+1}_{++}. In particular, the fractions $\bar{p}_i x_i(\boldsymbol{p})/f(\boldsymbol{p})$, $i = 1,...,n+1$ invested in the assets equal a.e. to the partial elasticities of $\bar{f}$ at $\bar{\boldsymbol{p}}$. Due to this observation, in what follows we describe self-financing portfolios by their value functions, treating portfolios with the same value function as identical or, equivalently, identifying self-financing portfolios that coincide a.e.

The definition of a self-financing portfolio used here implicitly assumes non-stochastic asset price movements. Indeed, one can arrive at (2) in the more commonly used dynamic setting (Björk, 2009, sections 6.1) assuming portfolio structure to be a function of the current price vector only and prices to be differentiable with respect to time, while in continuous-time stochastic calculus prices are a.e. non-differentiable. This leads to the results that are non-inherent to stochastic finance. In particular, in our model there is (deterministic) arbitrage: there are two self-financing portfolios f and f' ($f \neq f'$) such that both are admissible ($f(\boldsymbol{p}_0) = f'(\boldsymbol{p}_0) = M$) at some price vector $\boldsymbol{p}_0$ to an investor with initial wealth M (measured in the units of the $(n+1)$th asset), but $f(\boldsymbol{p}) \leq f'(\boldsymbol{p})$ for any $\boldsymbol{p} \in \mathrm{P}$. For instance, the constant (price weighted) portfolio $f'(\boldsymbol{p}) = (\bar{\boldsymbol{p}} \cdot \mathbf{1}_{n+1})/(n+1)$ always beats the continuously rebalanced equally weighted portfolio $f(\boldsymbol{p}) = \left(\prod_{i=1}^{n} p_i \right)^{\frac{1}{n+1}}$ due to the arithmetic-geometric mean inequality, and $f(\mathbf{1}_n) = f'(\mathbf{1}_n)$ (Rothstein, 1972).

More generally, consider two active portfolio management strategies: "buying in rising markets – selling in falling markets" (BR, for short) and "buying in falling markets – selling in rising markets" (BF, for short). We shall say that a self-financing portfolio is *BR* (resp. *BF*) if the homogeneous extension of its value function to R^{n+1}_{++} is sub- (resp. super-) modular. Recall that a real-valued function g on R^{n+1}_{++} is sub- (resp. super-) modular, if $g(\boldsymbol{z} \vee \boldsymbol{z}') + g(\boldsymbol{z} \wedge \boldsymbol{z}') \leq$ (resp. $\geq$) $g(\boldsymbol{z}) + g(\boldsymbol{z}')$ for all $\boldsymbol{z}, \boldsymbol{z}' \in \mathrm{R}^{n+1}_{++}$, where $\boldsymbol{z} \vee \boldsymbol{z}'$ and $\boldsymbol{z} \wedge \boldsymbol{z}'$ are the component-wise maximum and minimum, respectively. By that definition, the quantity of an asset in a BR (resp. BF) portfolio depends positively (resp. negatively) on the price of that asset and negatively (resp. positively) on prices of the other assets in the market. Namely, if $\boldsymbol{x}$ is a BR portfolio, then

$$\frac{\partial x_i(\boldsymbol{p})}{\partial p_i} \geq 0, \ \frac{\partial x_i(\boldsymbol{p})}{\partial p_j} \leq 0, \ \frac{\partial x_i(\lambda \boldsymbol{p})}{\partial \lambda} \geq 0, \ \frac{\partial x_{n+1}(\lambda \boldsymbol{p})}{\partial \lambda} \leq 0, \ \frac{\partial x_{n+1}(\boldsymbol{p})}{\partial p_j} \leq 0 \ \text{a.e.} \tag{4}$$

for any distinct indices $i, j \in \{1,...,n\}$. Note that since a homogeneous sub- (resp. super-) modular function is convex (resp. concave) (Marinacci and Montrucchio, 2008, theorem 3) and hence, twice differentiable a.e., the partial derivatives in (4) exists a.e. Moreover, for each BF portfolio f there is a BR portfolio f' such $f(\boldsymbol{p}_0) = f'(\boldsymbol{p}_0)$ at some price vector $\boldsymbol{p}_0$ and $f(\boldsymbol{p}) \le f'(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathrm{P}$. The converse does not hold so that the class of BR portfolios is to some extent preferable to the class of BF. The consequences of the existence of an arbitrage opportunity to the problem of optimal portfolio selection under uncertainty are studied in section 5.

While our definition of a self-financing portfolio may seem unreasonable for problems of dynamic stochastic portfolio optimization, it is worth stressing that our paper tackles a very different problem. The goal of the present paper is to provide an axiomatic derivation of the relative return index and justify the use of benchmarks in portfolio evaluations. We believe that our assumptions are well-suited for this particular goal. As will be shown further, they allow for an intuitive and concise characterization of the index by means of axiomatic and economic approaches to index number theory.

The following proposition characterizes value functions of self-financing portfolios.

Proposition 1.

Let f be a positive function on P. The following statements are equivalent:

(a) f is the value function of a self-financing portfolio;

(b) f is non-decreasing in its arguments, locally Lipschitz and

$$\boldsymbol{p} \cdot \nabla f(\boldsymbol{p}) \le f(\boldsymbol{p}) \text{ a.e.;} \tag{5}$$

(c) f is non-decreasing and subhomogeneous, i.e.

$$f(\alpha \boldsymbol{p}) \le \alpha f(\boldsymbol{p}) \text{ for any } \boldsymbol{p} \in \mathrm{P},\ \alpha > 1; \tag{6}$$

(d) the function $\bar{f}$ defined in (3) is non-decreasing in its arguments;

(e) for any $\boldsymbol{p}, \boldsymbol{p}' \in \mathrm{P}$

$$\frac{f(\boldsymbol{p}')}{f(\boldsymbol{p})} \le \max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right). \tag{7}$$

From part (e) of Proposition 1 it follows that self-financing portfolios are exactly those portfolios, whose value does not increase faster than the price of the most profitable asset in the market (equivalently, whose value does not increase more slowly than the price of the least profitable asset). By part (d), given a market with $n+1$ assets, there is a one-to-one correspondence between the set of self-financing portfolios and the set of non-decreasing, homogeneous, and positive functions on R_{++}^{n+1}. The equivalence of parts (b) and (c) can be considered as a version of

Euler's theorem for subhomogeneous functions (e.g., see Martínez-Legaz et al., 2005, preposition 1 and corollary 2).

If f is a value function of a self-financing portfolio, then there is a set $\Lambda \subset \mathrm{R}_{+}^{n+1}$ such that the representation

$$\frac{\bar{\boldsymbol{p}} \cdot \boldsymbol{x}}{f(\boldsymbol{p})} = \inf_{\lambda \in \Lambda} \max\{\boldsymbol{\lambda} \cdot \boldsymbol{y} : \boldsymbol{y} \in \mathrm{X},\ \bar{\boldsymbol{p}} \cdot \boldsymbol{y} = \bar{\boldsymbol{p}} \cdot \boldsymbol{x}\}$$

holds for all $\boldsymbol{x} \in \mathrm{X}$ and $\boldsymbol{p} \in \mathrm{P}$ (Martínez-Legaz et al., 2005, theorem 7). Thus, the ratios of the form $(\bar{\boldsymbol{p}} \cdot \boldsymbol{x})/f(\boldsymbol{p})$ in (1) admit the following interpretation. Given a portfolio with a structure $\boldsymbol{x}$, an investor is allowed to rebalance and sell it in third markets at an unknown price $\boldsymbol{\lambda} \in \Lambda$. Then $(\bar{\boldsymbol{p}} \cdot \boldsymbol{x})/f(\boldsymbol{p})$ is the minimum gain he/she obtains.

Let F be the set of all positive functions on P and let $\mathrm{S} \subset \mathrm{F}$ be the set of value functions of self-financing portfolios. Proposition 1 implies that S forms a convex cone in F, so that the set of self-financing portfolios is closed under the operation of union that assigns to portfolios with value functions f_1 and f_2 the portfolio with the value function $f_1 + f_2$ and the operation of scaling of investments that assigns to a portfolio with a value function f and a scalar $\lambda > 0$ the portfolio with the value function λf. See Martínez-Legaz et al. (2005) and Rubinov (2000, section 3.2) for other properties of the set S. Since it is agreed to identify self-financing portfolios with the same value function, the fact that a portfolio $\boldsymbol{x}$ is self-financing will also be denoted as $\boldsymbol{x} \in \mathrm{S}$.

Given a set $\mathrm{G} \subseteq \mathrm{F}$, put $\mathrm{I(G)} := \{I_f : f \in \mathrm{G}\}$, where I_f (1) is defined on the set $\mathrm{V} := \{\boldsymbol{v} = (\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}', t') \in (\mathrm{X} \times \mathrm{P} \times \mathrm{R})^2 : t < t'\}$ of ordered pairs of dated portfolio observations. The object of our analysis is the family $\mathrm{I(S)}$ and some of its subfamilies. Here are a few examples of indices from $\mathrm{I(S)}$ related to some basic benchmarks.

The simplest and commonly used indices are

$$I_i(\boldsymbol{v}) := \left(\frac{\bar{p}_i}{\bar{p}'_i} \frac{\bar{\boldsymbol{p}}' \cdot \boldsymbol{x}'}{\bar{\boldsymbol{p}} \cdot \boldsymbol{x}} \right)^{\frac{1}{t'-t}},\ i = 1,\ldots,n+1, \tag{8}$$

which in what follows will be referred to as *elementary* indices of return. They correspond to the case when the ith asset is chosen as a benchmark, $f(\boldsymbol{p}) = \bar{p}_i$. In the important special case when the assets $1,\ldots,n+1$ are currencies, I_i is the usual absolute return index for a portfolio, whose value is measured in currency i. We denote the set $\{\boldsymbol{p} \mapsto \bar{p}_i,\ i = 1,\ldots,n+1\}$ by E.

From part (e) of Proposition 1 it follows that $I \in \mathrm{I(S)}$ has obvious bounds in terms of the elementary indices of return:

$$\min_{i=1,\dots,n+1} I_i(\boldsymbol{v}) \le I(\boldsymbol{v}) \le \max_{i=1,\dots,n+1} I_i(\boldsymbol{v}).$$

If, in addition, the investment strategy is self-financing, that is for the given $\boldsymbol{v} = (\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}', t')$ there exists a self-financing portfolio $\boldsymbol{z}$ such that $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{x}$ and $\boldsymbol{z}(\boldsymbol{p}') = \boldsymbol{x}'$, then

$$\left(\min\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right)\Big/\max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right)\right)^{\frac{1}{t'-t}} \le I(\boldsymbol{v}) \le \left(\max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right)\Big/\min\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right)\right)^{\frac{1}{t'-t}}.$$

Another class of widely used benchmarks is stock market indices. These benchmarks are usually related to portfolios with either fixed quantities of assets (e.g., capitalization-weighted indices, price-weighted indices) or fixed money shares allocated to assets (e.g., continuously rebalanced equally weighted indices). The former leads to an affine portfolio value function,

$$f(\boldsymbol{p}) = \bar{\boldsymbol{p}} \cdot \boldsymbol{z}, \ \boldsymbol{z} \in \mathrm{X}, \tag{9}$$

and corresponds to constant portfolios. The latter leads to a multiplicative portfolio value function,

$$f(\boldsymbol{p}) = c\prod_{i=1}^{n} p_i^{\lambda_i}, \ c > 0, \ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Delta^n, \tag{10}$$

where $\Delta^n := \{\boldsymbol{\lambda} \in \mathrm{R}_+^n : \boldsymbol{\lambda} \cdot \mathbf{1}_n \le 1\}$ is the n-dimensional unit simplex. To deduce (10) assume that $f \in \mathrm{S}$ and let $\boldsymbol{\lambda} \in \Delta^n$ be the money shares allocated to the first n assets: $p_i x_i(\boldsymbol{p}) = \lambda_i f(\boldsymbol{p})$, $i = 1, \dots, n$. Then the function $\ln f$ is locally Lipschitz and, by (2), $\nabla(\ln f(\boldsymbol{p})) = \boldsymbol{\lambda}/\boldsymbol{p}$ a.e. Thus, (10) holds. Note that if the assets under consideration are currencies, then (10) corresponds to the case when the effective exchange rate is chosen as a benchmark. Index I_f with the function f of the form (10) is the weighted geometric mean of the elementary indices of return:

$$I_{\boldsymbol{\lambda}}(\boldsymbol{v}) := \prod_{i=1}^{n+1} I_i^{\lambda_i}(\boldsymbol{v}), \text{ where } \lambda_{n+1} := 1 - \boldsymbol{\lambda} \cdot \mathbf{1}_n. \tag{11}$$

In what follows, L and M are the sets of functions on P of the form (9) and (10), respectively.

Finally, one more class of benchmarks to be considered in this paper is portfolios with concave value functions. The importance of this class will be shown in sections 3.2 and 5. Let K be the set of all positive and concave functions on P. Let us show that $\mathrm{K} \subset \mathrm{S}$. Indeed, a function $f \in \mathrm{K}$ is non-decreasing: if $\boldsymbol{p}' - \boldsymbol{p} \in \mathrm{R}_+^n$ and $f(\boldsymbol{p}') < f(\boldsymbol{p})$, then $\partial f(\boldsymbol{p}') \cap \mathrm{R}_+^n = \varnothing$, where $\partial f(\boldsymbol{p}')$ is the superdifferential of f at $\boldsymbol{p}'$, which contradicts the boundedness from below of f. By concavity of f, the inequality

$$f\big(\lambda \boldsymbol{p} + (1-\lambda)(\varepsilon, \dots, \varepsilon)\big) \ge \lambda f(\boldsymbol{p}) + (1-\lambda) f(\varepsilon, \dots, \varepsilon), \ \lambda \in (0,1) \tag{12}$$

holds for any $\varepsilon > 0$. Since f is bounded from below, continuous (follows from concavity), and non-decreasing, tending $\varepsilon \to 0+$ in (12), we get

$$f(\lambda \boldsymbol{p}) \geq \lambda f(\boldsymbol{p}) + (1-\lambda) \lim_{\varepsilon \to 0+} f(\varepsilon,...,\varepsilon) \geq \lambda f(\boldsymbol{p}),\ \lambda \in (0,1). \tag{13}$$

Inequality (13) is equivalent to subhomogeneity of f, hence, by Proposition 1 (part (c)), $f \in \mathrm{S}$.

Note that all the introduced families of indices $\mathrm{I(S)}$, $\mathrm{I(E)}$, $\mathrm{I(L)}$, $\mathrm{I(M)}$, and $\mathrm{I(K)}$ are invariant with respect to the choice of a numéraire asset in the following sense: for any $I \in \mathrm{I(S)}$ (resp. $\mathrm{I(E)}$, $\mathrm{I(L)}$, $\mathrm{I(M)}$, $\mathrm{I(K)}$) and $k \in \{1,...,n\}$ there exists $I' \in \mathrm{I(S)}$ (resp. $\mathrm{I(E)}$, $\mathrm{I(L)}$, $\mathrm{I(M)}$, $\mathrm{I(K)}$) such that

$$I(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}', t') =$$

$$= I'\left(x_1,...,x_{k-1},x_{k+1},...,x_{n+1},x_k,\frac{p_1}{p_k},...,\frac{p_{k-1}}{p_k},\frac{p_{k+1}}{p_k},...,\frac{1}{p_k},t; \right.$$

$$\left. x'_1,...,x'_{k-1},x'_{k+1},...,x'_{n+1},x'_k,\frac{p'_1}{p'_k},...,\frac{p'_{k-1}}{p'_k},\frac{p'_{k+1}}{p'_k},...,\frac{1}{p'_k},t' \right).$$

This observation can be viewed as a very special (deterministic) case of the numéraire invariance property of the self-financing condition in stochastic finance (e.g. see Björk, 2009, lemma 26.1, p. 398).

3. Characterizations of the indices

The introduced family $\mathrm{I(S)}$ and some of its subfamilies admit simple characterizations by means of axiomatic (or test) and economic approaches to index number theory (see Balk (2008) and Diewert (1993) for reviews of these approaches). Neither characterization explicitly assumes benchmarking. The functional form of the index I_f and the interpretation of f as a benchmark value function are delineated by a few axioms and conditions that seem to be desirable for an index of return.

3.1 Axiomatic derivations

In this section we provide an axiomatic characterization of the families $\mathrm{I(S)}$ and $\mathrm{I(M)}$. A number of related axiomatizations of the indices of this type are given by Gray and Dewar (1971), Ebert (1984), Promislow and Spring (1996), Vilenskii and Smolyak (1998), Alexeev and Sokolov (2014).

An *index of return* is a positive function on V. It is natural to assume an index of return to satisfy the following conditions.

Internality:

(i) $\min_{i=1,...,n+1} I_i(\boldsymbol{v}) \leq I(\boldsymbol{v}) \leq \max_{i=1,...,n+1} I_i(\boldsymbol{v})$.

According to (i) an index of return is consistent with elementary indices: $I(\boldsymbol{v})$ is a mean of $I_i(\boldsymbol{v})$, $i = 1,...,n+1$.

Circular test:

(ii) $I^{t'-t}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t')I^{t''-t'}(\boldsymbol{x}',\boldsymbol{p}',t';\boldsymbol{x}'',\boldsymbol{p}'',t'') = I^{t''-t}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}'',\boldsymbol{p}'',t'')$.

Condition (ii) is a version of a circular (transitivity) test (Balk, 2008, section 3.4.1) in index number theory for unequal time intervals. This shows how an index of return over a consolidated period is related to indices of return over its subperiods.

Functional dependence on the elementary indices of return:

(iii) there exists a function G such that $I(\boldsymbol{v}) = G(I_1(\boldsymbol{v}),...,I_{n+1}(\boldsymbol{v}))$.

According to this condition, I is a function of the elementary indices of return I_i, $i = 1,...,n+1$ (equivalently, I is a function of the absolute portfolio return index I_{n+1} and the absolute return indices for individual assets $(p_i'/p_i)^{\frac{1}{t'-t}}$, $i = 1,...,n$).

Dimensional invariance:

(iv) $I(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') = I\left((\boldsymbol{\alpha},\alpha_{n+1})\boldsymbol{x},\frac{\alpha_{n+1}}{\boldsymbol{\alpha}}\boldsymbol{p},t;(\boldsymbol{\alpha},\alpha_{n+1})\boldsymbol{x}',\frac{\alpha_{n+1}}{\boldsymbol{\alpha}}\boldsymbol{p}',t'\right)$ for any $(\boldsymbol{\alpha},\alpha_{n+1}) = (\alpha_1,...,\alpha_n,\alpha_{n+1}) \in \mathrm{R}^{n+1}_{++}$.

Condition (iv) states that an index of return is invariant with respect to changes in the units of measurement of assets.

To formulate the next condition we introduce the following notation: given a vector $\boldsymbol{p} = (p_1,...,p_n)$, define $(p_i',\boldsymbol{p}_{-i}) := (p_1,...,p_{i-1},p_i',p_{i+1},...,p_n)$.

Independence from prices of irrelevant assets:

(v) if $x_i = x_i' = 0$, $i \in \{1,...,n\}$, then $I(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',(p_i,\boldsymbol{p}'_{-i}),t') = I(\boldsymbol{x},(p_i',\boldsymbol{p}_{-i}),t;\boldsymbol{x}',\boldsymbol{p}',t')$; if $x_{n+1} = x_{n+1}' = 0$, then $I(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') = I(\boldsymbol{x},\alpha\boldsymbol{p},t;\boldsymbol{x}',\alpha\boldsymbol{p}',t')$ for any $\alpha > 0$.

The condition states that an index of return does not depend on the prices of assets not included in a portfolio.

Now we give a characterization of the family $\mathrm{I(S)}$.

Theorem 1.

Let I be an index of return. The following two statements are equivalent:

(a) (i) and (ii) hold;

(b) $I \in \mathrm{I(S)}$.

The given characterization admits an obvious generalization to the case when only assets from a set $\{n+1\} \subseteq \mathrm{J} \subseteq \{1,...,n+1\}$ can serve as numéraires (for instance, J may contain only assets that are currencies). Condition (i) then reduces to

(i)' $\min_{i \in \mathrm{J}} I_i(\boldsymbol{v}) \le I(\boldsymbol{v}) \le \max_{i \in \mathrm{J}} I_i(\boldsymbol{v})$,

and any index of return that satisfies (i)' and (ii) has the form (1), where f is a non-decreasing and subhomogeneous function of the elements of a price vector with indices from the set $\mathrm{J} \setminus \{n+1\}$. This observation shows that the family $\mathrm{I(S)}$ is a natural generalization of the index of absolute return I_{n+1} to the case when multiple numéraires are possible. From the proof of Theorem 1 it also follows that the conditions

(i)" $I(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}, t') = I_{n+1}(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}, t')$

and (ii) characterize the family $\mathrm{I(F)}$.

One can prove that each system of the axioms {(i), (ii), (iii)}, {(i), (ii), (iv)}, and {(i), (ii), (v)} is independent (i.e., any two of the axioms do not imply the third) whenever $n \ge 1$. According to the next result, the systems are equivalent and characterize $\mathrm{I(M)}$.

Theorem 2.

Let $I \in \mathrm{I(S)}$. The following statements are equivalent:

(a) (iii) holds;

(b) (iv) holds;

(c) (v) holds;

(d) $I \in \mathrm{I(M)}$.

An index of return can be used to simply rank performances of various portfolios, or the same portfolio at different states. In that case it is sufficient to define the index up to an order-preserving transformation. For example, a wide range of portfolio performance measures used in finance are order-preserving transformations of I_f and thus define the same ordering: $I_f/(1+r)$, where r stands for the inflation rate, is the real index of growth, $I_f - 1$ is the compound annual growth rate, $\ln I_f$ is the logarithmic rate of return, etc. To formalize this case, a total preorder (a complete and transitive binary relation) $\succeq$ is assumed to be defined on V and ranks elements of V by their return. An index of return I represents $\succeq$ if

$$\boldsymbol{v} \succeq \boldsymbol{v}' \Leftrightarrow I(\boldsymbol{v}) \ge I(\boldsymbol{v}').$$

We require $\succeq$ (with $\sim$ and $\succ$ being the symmetric and asymmetric parts of $\succeq$) to satisfy the following natural conditions.

Monotonicity and solvability:

(I) (A) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \succ (\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}'',\boldsymbol{p}',t')$ whenever $\boldsymbol{x}'-\boldsymbol{x}''\in \mathrm{X}$; (B) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{0}_n,x'_{n+1},\boldsymbol{p}',t') \succeq (\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{0}_n,x'_{n+1},\boldsymbol{p}'',t')$ whenever $\boldsymbol{p}''-\boldsymbol{p}'\in \mathrm{R}^n_+$; $(\boldsymbol{x},\boldsymbol{p},t;x'_1,\dots,x'_n,0,\boldsymbol{p}',t') \succeq (\boldsymbol{x},\boldsymbol{p},t;x'_1,\dots,x'_n,0,\alpha\boldsymbol{p}',t')$ for any $\alpha\in(0,1)$; (C) for every $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t'),\boldsymbol{w}\in \mathrm{V}$ there exist $\boldsymbol{y},\boldsymbol{y}'\in \mathrm{X}$ such that $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{p}',t') \succeq \boldsymbol{w} \succeq (\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y}',\boldsymbol{p}',t')$.

Averaging:

(II) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \sim (\boldsymbol{x}',\boldsymbol{p}',t';\boldsymbol{x}'',\boldsymbol{p}'',t'')$ implies $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \sim (\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}'',\boldsymbol{p}'',t'')$.

Scale invariance:

(III) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \sim (\alpha\boldsymbol{x},\boldsymbol{p},t;\alpha\boldsymbol{x}',\boldsymbol{p}',t')$ for any $\alpha>0$.

Time-shift invariance:

(IV) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \sim (\boldsymbol{x},\boldsymbol{p},t+\tau;\boldsymbol{x}',\boldsymbol{p}',t'+\tau)$ for any $\tau\in \mathrm{R}$.

Continuity:

(V) $\succeq$ is a closed subset of V^2.

Value dependence:

(VI) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \sim (\boldsymbol{y},\boldsymbol{p},t;\boldsymbol{y}',\boldsymbol{p}',t')$ whenever $\bar{\boldsymbol{p}}\cdot\boldsymbol{x}=\bar{\boldsymbol{p}}\cdot\boldsymbol{y}$ and $\bar{\boldsymbol{p}}'\cdot\boldsymbol{x}'=\bar{\boldsymbol{p}}'\cdot\boldsymbol{y}'$.

Dimensional covariance:

(VII) $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \succeq (\boldsymbol{y},\boldsymbol{q},\tau;\boldsymbol{y}',\boldsymbol{q}',\tau')$ implies $\left((\boldsymbol{\alpha},\alpha_{n+1})\boldsymbol{x},\frac{\alpha_{n+1}}{\boldsymbol{\alpha}}\boldsymbol{p},t;(\boldsymbol{\alpha},\alpha_{n+1})\boldsymbol{x}',\frac{\alpha_{n+1}}{\boldsymbol{\alpha}}\boldsymbol{p}',t'\right) \succeq \left((\boldsymbol{\alpha},\alpha_{n+1})\boldsymbol{y},\frac{\alpha_{n+1}}{\boldsymbol{\alpha}}\boldsymbol{q},\tau;(\boldsymbol{\alpha},\alpha_{n+1})\boldsymbol{y}',\frac{\alpha_{n+1}}{\boldsymbol{\alpha}}\boldsymbol{q}',\tau'\right)$ for any vector $(\boldsymbol{\alpha},\alpha_{n+1})=(\alpha_1,\dots,\alpha_n,\alpha_{n+1})\in \mathrm{R}^{n+1}_{++}$.

Interpretations of the conditions (I)–(VII) are as follows.

Condition (I) (part (A)) states that return is strictly increasing in assets' quantities at the final state. Condition (I) (part (C)) is a version of the solvability axiom (e.g., see Krantz et al., 1971, section 4.2; Alexeev and Sokolov, 2014). This means that return can be set arbitrarily low/high by changing the assets' quantities at the final state. By condition (I) (part (B)), an increase in the prices of assets not included in the portfolio reduces the return. According to condition (II), if a portfolio is equally profitable for two subsequent periods, the return over the consolidated period will be the same as the return in each of the subperiods. This is a weak form of the internality axiom used by Vilenskii and Smolyak (1998) (see also Alexeev and Sokolov (2014)) to characterize the internal rate of return. The axiom asserts that the average return over the consolidated period must fall between the returns over its subperiods. The axiom validates the following obvious guidance: to guarantee a target level of return over a given period it suffices to keep the target on each subperiod. Condition (III) claims that return takes no account of the investment size and hence is a relative measure. Most of the known measures of return satisfy this property. Condition (IV) is a stationary-like assumption. This states that return is invariant with respect to a time shift, it depends on dates t and t' through their difference $t'-t$. Condition (V) is a standard technical assumption, this means that $\succeq$ is preserved under limits. Condition (VI) states that return depends on a portfolio structure through its value. The condition assumes no transaction costs to rebalance a portfolio, so that portfolios that share common values at time t and t' must have the same return since they can be transformed into each other by a chain of deals with zero transaction costs. Finally, according to (VII), $\succeq$ is preserved under changes in the units of measurement of the assets.

Our next result provides an ordinal characterization of $\mathrm{I(S)}$. It states that conditions (I)–(VI) specify the total preorders represented by the elements of $\mathrm{I(S)}$.

Theorem 3.

For a total preorder $\succeq$ on V the following statements are equivalent:

(a) (I)–(VI) holds;

(b) there exists an index $I \in \mathrm{I(S)}$ that represents $\succeq$.

Note that the characterizations of $\mathrm{I(S)}$ in Theorem 1 and Theorem 3 do not explicitly assume benchmarking. The existence of a self-financing benchmark naturally arises from a few basic axioms that seem to be reasonable when measuring portfolio return.

Now we give an ordinal characterization of the family $\mathrm{I(M)}$.

Theorem 4.

For a total preorder $\succeq$ on V the following two statements are equivalent:

(a) (I)–(VII) holds;

(b) there exists an index $I \in \mathrm{I(M)}$ that represents $\succeq$.

Theorem 3 and Theorem 4 describe the limits of applicability of indices $\mathrm{I(S)}$ and $\mathrm{I(M)}$. For instance, they cannot be applied to the cases of time-varying inflation (condition (IV) is violated). They cannot be used if one accepts a time-inconsistent model of discounting (violation of conditions (II) and (IV)), or expects return to be dependent on an investment size (violation of condition (III)), etc.

3.2 Economic derivations

In this section we use the economic approach to index numbers theory to characterize the family $\mathrm{I(K)}$. We establish the equivalence of $\mathrm{I(K)}$ and the two well-known indices of welfare change that have a number of appealing properties. Namely, $\mathrm{I(K)}$ coincides with the set of Allen welfare indices for economic agents with homothetic preferences on the set X, and with the set of Konüs-Pollak implicit quantity indices.

Consider an investor with strongly monotonic, upper semicontinuous, and convex preferences (a total preorder) over X. By Rader's theorem (Rader, 1963), the preferences can be represented by a utility function $u:\mathrm{X}\to\mathrm{R}_+$ satisfying the following properties: (A) u is strictly increasing ($\boldsymbol{x}-\boldsymbol{x}'\in\mathrm{R}^{n+1}_{++} \;\Rightarrow\; u(\boldsymbol{x})>u(\boldsymbol{x}')$); (B) u is upper semicontinuous (for any y the set $\{\boldsymbol{x}\in\mathrm{X}:u(\boldsymbol{x})\ge y\}$ is closed in X), (C) u is quasi-concave (the set $\{\boldsymbol{x}\in\mathrm{X}:u(\boldsymbol{x})\ge y\}$ is convex for every y). Extend the domain of u to R^{n+1}_{+} by putting $u(\mathbf{0}_{n+1}):=\max\{y:(\mathbf{0}_{n+1},y)\in\mathrm{cl}(\mathrm{Hyp}\,u)\}$, where $\mathrm{cl}(\mathrm{Hyp}\,u)$ is the closure of the hypograph $\mathrm{Hyp}\,u:=\{(\boldsymbol{x},y):u(\boldsymbol{x})\ge y\}$ of u in R^{n+2}. The constructed extension is strictly increasing, upper semicontinuous, and quasi-concave on R^{n+1}_{+}.[6] Since u is defined up to a right-continuous order preserving transformation, without loss of generality, we assume that (D) u is unbounded above and (E) $u(\mathbf{0}_{n+1})=0$ (Shah, 2007, remark 2.2). In what follows, U is the set of all real function on R^{n+1}_{+} satisfying (A)–(E).

Recall that a utility function $u\in\mathrm{U}$ is homothetic if there exist a strictly increasing function g and a homogeneous utility function $\tilde{u}\in\mathrm{U}$ such that $u=g\circ\tilde{u}$. From the definition it follows that $\tilde{u}$ must be concave, g must be right-continuous and $g(0)=0$. We define $\tilde{\mathrm{U}}:=\{u\in\mathrm{U}:u \text{ is homothetic}\}$.

Let $\mathrm{B}(\boldsymbol{p},M):=\{\boldsymbol{x}\in\mathrm{R}^{n+1}_{+}:\bar{\boldsymbol{p}}\cdot\boldsymbol{x}=M\}$ be the budget set for the investor who faces a price vector $\boldsymbol{p}\in\mathrm{P}$ and has capital $M\ge 0$ (measured in the units of the asset $n+1$). The indirect utility function and the expenditure function of the investor are given by $v(\boldsymbol{p},M):=\max\limits_{\boldsymbol{x}\in\mathrm{B}(\boldsymbol{p},M)}u(\boldsymbol{x})$ and $e(\boldsymbol{p},y):=\inf\{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}:u(\boldsymbol{x})\ge y,\boldsymbol{x}\in\mathrm{R}^{n+1}_{+}\}$, respectively. In the sequel, the utility maximization problem in the definition of $v(\boldsymbol{p},M)$ is denoted by $\langle u,\boldsymbol{p},M\rangle$.

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[6] Indeed, since $u(\boldsymbol{x})=\max\{y:(\boldsymbol{x},y)\in\mathrm{cl}(\mathrm{Hyp}\,u)\}$, then the hypograph of the extension is $\mathrm{cl}(\mathrm{Hyp}\,u)$ and, therefore, the extension is upper semicontinuous. The extension is quasi-concave since the closure of the convex set $\{\boldsymbol{x}\in\mathrm{X}:u(\boldsymbol{x})\ge y\}$ is convex. Finally, if $u(\boldsymbol{x})\le u(\mathbf{0}_{n+1})$ for some $\boldsymbol{x}\in\mathrm{R}^{n+1}_{++}$, then, by quasi-concavity, $u(\lambda\boldsymbol{x})=u(\lambda\boldsymbol{x}+(1-\lambda)\mathbf{0}_{n+1})\ge\min\{u(\boldsymbol{x}),u(\mathbf{0}_{n+1})\}=u(\boldsymbol{x})$, $\lambda\in(0,1)$ which contradicts property (A) on the set X.

A well-known measure of welfare change over the situation $\boldsymbol{v} = (\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}', t') \in \mathrm{V}$ for an agent with preferences represented by a utility function $u \in \mathrm{U}$, the Konüs–Pollak implicit quantity index (Diewert, 1993, section 3), is given by

$$KP_u^{(u_0)}(\boldsymbol{v}) := \left(\frac{\overline{\boldsymbol{p}}' \cdot \boldsymbol{x}'}{\overline{\boldsymbol{p}} \cdot \boldsymbol{x}} \Big/ P_u^{(u_0)}(\boldsymbol{p}, \boldsymbol{p}') \right)^{\frac{1}{t'-t}},$$

where $P_u^{(u_0)}(\boldsymbol{p}, \boldsymbol{p}') = e(\boldsymbol{p}', u_0) / e(\boldsymbol{p}, u_0)$ is the Konüs cost-of-living index, e is the expenditure function generated by u, and $u_0 \in \mathrm{R}_{++}$ is a reference utility level. The case of $u_0 = u(\boldsymbol{x})$ (resp. $u_0 = u(\boldsymbol{x}')$) corresponds to the Laspeyres– (resp. Paasche–) Konüs cost-of-living index.

Another important welfare change measure is the Allen welfare index (Diewert, 1993, section 3; Ebert, 1984, section 4)

$$A_u^{(\boldsymbol{p}_0)}(\boldsymbol{v}) := \left(\frac{e(\boldsymbol{p}_0, v(\boldsymbol{p}', \overline{\boldsymbol{p}}' \cdot \boldsymbol{x}'))}{e(\boldsymbol{p}_0, v(\boldsymbol{p}, \overline{\boldsymbol{p}} \cdot \boldsymbol{x}))} \right)^{\frac{1}{t'-t}}, \tag{14}$$

where e and v are the expenditure and indirect utility functions generated by u, $\boldsymbol{p}_0 \in \mathrm{P}$ is a reference price vector. The index (14) is a relative counterpart of a difference-based welfare change measure, general variation (Hammond, 1994), so that the case of $\boldsymbol{p}_0 = \boldsymbol{p}$ (resp. $\boldsymbol{p}_0 = \boldsymbol{p}'$) corresponds to the equivalent (resp. compensating) variation based welfare change measures.

Our next result states that $\mathrm{I(K)}$, the set of Konüs-Pollak implicit quantity indices, and the set of Allen welfare indices for investors with homothetic preferences coincide.

Theorem 5.

$$\mathrm{I(K)} = \{KP_u^{(u_0)} : u_0 \in \mathrm{R}_{++}, u \in \mathrm{U}\} = \{A_u^{(\boldsymbol{p}_0)} : \boldsymbol{p}_0 \in \mathrm{P}, u \in \widetilde{\mathrm{U}}\}.$$

To prove the theorem we need the following proposition, which is of independent interest and clarifies the relation between $\mathrm{I(K)}$ and the Allen welfare indices.

Proposition 2.

Given functions $f \in \mathrm{F}$ and $u \in \mathrm{U}$, the following statements are equivalent:

(a) there is $\boldsymbol{p}_0 \in \mathrm{P}$ such that $A_u^{(\boldsymbol{p}_0)} = I_f$;

(b) $f \in \mathrm{K}$ and $u = g \circ \widetilde{u}$, where $g : \mathrm{R}_+ \to \mathrm{R}_+$ is a strictly increasing right-continuous function with $g(0) = 0$ and

$$\widetilde{u}(\boldsymbol{x}) = \inf_{\boldsymbol{p} \in \mathrm{P}} \frac{\overline{\boldsymbol{p}} \cdot \boldsymbol{x}}{f(\boldsymbol{p})}; \tag{15}$$

(c) $u \in \widetilde{\mathrm{U}}$, that is there exist a strictly increasing function g and a homogeneous utility function $\widetilde{u} \in \mathrm{U}$ such that $u = g \circ \widetilde{u}$, and

$$f(\boldsymbol{p}) = \inf_{\boldsymbol{x} \in \mathrm{R}_{++}^{n+1}} \frac{\overline{\boldsymbol{p}} \cdot \boldsymbol{x}}{\widetilde{u}(\boldsymbol{x})}. \tag{16}$$

As a consequence of Proposition 2, we obtain that the elementary indices of return correspond to Allen welfare indices for investors who hold their wealth in a particular asset: $\mathrm{I(E)} = \{A_u^{(\boldsymbol{p}_0)} : \boldsymbol{p}_0 \in \mathrm{P},\ u(\boldsymbol{x}) = x_i,\ i \in \{1,\ldots,n+1\}\}$. The sets $\mathrm{I(L)}$ and $\mathrm{I(M)}$ correspond to investors with the Leontief and Cobb–Douglas preferences,

$$\mathrm{I(L)} = \{A_u^{(\boldsymbol{p}_0)} : \boldsymbol{p}_0 \in \mathrm{P},\ u(\boldsymbol{x}) = \min_{i: z_i > 0} x_i / z_i\ ,\ \boldsymbol{z} \in \mathrm{X}\},$$

$$\mathrm{I(M)} = \{A_u^{(\boldsymbol{p}_0)} : \boldsymbol{p}_0 \in \mathrm{P},\ u(\boldsymbol{x}) = \prod_{i=1}^{n+1} x_i^{\lambda_i},\ \boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n) \in \Delta^n,\ \lambda_{n+1} = 1 - \boldsymbol{\lambda} \cdot \mathbf{1}_n\}.$$

By Theorem 5 and Proposition 2, given $f \in \mathrm{K}$, there is a homogeneous utility function $\widetilde{u} \in \mathrm{U}$, $u_0 \in \mathrm{R}_{++}$, and $\boldsymbol{p}_0 \in \mathrm{P}$, such that $I_f = KP_{\widetilde{u}}^{(u_0)} = A_{\widetilde{u}}^{(\boldsymbol{p}_0)}$. Since the expenditure function generated by $\widetilde{u}$ is given by $\widetilde{e}(\boldsymbol{p}, y) = f(\boldsymbol{p})y$, i.e., f is the unit cost function for $\widetilde{u}$, this produces utility-based interpretations for the quantities $f(\boldsymbol{p})$ and $f(\boldsymbol{p}')$ in (1). Namely, one can interpret $f(\boldsymbol{p})$ (resp. $f(\boldsymbol{p}')$) as the minimum capital (measured in the units of the asset $n+1$) needed to receive a utility unit under the prices $\boldsymbol{p}$ (resp. $\boldsymbol{p}'$). We also have the following nonparametric (i.e. independent of the future/past price vector) bounds on I_f

$$\left(\frac{\widetilde{e}(\boldsymbol{p}, \widetilde{u}(\boldsymbol{x}'))}{\overline{\boldsymbol{p}} \cdot \boldsymbol{x}}\right)^{\frac{1}{t'-t}} \le I_f(\boldsymbol{v}) \le \left(\frac{\overline{\boldsymbol{p}}' \cdot \boldsymbol{x}'}{\widetilde{e}(\boldsymbol{p}', \widetilde{u}(\boldsymbol{x}))}\right)^{\frac{1}{t'-t}}. \tag{17}$$

Note that the left-hand (resp. right-hand) side of (17) can be considered as a relative version of the equivalent (resp. compensating) variation.

4. Price and quantity indices

A well-known result in index number theory asserts that the price/quantity indices of Laspeyres and Paasche are the only basket-type indices that are consistent with the product test. In the present context, this test states that portfolio performance can be decomposed into the effect of price changes (measured by the price index) and the effect of portfolio rebalancing (measured by the quantity index). In this section we show that a similar result holds for price and quantity indices induced by $I \in \mathrm{I(S)}$.

An index of return I induces the *price index*

$$P(\boldsymbol{p},t;\boldsymbol{p}',t';\boldsymbol{x}_0) := I(\boldsymbol{x}_0,\boldsymbol{p},t;\boldsymbol{x}_0,\boldsymbol{p}',t')$$

and the *quantity* (or *volume*) *index*

$$Q(\boldsymbol{x},t;\boldsymbol{x}',t';\boldsymbol{p}_0) := I(\boldsymbol{x},\boldsymbol{p}_0,t;\boldsymbol{x}',\boldsymbol{p}_0,t'),$$

where $\boldsymbol{x}_0$ and $\boldsymbol{p}_0$ are reference quantity and price vectors. The price (resp. quantity) index captures the effect of price changes (resp. portfolio rebalancing) on portfolio performance. A price (resp. quantity) index is the *Laspeyres index* if $\boldsymbol{x}_0 = \boldsymbol{x}$ (resp. $\boldsymbol{p}_0 = \boldsymbol{p}$). A price (resp. quantity) index is the *Paasche index* if $\boldsymbol{x}_0 = \boldsymbol{x}'$ (resp. $\boldsymbol{p}_0 = \boldsymbol{p}'$).

Let P and Q be the price and quantity indices induced by I. Given binary operations $*$, $\circ$, and $\bullet$ on $I(\mathrm{V})$, X, and P, respectively, the *generalized product test* holds if

(a) $P(\boldsymbol{p},t;\boldsymbol{p}',t';\boldsymbol{x}\circ\boldsymbol{x}') * Q(\boldsymbol{x},t;\boldsymbol{x}',t';\boldsymbol{p}\bullet\boldsymbol{p}') = I(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t')$ for any $(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t') \in \mathrm{V}$;

(b) if $x_k = x'_k = 0$ for some $k \in \{1,...,n+1\}$, then the k-th element of the vector $\boldsymbol{x}\circ\boldsymbol{x}'$ equals 0.

Part (a) of the definition can be considered as a version of the usual product test in index number theory (Balk, 2008, section 3.4.2). The reference quantity (resp. price) vector in the price (resp. quantity) index in (a) is assumed to be a function of $\boldsymbol{x}$ and $\boldsymbol{x}'$ (resp. $\boldsymbol{p}$ and $\boldsymbol{p}'$). The part (b) of the definition states that whenever the portfolio does not contain a particular asset at the initial and the final states, the reference quantity vector $\boldsymbol{x}\circ\boldsymbol{x}'$ does not contain this asset as well.

Proposition 3.

Let P and Q be the price and quantity indices induced by $I \in \mathrm{I(S)}$. Then the generalized product test holds if and only if $*$ is the operation of multiplication and there is a function $\alpha : \mathrm{X}^2 \to \mathrm{R}_{++}$ such that either $\boldsymbol{x}\circ\boldsymbol{x}' \equiv \alpha(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{x}$, $\boldsymbol{p}\bullet\boldsymbol{p}' \equiv \boldsymbol{p}'$, or $\boldsymbol{x}\circ\boldsymbol{x}' \equiv \alpha(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{x}'$, $\boldsymbol{p}\bullet\boldsymbol{p}' \equiv \boldsymbol{p}$ (that is one of the indices P and Q is the Laspeyres index and the other one is the Paasche index).

A classical result on the consistency of the Laspeyres and Paasche indices among basket-type indices follows from Proposition 3 with $I = I_{n+1}$.

5. The index and the problem of portfolio choice under uncertainty

In this section, we study the problem of optimal portfolio selection under complete uncertainty about a future price system with the objective function I_f. Complete uncertainty about future prices means that, given a current price vector $\boldsymbol{p}$ and an investment period $[t,t')$, the investor knows the set $\{I_f(\boldsymbol{x}(\boldsymbol{p}),\boldsymbol{p},t;\boldsymbol{x}(\boldsymbol{p}'),\boldsymbol{p}',t'),\ \boldsymbol{p}' \in \mathrm{P}\}$ of possible outcomes for each investment strategy $\boldsymbol{x}$, but has no information about the probabilities of those outcomes. There are several approaches to such problems (e.g. see Barberà et al. (2004) and references therein; see also Guidolin and Rinaldi (2013) for a review of some of these approaches to portfolio choice theory). Some of them induce the decision rules closely related to Wald's maximin criterion, which we adopt here.

A desirable investment strategy under complete uncertainty is investing in a portfolio that does not decrease the investor's welfare under any price change. The following definition formalizes such portfolios. A portfolio $\boldsymbol{x} : \mathrm{P} \to \mathrm{X}$ is said to be *conservative* with respect to I (an index of return), $\boldsymbol{p}$ (a price vector), $M > 0$ (capital measured in the units of the asset $n+1$), t and t' $(> t)$ (an investment period), if $\boldsymbol{x}(\boldsymbol{p}) \in \mathrm{B}(\boldsymbol{p}, M)$[7] and $I(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t; \boldsymbol{x}(\boldsymbol{p}'), \boldsymbol{p}', t') \geq I(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t; \boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t')$ for any price vector $\boldsymbol{p}' \in \mathrm{P}$. Under mild conditions a conservative portfolio corresponds to Wald's maximin criterion and admits a simple game-theoretic interpretation. The following proposition is an immediate consequence of the definition and its proof is omitted.

Proposition 4.

Let G be a set of portfolios and let I be an index of return such that for any $\boldsymbol{x}$, $\boldsymbol{x}'$, $\boldsymbol{p}$, t, and t' $(> t)$

$$I(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}, \boldsymbol{p}, t') = I(\boldsymbol{x}', \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}, t') \text{ whenever } \bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \bar{\boldsymbol{p}} \cdot \boldsymbol{x}'. \tag{18}$$

A portfolio $\boldsymbol{x}^* \in \mathrm{G}$ is conservative with respect to I, $\boldsymbol{p}$, M, t, t' if and only if $(\boldsymbol{x}^*, \boldsymbol{p})$ is a pure-strategy Nash equilibrium in the zero-sum game $\langle \{\boldsymbol{x} \in \mathrm{G} : \boldsymbol{x}(\boldsymbol{p}) \in \mathrm{B}(\boldsymbol{p}, M)\}, \mathrm{P}, (\boldsymbol{x}; \boldsymbol{p}') \mapsto I(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t; \boldsymbol{x}(\boldsymbol{p}'), \boldsymbol{p}', t') \rangle$, where the strategy set of the first player (the investor) is the set of feasible portfolios, the strategy set of the second player (the Market) is the set P of possible price vectors, and the payoff function of the first player takes each pair $(\boldsymbol{x}; \boldsymbol{p}')$ of strategies to $I(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t; \boldsymbol{x}(\boldsymbol{p}'), \boldsymbol{p}', t')$.

Note that condition (18) is very mild. For instance, it holds whenever the index of return depends on the portfolio structure through its value (compare with condition (VI) in section 3.1) or whenever the index is equal to a fixed constant (the status quo) if neither the portfolio structure, nor the price vector changes during the investment period. Obviously, the condition (18) holds for I_f. In the rest of this section we determine necessary and sufficient conditions for the existence of a

[7] Recall that $\mathrm{B}(\boldsymbol{p}, M)$ is the investor's budget set.

conservative portfolio with respect to I_f within a given portfolio class (primarily, L, M, K, S). We also describe the structure of such portfolios.

If $I = I_f$ in the definition of a conservative portfolio, then the investment period becomes irrelevant and the notion of conservativity can be reformulated in terms of the portfolio value function. Namely, a portfolio with a value function g is conservative with respect to I_f, $\boldsymbol{p}$, M, t, t' if and only if $g(\boldsymbol{p}) = M$ and $g(\boldsymbol{p}')/g(\boldsymbol{p}) \geq f(\boldsymbol{p}')/f(\boldsymbol{p})$ for any $\boldsymbol{p}'$. A trivial example of a conservative portfolio is the benchmark-tracking portfolio with the value function $g(\boldsymbol{p}') = Mf(\boldsymbol{p}')/f(\boldsymbol{p})$. We denote the set of value functions of all conservative portfolios with respect to I_f, $\boldsymbol{p}$, M by $\mathrm{C}(f, \boldsymbol{p}, M)$. Given f, $\boldsymbol{p}$, M, and a set $\mathrm{G} \subseteq \mathrm{F}$, the question we investigate is whether there exists a conservative portfolio with respect to I_f, $\boldsymbol{p}$, M within the class of portfolios with value functions from the set G, or, in symbols, $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$. It will be convenient to formulate the answer in terms of abstract convexity theory. Recall the following definitions (Rubinov, 2000, section 1.6). Given a nonempty set G of real-valued functions on P, a function $g \in \mathrm{G}$ is called a G-*supergradient* of a function f on P at a point $\boldsymbol{p}$ if $g(\boldsymbol{p}') - g(\boldsymbol{p}) \geq f(\boldsymbol{p}') - f(\boldsymbol{p})$ for all $\boldsymbol{p}' \in \mathrm{P}$. The set $\partial_{\mathrm{G}} f(\boldsymbol{p})$ of all G-supergradients of f at $\boldsymbol{p}$ is referred to as the G-*superdifferential* of the function f at the point $\boldsymbol{p}$. A real-valued function f on P is said to be G-*concave* if $f(\boldsymbol{p}) = \min\{g(\boldsymbol{p}) + c : g \in \mathrm{G},\ c \in \mathrm{R},\ f(\boldsymbol{p}') \leq g(\boldsymbol{p}') + c \text{ for all } \boldsymbol{p}' \in \mathrm{P}\}$ for all $\boldsymbol{p} \in \mathrm{P}$. Clearly, if G is the set of all linear functions on P, then the introduced definitions correspond to the conventional notions of supergradient, superdifferential, and concavity. Given a set $\mathrm{G} \subseteq \mathrm{F}$, the set of the homogeneous extensions of its elements to R^{n+1}_{++} is denoted by $\overline{\mathrm{G}} := \{\bar{g} : \mathrm{R}^{n+1}_{++} \to \mathrm{R}_{++},\ g \in \mathrm{G}\}$.

From these definitions we obtain the following proposition.

Proposition 5.

Let $f \in \mathrm{F}$ and let $\mathrm{G} \subseteq \mathrm{F}$ be a cone ($\lambda g \in \mathrm{G}$ whenever $\lambda > 0$ and $g \in \mathrm{G}$). $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$ if and only if $\partial_{\ln \mathrm{G}} \ln f(\boldsymbol{p}) \neq \varnothing$, where $\ln \mathrm{G} := \{\ln g,\ g \in \mathrm{G}\}$. If, in addition, G is closed under vertical shifts (that is $g + c \in \mathrm{G}$ whenever $c \in \mathrm{R}$, $g \in \mathrm{G}$, $g + c \in \mathrm{F}$), then $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$ if and only if $\partial_{\mathrm{G}} f(\boldsymbol{p}) \neq \varnothing$; moreover, the map $g \mapsto M^{-1} f(\boldsymbol{p}) \bar{g}$ defines a one-to-one correspondence between $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M)$ and $\partial_{\overline{\mathrm{G}}} \bar{f}(\bar{\boldsymbol{p}})$.

As a direct consequence of Proposition 5 we get that $\mathrm{L} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$ if and only if $\partial f(\boldsymbol{p}) \neq \varnothing$; moreover, the map

$$a \mapsto \frac{M}{f(p)}(a, f(p) - a \cdot p) \tag{19}$$

defines a one-to-one correspondence between $\partial f(\boldsymbol{p})$ and the structures of constant portfolios conservative with respect to I_f, $\boldsymbol{p}$, M.

Since a function f is G-concave if and only if $\partial_{\mathrm{G}} f(\boldsymbol{p}) \neq \varnothing$ for all $\boldsymbol{p}$ (Rubinov, 2000, proposition 1.2, p. 10), we obtain the following result.

Corollary 1.

Let $f \in \mathrm{F}$ and $\mathrm{G} \subseteq \mathrm{F}$ be a cone. $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$ for all $\boldsymbol{p}$ and M if and only if $\ln f$ is $\ln \mathrm{G}$-concave. If, in addition, G is closed under vertical shifts, then $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$ for all $\boldsymbol{p}$ and M if and only if f is G-concave.

In particular, if $\mathrm{G} = \mathrm{L}$ (resp. M, K, $\{\boldsymbol{p} \mapsto \max(\boldsymbol{x}\bar{\boldsymbol{p}}), \boldsymbol{x} \in \mathrm{X}\}$, S) then $\mathrm{G} \cap \mathrm{C}(f, \boldsymbol{p}, M) \neq \varnothing$ for all $\boldsymbol{p}$ and M if and only if $f \in \mathrm{K}$ (resp. f is multiplicatively concave (that is the function $\log \circ f \circ \exp$ is concave), $f \in \mathrm{K}$, $f \in \mathrm{S}$ (Rubinov, 2000, p. 82), $f \in \mathrm{S}$).

Now we give a game-theoretic characterization of the set $\mathrm{L} \cap \mathrm{C}(f, \boldsymbol{p}, M)$ for $f \in \mathrm{K}$ and a characterization through the lens of the economic approach to index number theory.

Proposition 6.

Let $f \in \mathrm{K}$. The following statements are equivalent:

(a) a constant portfolio with a structure $\boldsymbol{x}^* \in \mathrm{X}$ is conservative with respect to I_f, $\boldsymbol{p}$, M, t, t';

(b) $\boldsymbol{x}^*$ is a solution of the utility maximization problem $\langle \tilde{u}, \boldsymbol{p}, M \rangle$, where $\tilde{u}$ is defined in (15);

(c) the pair $(\boldsymbol{x}^*; \boldsymbol{p})$ is a pure-strategy Nash equilibrium in the zero-sum game $\langle \mathrm{B}(\boldsymbol{p}, M), \mathrm{P}, (\boldsymbol{x}, \boldsymbol{p}') \mapsto I_f(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}, \boldsymbol{p}', t') \rangle$, where the strategy set of the first player (the investor) is the set $\mathrm{B}(\boldsymbol{p}, M)$ of feasible portfolio structures, the strategy set of the second player (the Market) is the set P of possible price vectors, and the payoff function of the first player takes each pair $(\boldsymbol{x}; \boldsymbol{p}')$ of strategies to $I_f(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}, \boldsymbol{p}', t')$;

(d) $M^{-1} f(\boldsymbol{p}) \boldsymbol{x}^* \in \partial \bar{f}(\bar{\boldsymbol{p}})$.

$\mathrm{L} \cap \mathrm{C}(f, \boldsymbol{p}, M)$ is a singleton if and only if f is differentiable at $\boldsymbol{p}$; the structure of this portfolio (if any) is given by

$$\boldsymbol{x}^* = M \nabla \ln \bar{f}(\bar{\boldsymbol{p}}). \tag{20}$$

Under the conditions of Proposition 6, we have the following.

– Given f and M, the set $\mathrm{L} \cap \mathrm{C}(f, \boldsymbol{p}, M)$ is a singleton a.e. with respect to the Lebesgue measure on P;

– Constant portfolios that are conservative with respect to I_f, $\boldsymbol{p}$, M, t, t' correspond to Wald's maximin decision rule. Namely, the structures of these portfolios are the solutions of the problem

$$\max_{\boldsymbol{x} \in \mathrm{B}(\boldsymbol{p}, M)} \inf_{\boldsymbol{p}' \in \mathrm{P}} I_f(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}, \boldsymbol{p}', t').$$

– If f is differentiable at $\boldsymbol{p}$, then a constant portfolio with a structure $\boldsymbol{x}^* \in \mathrm{X}$ is conservative with respect to I_f, $\boldsymbol{p}$, M if and only if the budget shares $\bar{p}_i x_i^* / M$ invested in assets $i = 1, \ldots, n+1$ equal to the partial elasticities of $\bar{f}$ at $\bar{\boldsymbol{p}}$. For instance, given $\boldsymbol{p}$ and M, the constant portfolio with the structure $\boldsymbol{x}^* = M(\boldsymbol{\lambda}, \lambda_{n+1}) / \bar{\boldsymbol{p}}$ is conservative with respect to $I_{\boldsymbol{\lambda}}$ (11); the constant portfolio with the structure $\boldsymbol{x}^* = M\boldsymbol{z} / (\bar{\boldsymbol{p}} \cdot \boldsymbol{z})$ is conservative with respect to I_f, where $f(\boldsymbol{p}') = \bar{\boldsymbol{p}}' \cdot \boldsymbol{z}$, $\boldsymbol{z} \in \mathrm{X}$; investing money entirely in asset i is a conservative strategy with respect to I_i (8).

– Though conservativity is a "global" property, it admits the following "local" characterization: if f is differentiable at $\boldsymbol{p}$, then a constant portfolio with a structure $\boldsymbol{x}^* \in \mathrm{B}(\boldsymbol{p}, M)$ is conservative with respect to I_f, $\boldsymbol{p}$, M, t, t' if and only if its return is locally independent of price movements:

$$\left. \frac{\partial I_f(\boldsymbol{x}^*, \boldsymbol{p}, t; \boldsymbol{x}^*, \boldsymbol{p}', t')}{\partial \boldsymbol{p}'} \right|_{\boldsymbol{p}' = \boldsymbol{p}} = \boldsymbol{0}_n .$$

Let us consider a robust version of the described problem. Assume now that the investor has in mind a collection of possible benchmarks G and seeks a portfolio that is conservative with respect to all the benchmarks in G. Given $\mathrm{G} \subseteq \mathrm{F}$, we define $\mathrm{C}(\mathrm{G}, \boldsymbol{p}, M) := \bigcap_{g \in \mathrm{G}} \mathrm{C}(g, \boldsymbol{p}, M)$.

Proposition 7.

Let a set G be such that $\mathrm{E} \subseteq \mathrm{G} \subseteq \mathrm{S}$ and let f be the value function of a portfolio $\boldsymbol{x}^* \in \mathrm{S}$. The following statements are equivalent:

(a) $f \in \mathrm{C}(\mathrm{G}, \boldsymbol{p}, M)$;

(b) $\boldsymbol{x}^*$ is a dominant strategy of the first player in the zero-sum game $\langle \{\boldsymbol{x} \in \mathrm{S} : \boldsymbol{x}(\boldsymbol{p}) \in \mathrm{B}(\boldsymbol{p}, M)\}, \mathrm{G} \times \mathrm{P}, (\boldsymbol{x}; g, \boldsymbol{p}') \mapsto I_g(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t; \boldsymbol{x}(\boldsymbol{p}'), \boldsymbol{p}', t') \rangle$, where the strategy set of the first player (the investor) is the set of feasible self-financing portfolios, the strategy set of the second player (the Market) is the Cartesian product of set G of benchmarks and the set P of possible price vectors, and the payoff function of the first player takes each triple $(\boldsymbol{x}; g, \boldsymbol{p}')$ to $I_g(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, t; \boldsymbol{x}(\boldsymbol{p}'), \boldsymbol{p}', t')$;

(c) $f(\boldsymbol{p}') = M \max(\bar{\boldsymbol{p}}' / \bar{\boldsymbol{p}})$.

Proposition 7 shows that if a collection of possible benchmarks $\mathrm{G} \subseteq \mathrm{S}$ is "large enough," then within the class S there is a unique portfolio that is conservative with respect to all the benchmarks in G. Unsurprisingly, this portfolio holds wealth entirely in the most profitable asset in the market.

6. Conclusion

The use of the relative return index to measure the performance of an active portfolio management is a universal practice in modern finance. Despite this fact, the theoretical justification of the use of benchmarks in portfolio evaluations remains an open question. The present study provides such a justification and describes the properties of the relative return index.

We show that a reasonable restriction on a benchmark, the self-financing condition, corresponds to the case of a non-decreasing and subhomogeneous function f in the definition of the relative return index I_f. The relative return index admits simple characterizations by means of axiomatic and economic approaches to index number theory. Without any explicit assumptions about benchmarking, the benchmark-based functional form of the index follows from a few general axioms that are desirable when measuring portfolio return. Provided that a benchmark value is a concave function of asset prices, the relative return index coincides with the Allen welfare index for an economic agent with neoclassical preferences over the set of assets, which is a well-known measure of welfare change with many appealing properties. The value of a benchmark as a function of asset prices, in this case, has a genuine interpretation of the minimum capital that is needed to receive a utility unit under that prices.

The price and quantity indices induced by the relative return index are consistent with the generalized product test (i.e. they decompose portfolio performance into the effect of price changes and the effect of portfolio rebalancing), if and only if one of them is the Laspeyres index, and the other one is the Paasche index. An application of the relative return index to the problem of portfolio choice under uncertainty leads to strategies that can be easily described in terms of abstract convexity. These strategies admit a genuine game-theoretic interpretation and a characterization through the lens of the economic approach.

The results obtained from our analysis justify the use of benchmark-based portfolio evaluation and determine the limits of its applicability. The simplicity and conciseness of the obtained results are, however, due to a simplified portfolio model we adopt. In particular, the portfolio value function is assumed to be Lipschitz continuous, and the portfolio structure is assumed to be a function of the current price vector only. These assumptions are rather restrictive. They do not allow for stochastic price movements, permit an arbitrage opportunity in the market, and exclude some usual portfolio strategies from consideration. An important next step would be to consider a less restrictive portfolio model to extend the present analysis to a stochastic setting used in mathematical finance.

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Appendix: Proofs

Proof of Proposition 1.

(a)$\Rightarrow$(b). Let $f(\boldsymbol{p}) := \bar{\boldsymbol{p}} \cdot \boldsymbol{x}(\boldsymbol{p})$ be the value function of a self-financing portfolio $\boldsymbol{x}$. Since $f(\boldsymbol{p}) - \boldsymbol{p} \cdot \nabla f(\boldsymbol{p}) = x_{n+1}(\boldsymbol{p}) \geq 0$ a.e., (5) holds. Let us show that f is non-decreasing. By (2) for almost all $p_2,...,p_n$ the function $f(\cdot, p_2,...,p_n)$ has a.e. non-negative derivative. The restriction of $f(\cdot, p_2,...,p_n)$ to an arbitrary closed bounded interval is globally Lipschitz and hence absolutely continuous. An absolutely continuous function with a.e. non-negative derivative is non-decreasing. Therefore, $f(\cdot, p_2,...,p_n)$ is non-decreasing for almost all $p_2,...,p_n$, and, thus, for all $p_2,...,p_n$, since f is continuous.

(b)$\Rightarrow$(c). Let G be the set of non-differentiability points of f and let 1_{G} be the characteristic function of G. By Rademacher's theorem, G is a set of Lebesgue measure zero. Converting to polar coordinates in the Lebesgue integral $\int_{\mathrm{P}} 1_{\mathrm{G}}(\boldsymbol{p}) d\boldsymbol{p} = 0$ yields that for almost all $\boldsymbol{p}$ the function f is differentiable a.e. on the ray $\{t\boldsymbol{p} : t \in \mathrm{R}_{++}\}$. Put $g_{\boldsymbol{p}}(t) := \ln f(t\boldsymbol{p})$. By (5), for almost all $\boldsymbol{p}$

$$g'_{\boldsymbol{p}}(t) = \frac{\boldsymbol{p} \cdot \nabla f(t\boldsymbol{p})}{f(t\boldsymbol{p})} \leq 1/t \text{ a.e.} \tag{21}$$

The restriction of the function $g_{\boldsymbol{p}}$ to the interval $[1, \alpha]$ is absolutely continuous. Integrating the inequality (21) over the interval $[1, \alpha]$ yields (6).

(c)$\Leftrightarrow$(d). Proved in Rubinov (2000, theorem 3.1, p. 80).

(d)$\Rightarrow$(e). If $\bar{f}$ is non-decreasing in its arguments, then

$$\frac{f(\boldsymbol{p}')}{f(\boldsymbol{p})}=\frac{\bar{f}(\bar{\boldsymbol{p}}')}{\bar{f}(\bar{\boldsymbol{p}})}\le\frac{\bar{f}\left(\max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right)\bar{\boldsymbol{p}}\right)}{\bar{f}(\bar{\boldsymbol{p}})}=\max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right).$$

(e)$\Rightarrow$(a). A positive function f, satisfying (7), is subhomogeneous,

$$f(\alpha\boldsymbol{p})\le\max\left(\frac{(\alpha\boldsymbol{p},1)}{\bar{\boldsymbol{p}}}\right)f(\boldsymbol{p})=\alpha f(\boldsymbol{p}),\ \alpha>1,$$

and is non-decreasing: if $\boldsymbol{p}-\boldsymbol{p}'\in\mathrm{R}^n_+$ then

$$f(\boldsymbol{p}')\le\max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right)f(\boldsymbol{p})=f(\boldsymbol{p}).$$

Thus, (c) and (d) hold. A function f satisfying (c) is continuous Rubinov (2000, p. 78). Let $\mathrm{B}\subset\mathrm{R}^n_{++}$ be a closed ball in R^n. For any $\boldsymbol{p},\boldsymbol{p}'\in\mathrm{B}$ we have

$$|f(\boldsymbol{p}')-f(\boldsymbol{p})|\le\max\left(\frac{|\boldsymbol{p}'-\boldsymbol{p}|}{\boldsymbol{p}}\right)f(\boldsymbol{p})\le\max(|\boldsymbol{p}'-\boldsymbol{p}|)\frac{f(\boldsymbol{p})}{\min(\boldsymbol{p})}.$$

Since the function $\boldsymbol{p}\mapsto f(\boldsymbol{p})/\min(\boldsymbol{p})$ is continuous and thus bounded on B, f is locally Lipschitz.

Put $\boldsymbol{x}(\boldsymbol{p}):=\nabla\bar{f}(\bar{\boldsymbol{p}})$ if f is differentiable at $\boldsymbol{p}$, and $x_i(\boldsymbol{p}):=0$, $i=1,...,n$, $x_{n+1}(\boldsymbol{p}):=f(\boldsymbol{p})$ otherwise. Then $\bar{\boldsymbol{p}}\cdot\boldsymbol{x}(\boldsymbol{p})\equiv f(\boldsymbol{p})$ and (2) holds. Since $\bar{f}$ is non-decreasing, $\boldsymbol{x}(\boldsymbol{p})\in\mathrm{X}$. ■

Proof of Theorem 1.

(a)$\Rightarrow$(b). From internality (i) it follows that $I(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x},\boldsymbol{p},t')=1$. Applying (ii) with $(\boldsymbol{x},\boldsymbol{p})=(\boldsymbol{x}',\boldsymbol{p}')$ and with $(\boldsymbol{x}',\boldsymbol{p}')=(\boldsymbol{x}'',\boldsymbol{p}'')$, we obtain that for each fixed $\boldsymbol{x}'$, $\boldsymbol{p}'$, $\boldsymbol{x}''$, $\boldsymbol{p}''$ the value $I^{t'-t}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t')$ is independent of t and t'. Define $J(\boldsymbol{x},\boldsymbol{p};\boldsymbol{x}',\boldsymbol{p}'):=I^{t'-t}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p}',t')$. By (ii), J satisfies the Sincov functional equation, $J(z;z')J(z';z'')=J(z;z'')$, $z,z',z''\in\mathrm{X}\times\mathrm{P}$. Its general solution is given by (Aczél, 1966, section 8.1.3)

$$J(\boldsymbol{x},\boldsymbol{p};\boldsymbol{x}',\boldsymbol{p}')=\frac{g(\boldsymbol{x}',\boldsymbol{p}')}{g(\boldsymbol{x},\boldsymbol{p})},$$

where g is an arbitrary positive function. Applying (i) with $\boldsymbol{p}'=\boldsymbol{p}$, we get

$$\frac{g(\boldsymbol{x}',\boldsymbol{p})}{g(\boldsymbol{x},\boldsymbol{p})}=J(\boldsymbol{x},\boldsymbol{p};\boldsymbol{x}',\boldsymbol{p})=I^{t'-t}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{x}',\boldsymbol{p},t')=\frac{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}'}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}. \tag{22}$$

Since (22) holds for any $\boldsymbol{x}$ and $\boldsymbol{x}'$, there exists a function $f\in\mathrm{F}$ such that $g(\boldsymbol{x},\boldsymbol{p})=(\bar{\boldsymbol{p}}\cdot\boldsymbol{x})/f(\boldsymbol{p})$. By (i), $f\in\mathrm{S}$.

(b)$\Rightarrow$(a). Straightforward. ■

Proof of Theorem 2.

Let $I = I_f$, $f \in \mathrm{S}$.

(a)$\Rightarrow$(d). Applying (iii) with $\boldsymbol{x} = 1/\bar{\boldsymbol{p}}$, $\boldsymbol{x}' = 1/\bar{\boldsymbol{p}}'$, $t' - t = 1$, we obtain the multiplicative Cauchy functional equation with respect to the function f:

$$\frac{f(\boldsymbol{p})}{f(\boldsymbol{p}')} = G\left(\frac{\boldsymbol{p}}{\boldsymbol{p}'}\right).$$

Its general positive non-decreasing solution is given by $f(\boldsymbol{p}) = c\prod_{i=1}^{n} p_i^{\lambda_i}$ (Aczél, 1966, section 8.1.1), where $c > 0$, $\boldsymbol{\lambda} = (\lambda_1,...,\lambda_n) \in \mathrm{R}_+^n$. Since f is subhomogeneous, $\boldsymbol{\lambda} \in \Delta^n$.

(b)$\Rightarrow$(d). From (iv) with $(\boldsymbol{\alpha}, \alpha_{n+1}) = \bar{\boldsymbol{p}}$, we again obtain the multiplicative Cauchy functional equation with respect to f.

(c)$\Rightarrow$(d). By (v), for any $i \in \{1,...,n\}$ the ratio $f(\boldsymbol{p})/f(1, \boldsymbol{p}_{-i})$ is a function of only p_i. Thus, there exist positive functions f_i, $i = 1,...,n$ such that $f(\boldsymbol{p}) = \prod_{i=1}^{n} f_i(p_i)$. From (v) with $x_{n+1} = x'_{n+1} = 0$ it follows that the ratio $f(\alpha\boldsymbol{p})/f(\boldsymbol{p})$ is a function of only α and, therefore, there is a constant $\lambda \geq 0$ such that $f(\alpha\boldsymbol{p})/f(\boldsymbol{p}) = \alpha^{\lambda}$. Then

$$\prod_{i=1}^{n} f_i(\alpha p_i) = f(\alpha\boldsymbol{p}) = \alpha^{\lambda} f(\boldsymbol{p}) = \alpha^{\lambda}\prod_{i=1}^{n} f_i(p_i)$$

and $f_i(\alpha p_i)/f_i(p_i)$ is again a function of only α. Thus, there exist constants $c_i > 0$ and $\lambda_i \geq 0$ such that $f_i(p_i) = c_i p_i^{\lambda_i}$. $(\lambda_1,...,\lambda_n) \in \Delta^n$ since $f \in \mathrm{S}$.

(d)$\Rightarrow$(a), (b), (c). Trivial. ■

Proof of Theorem 3.

(a)$\Rightarrow$(b). We have

$$(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{x}', \boldsymbol{p}', t') \sim (\boldsymbol{0}_n, \bar{\boldsymbol{p}}\cdot\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{0}_n, \bar{\boldsymbol{p}}'\cdot\boldsymbol{x}', \boldsymbol{p}', t') \sim \left(\boldsymbol{0}_n, 1, \boldsymbol{p}, 0; \boldsymbol{0}_n, \frac{\bar{\boldsymbol{p}}'\cdot\boldsymbol{x}'}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}, \boldsymbol{p}', t' - t\right),$$

where the first equality follows from (VI) with $\boldsymbol{y} = (\boldsymbol{0}_n, \bar{\boldsymbol{p}}\cdot\boldsymbol{x})$ and $\boldsymbol{y}' = (\boldsymbol{0}_n, \bar{\boldsymbol{p}}'\cdot\boldsymbol{x}')$, while the second one follows from (III) and (IV).

Let q be the function that takes each vector $(d, \boldsymbol{p}, \boldsymbol{p}', \tau) \in \mathrm{R}_{++} \times \mathrm{P}^2 \times \mathrm{R}_{++}$ to a solution $x \in \mathrm{R}_{++}$ of the equation

$$(\boldsymbol{0}_n, 1, \boldsymbol{p}, 0; \boldsymbol{0}_n, x, \boldsymbol{p}', \tau) \sim (\boldsymbol{0}_n, 1, \boldsymbol{1}_n, 0; \boldsymbol{0}_n, d, \boldsymbol{1}_n, 1).$$

By (I) (parts (A) and (C)) and (V) the function q is well defined and continuous by the implicit function theorem for continuous maps (Kumagai, 1980).

From (II) it follows that

$$q(d,\boldsymbol{p},\boldsymbol{p}',\tau)q(d,\boldsymbol{p}',\boldsymbol{p}'',\tau')=q(d,\boldsymbol{p},\boldsymbol{p}'',\tau+\tau'). \tag{23}$$

Equation (23) is a combination of the Cauchy and Sincov functional equations (Aczél, 1966, sections 2.1.2, 8.1.3). Its general solution that is continuous in the last argument is given by

$$q(d,\boldsymbol{p},\boldsymbol{p}',\tau)=\alpha(d)^{\tau}\frac{\beta(d,\boldsymbol{p}')}{\beta(d,\boldsymbol{p})}$$

for some functions $\alpha:\mathrm{R}_{++}\to\mathrm{R}_{++}$ and $\beta:\mathrm{R}_{++}\times\mathrm{P}\to\mathrm{R}_{++}$. By the definition of q, $q(d;\mathbf{1}_n,\mathbf{1}_n;1)=d$ and, hence, α is the identity function. Let us prove that for every fixed $\boldsymbol{p},\boldsymbol{p}'\in\mathrm{P}$ the function $d\mapsto\beta(d,\boldsymbol{p}')/\beta(d,\boldsymbol{p})$ is identically equal to a constant. Indeed, for any $d<d'$ we have

$$d^{\tau}\frac{\beta(d,\boldsymbol{p}')}{\beta(d,\boldsymbol{p})}=q(d,\boldsymbol{p},\boldsymbol{p}',\tau)<q(d',\boldsymbol{p},\boldsymbol{p}',\tau)=d'^{\tau}\frac{\beta(d',\boldsymbol{p}')}{\beta(d',\boldsymbol{p})}. \tag{24}$$

Tending $\tau\to 0+$ in (24), we obtain

$$\frac{\beta(d,\boldsymbol{p}')}{\beta(d,\boldsymbol{p})}\le\frac{\beta(d',\boldsymbol{p}')}{\beta(d',\boldsymbol{p})}. \tag{25}$$

The inequality (25) holds for all $\boldsymbol{p},\boldsymbol{p}'\in\mathrm{P}$ if and only if it holds with equality.

Put $f(\boldsymbol{p}):=\beta(1,\boldsymbol{p})$. Then (I) (part (B)) implies $f\in\mathrm{S}$. Now (b) follows from the definition of the function q.

(b)$\Rightarrow$(a). Straightforward. ■

Proof of Theorem 4.

(a)$\Rightarrow$(b). By Theorem 3, there exists $f\in\mathrm{S}$ such that I_f represents $\succeq$. From (VII) with $\boldsymbol{x}=\boldsymbol{x}'=\boldsymbol{y}=\boldsymbol{y}'=(\mathbf{0}_n,1)$, $\alpha_{n+1}=1$, and $t'-t=\tau'-\tau=1$ it follows that

$$\frac{f(\boldsymbol{p}')}{f(\boldsymbol{p})}\ge\frac{f(\boldsymbol{q}')}{f(\boldsymbol{q})}\ \Rightarrow\ \frac{f(\boldsymbol{\alpha p}')}{f(\boldsymbol{\alpha p})}\ge\frac{f(\boldsymbol{\alpha q}')}{f(\boldsymbol{\alpha q})} \tag{26}$$

for any $\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q},\boldsymbol{q}',\boldsymbol{\alpha}\in\mathrm{P}$. The general positive and continuous solution of the system of functional inequalities (26) is given by (Aczél, 1990, corollary 10)

$$f(\boldsymbol{p})=\exp\left(a\prod_{i=1}^{n}p_i^{\lambda_i}+b\right)\text{ and}$$

$$f(\boldsymbol{p})=c\prod_{i=1}^{n}p_i^{\lambda_i}, \tag{27}$$

where a, b, $c>0$, $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)$ are some constants. The inclusion $f\in \mathrm{S}$ holds only for the solution (27) with $\boldsymbol{\lambda}\in\Delta^n$.

(b)$\Rightarrow$(a). Trivial. ■

Proof of Proposition 2.

(a)$\Rightarrow$(b). The identities $A_u^{(p_0)}=I_f$ and $e(\boldsymbol{p}_0, v(\boldsymbol{p}_0, M))=M$ (Shah, 2007, theorem 4.3) imply $e(\boldsymbol{p}_0, v(\boldsymbol{p}, M))=Mf(\boldsymbol{p}_0)/f(\boldsymbol{p})$. Since $e(\boldsymbol{p}_0,\cdot)$ is non-decreasing and continuous (Shah, 2007, theorem 2.4) and $v(\boldsymbol{p},\cdot)$ is upper semicontinuous (Shah, 2007, theorem 3.2), we have

$$v(\boldsymbol{p},M)=g\left(\frac{M}{f(\boldsymbol{p})}\right),$$

where the function $g(z):=\sup\{y: e(\boldsymbol{p}_0,y)\le f(\boldsymbol{p}_0)z\}$ is strictly increasing, right-continuous, and $g(0)=0$. Since an indirect utility function is quasi-convex (Shah, 2007, theorem 3.2), the function $\bar{f}$, defined by (3), is quasi-concave and hence (since $\bar{f}$ is homogeneous) concave. Therefore, $f\in \mathrm{K}$. Given the indirect utility function v, there is a unique direct utility function $u\in \mathrm{U}$ that induces v (Shah, 2007, theorem 3.2), namely $u(\boldsymbol{x})=\sup\bigcap_{\boldsymbol{p}\in \mathrm{P}}\{y: v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x})\ge y\}$. We get

$$\begin{aligned} u(\boldsymbol{x}) &= \sup\bigcap_{\boldsymbol{p}\in \mathrm{P}}\{y: v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x})\ge y\} = \\ &= \sup\left\{\mathrm{R}\setminus\bigcup_{\boldsymbol{p}\in \mathrm{P}}\{y: v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x})< y\}\right\} = \inf\left\{\bigcup_{\boldsymbol{p}\in \mathrm{P}}\{y: v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x})< y\}\right\} = \\ &= \inf_{\boldsymbol{p}\in \mathrm{P}}\inf\{y: v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x})< y\} = \inf_{\boldsymbol{p}\in \mathrm{P}} v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x}) = \inf_{\boldsymbol{p}\in \mathrm{P}} g\left(\frac{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}{f(\boldsymbol{p})}\right) = g\left(\inf_{\boldsymbol{p}\in \mathrm{P}}\frac{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}{f(\boldsymbol{p})}\right), \end{aligned} \tag{28}$$

where we use the facts that $\bigcup_{\boldsymbol{p}\in \mathrm{P}}\{y: v(\boldsymbol{p},\bar{\boldsymbol{p}}\cdot\boldsymbol{x})< y\}$ is the set of the form $(a,+\infty)$, g is strictly increasing and right-continuous.

(b)$\Rightarrow$(c). The function $\tilde{u}$ defined in (15) is homogeneous, unbounded above, concave, and, therefore, continuous, $\tilde{u}(\mathbf{0}_{n+1})=0$. Since u is strictly increasing, then so is $\tilde{u}$. Thus, $\tilde{u}\in \mathrm{U}$ and $u\in\tilde{\mathrm{U}}$. For any $\boldsymbol{p}$ the function

$$(\boldsymbol{x},\boldsymbol{p}')\mapsto\frac{1}{f(\boldsymbol{p}')}\frac{\bar{\boldsymbol{p}}'\cdot\boldsymbol{x}}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}$$

is quasi-concave with respect to $\boldsymbol{x}$ for each fixed $\boldsymbol{p}'$ and quasi-convex with respect to $\boldsymbol{p}'$ for each fixed $\boldsymbol{x}$ (since $f\in \mathrm{K}$). By the minimax theorem (Sion, 1958, corollary 3.3),

$$\sup_{\boldsymbol{x}\in \mathrm{R}^{n+1}_{++}}\frac{\tilde{u}(\boldsymbol{x})}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}=\sup_{\boldsymbol{x}\in \mathrm{X}}\frac{\tilde{u}(\boldsymbol{x})}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}=\sup_{\boldsymbol{x}\in \mathrm{X}:\|\boldsymbol{x}\|=1}\frac{\tilde{u}(\boldsymbol{x})}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}=\sup_{\boldsymbol{x}\in \mathrm{X}:\|\boldsymbol{x}\|=1}\inf_{\boldsymbol{p}'\in \mathrm{P}}\frac{1}{f(\boldsymbol{p}')}\frac{\bar{\boldsymbol{p}}'\cdot\boldsymbol{x}}{\bar{\boldsymbol{p}}\cdot\boldsymbol{x}}= \tag{29}$$

$$= \inf_{p' \in \mathrm{P}} \sup_{x \in \mathrm{X}: \|x\|=1} \frac{1}{f(\boldsymbol{p}')} \frac{\bar{\boldsymbol{p}}' \cdot \boldsymbol{x}}{\bar{\boldsymbol{p}} \cdot \boldsymbol{x}} = \inf_{p' \in \mathrm{P}} \frac{1}{f(\boldsymbol{p}')} \max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}} \right) = \frac{1}{f(\boldsymbol{p})},$$

where $\|\cdot\|$ is the Euclidean norm on R^{n+1}. The first equality in (29) follows from continuity and homogeneity of $\tilde{u}$ and the last equality follows from part (e) of Proposition 1. Identity (29) implies (16).

(c)$\Rightarrow$(a). Let v ($\tilde{v}$) and e ($\tilde{e}$) be the indirect utility function and the expenditure function generated by u ($\tilde{u}$). Since $\tilde{u}$ is homogeneous, $\tilde{v}(\boldsymbol{p}, M) = M/f(\boldsymbol{p})$ and $\tilde{e}(\boldsymbol{p}, y) = f(\boldsymbol{p})y$, where f is defined in (16). Then $e(\boldsymbol{p}_0, v(\boldsymbol{p}, \bar{\boldsymbol{p}} \cdot \boldsymbol{x})) = \tilde{e}(\boldsymbol{p}_0, \tilde{v}(\boldsymbol{p}, \bar{\boldsymbol{p}} \cdot \boldsymbol{x})) = (\bar{\boldsymbol{p}} \cdot \boldsymbol{x}) f(\boldsymbol{p}_0)/f(\boldsymbol{p})$ and $I_f = A_u^{(p_0)}$ for any $\boldsymbol{p}_0$. ■

Proof of Theorem 5.

Let $f \in \mathrm{K}$. Then the function $\tilde{u}$ defined in (15) is concave and hence continuous. $\tilde{u}(\mathbf{0}_{n+1}) = 0$ and $\tilde{u}$ is positive on R_{++}^{n+1}:

$$\tilde{u}(\boldsymbol{x}) = \inf_{p \in \mathrm{P}} \frac{\bar{\boldsymbol{p}} \cdot \boldsymbol{x}}{f(\boldsymbol{p})} \geq \inf_{p \in \mathrm{P}} \frac{\bar{\boldsymbol{p}} \cdot \boldsymbol{x}}{f(\mathbf{1}_n) \max(\bar{\boldsymbol{p}})} = \frac{\min(\boldsymbol{x})}{f(\mathbf{1}_n)} > 0, \ \boldsymbol{x} \in \mathrm{R}_{++}^{n+1}, \tag{30}$$

where the first inequality is implied by $f \in \mathrm{S}$. $\tilde{u}$ is homogeneous and hence unbounded above. Finally, $\tilde{u}$ is strictly increasing: for any $\boldsymbol{z} \in \mathrm{R}_{++}^{n+1}$

$$\tilde{u}(\boldsymbol{x} + \boldsymbol{z}) = \inf_{p \in \mathrm{P}} \frac{\bar{\boldsymbol{p}} \cdot (\boldsymbol{x} + \boldsymbol{z})}{f(\boldsymbol{p})} \geq \tilde{u}(\boldsymbol{x}) + \tilde{u}(\boldsymbol{z}) > \tilde{u}(\boldsymbol{x}).$$

Therefore, $\tilde{u} \in \mathrm{U}$ and, by Proposition 2, $I_f = A_{\tilde{u}}^{(p_0)}$ for any $\boldsymbol{p}_0 \in \mathrm{P}$. Since the expenditure function generated by $\tilde{u}$ is given by $e(\boldsymbol{p}, y) = f(\boldsymbol{p})y$, $I_f = KP_{\tilde{u}}^{(u_0)}$ for any $u_0 \in \mathrm{R}_{++}$.

Now let $\boldsymbol{p}_0 \in \mathrm{P}$, $u \in \tilde{\mathrm{U}}$, and let g and $\tilde{u} \in \mathrm{U}$ be a strictly increasing function and a homogeneous utility function such that $u = g \circ \tilde{u}$. Then f defined by (16) is positive by arguments similar to those used in (30). Thus, by Proposition 2, $f \in \mathrm{K}$ and $A_u^{(p_0)} = I_f$.

Finally, given $u \in \mathrm{U}$ and $u_0 \in \mathrm{R}_{++}$, put $f(\boldsymbol{p}) := e(\boldsymbol{p}, u_0)$, where e is the expenditure function generated by u. Then $KP_u^{(u_0)} = I_f$ and, by the regularity properties of an expenditure function (Shah, 2007, theorem 2.4), $f \in \mathrm{K}$. ■

Proof of Proposition 3.

Given $I = I_f \in \mathrm{I(S)}$, assume that the generalized product test holds. Define the vector-valued functions $\boldsymbol{g} = (g_1, ..., g_{n+1}) : \mathrm{X}^2 \to \mathrm{X}$ and $\bar{\boldsymbol{h}} = (\bar{h}_1, ..., \bar{h}_{n+1}) : \mathrm{P}^2 \to \mathrm{P} \times \{1\}$ by $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{x}') := \boldsymbol{x} \circ \boldsymbol{x}'$, $\bar{\boldsymbol{h}}(\boldsymbol{p}, \boldsymbol{p}') := (\boldsymbol{p} \bullet \boldsymbol{p}', 1)$.

From the generalized product test with $t'-t=1$, we have

$$\left(\frac{\bar{\boldsymbol{p}}'\cdot \boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}')}{f(\boldsymbol{p}')}\Big/\frac{\bar{\boldsymbol{p}}\cdot \boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}')}{f(\boldsymbol{p})}\right)*\left(\frac{\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}')\cdot \boldsymbol{x}'}{\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}')\cdot \boldsymbol{x}}\right)=\frac{\bar{\boldsymbol{p}}'\cdot \boldsymbol{x}'}{f(\boldsymbol{p}')}\Big/\frac{\bar{\boldsymbol{p}}\cdot \boldsymbol{x}}{f(\boldsymbol{p})}. \tag{31}$$

Let $\mathbf{1}^j := (0,...,0,1,0,...,0)$ be the $(n+1)$-dimensional vector, where 1 stands at the j-th position. Put $\mathrm{Y} := \ln f(\mathrm{P})$. Since $f \in \mathrm{S}$ and, hence, continuous, Y is a (possibly unbounded) real interval. We consider two cases:

1) Y is unbounded. Setting $\boldsymbol{x}=\mathbf{1}^{n+1}$, $\boldsymbol{x}'=(0,...,0,x')$ in (31), we obtain

$$\left(\frac{f(\boldsymbol{p})}{f(\boldsymbol{p}')}\right)*x'=\frac{f(\boldsymbol{p})}{f(\boldsymbol{p}')}x'.$$

Since Y is unbounded, the function $(\boldsymbol{p},\boldsymbol{p}')\mapsto f(\boldsymbol{p})/f(\boldsymbol{p}')$ is onto R_{++}.

2) Y is bounded. Setting in (31) $\boldsymbol{p}'=\alpha\boldsymbol{p}$, $\boldsymbol{x}=(x_1,...,x_n,0)$, $\boldsymbol{x}'=\beta\boldsymbol{x}$, where α and β are positive scalars, we have

$$\left(\alpha\frac{f(\boldsymbol{p})}{f(\alpha\boldsymbol{p})}\right)*\beta=\alpha\beta\frac{f(\boldsymbol{p})}{f(\alpha\boldsymbol{p})}.$$

Since Y is bounded, the function $(\alpha,\boldsymbol{p})\mapsto \alpha f(\boldsymbol{p})/f(\alpha\boldsymbol{p})$ is onto R_{++}.

Therefore, the binary operation $*$ defined on $I_f(\mathrm{V})=\mathrm{R}_{++}$ is multiplication and the identity (31) takes the form

$$\frac{\bar{\boldsymbol{p}}'\cdot \boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}')}{\bar{\boldsymbol{p}}\cdot \boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}')}\frac{\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}')\cdot \boldsymbol{x}'}{\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}')\cdot \boldsymbol{x}}=\frac{\bar{\boldsymbol{p}}'\cdot \boldsymbol{x}'}{\bar{\boldsymbol{p}}\cdot \boldsymbol{x}}. \tag{32}$$

Setting in (32) $\boldsymbol{x}=\mathbf{1}^i$, $\boldsymbol{x}'=\mathbf{1}^j$ and $\boldsymbol{x}=\mathbf{1}^j$, $\boldsymbol{x}'=\mathbf{1}^i$, $i\neq j$, we get

$$\frac{\bar{p}_i' g_i(\mathbf{1}^i,\mathbf{1}^j)+\bar{p}_j' g_j(\mathbf{1}^i,\mathbf{1}^j)}{\bar{p}_i g_i(\mathbf{1}^i,\mathbf{1}^j)+\bar{p}_j g_j(\mathbf{1}^i,\mathbf{1}^j)}\frac{\bar{h}_j(\boldsymbol{p},\boldsymbol{p}')}{\bar{h}_i(\boldsymbol{p},\boldsymbol{p}')}=\frac{\bar{p}_j'}{\bar{p}_i},\quad \frac{\bar{p}_i' g_i(\mathbf{1}_j,\mathbf{1}_i)+\bar{p}_j' g_j(\mathbf{1}_j,\mathbf{1}_i)}{\bar{p}_i g_i(\mathbf{1}_j,\mathbf{1}_i)+\bar{p}_j g_j(\mathbf{1}_j,\mathbf{1}_i)}\frac{\bar{h}_i(\boldsymbol{p},\boldsymbol{p}')}{\bar{h}_j(\boldsymbol{p},\boldsymbol{p}')}=\frac{\bar{p}_i'}{\bar{p}_j}. \tag{33}$$

Combining the equalities (33), we obtain

$$\begin{aligned}&\frac{\left(\bar{p}_i g_i(\mathbf{1}^i,\mathbf{1}^j)+\bar{p}_j g_j(\mathbf{1}^i,\mathbf{1}^j)\right)\left(\bar{p}_i g_i(\mathbf{1}^j,\mathbf{1}^i)+\bar{p}_j g_j(\mathbf{1}^j,\mathbf{1}^i)\right)}{\bar{p}_i\bar{p}_j}=\\&=\frac{\left(\bar{p}_i' g_i(\mathbf{1}^i,\mathbf{1}^j)+\bar{p}_j' g_j(\mathbf{1}^i,\mathbf{1}^j)\right)\left(\bar{p}_i' g_i(\mathbf{1}^j,\mathbf{1}^i)+\bar{p}_j' g_j(\mathbf{1}^j,\mathbf{1}^i)\right)}{\bar{p}_i'\bar{p}_j'}.\end{aligned} \tag{34}$$

The equality (34) holds for any $\boldsymbol{p}$ and $\boldsymbol{p}'$ if and only if either $g_i(\mathbf{1}^i,\mathbf{1}^j)=g_j(\mathbf{1}^j,\mathbf{1}^i)=0$ and $g_j(\mathbf{1}^i,\mathbf{1}^j)>0$, $g_i(\mathbf{1}^j,\mathbf{1}^i)>0$ or $g_j(\mathbf{1}^i,\mathbf{1}^j)=g_i(\mathbf{1}^j,\mathbf{1}^i)=0$ and $g_i(\mathbf{1}^i,\mathbf{1}^j)>0$, $g_j(\mathbf{1}^j,\mathbf{1}^i)>0$. If the former alternative holds then $\bar{h}_i(\boldsymbol{p},\boldsymbol{p}')/\bar{h}_j(\boldsymbol{p},\boldsymbol{p}')=\bar{p}_i/\bar{p}_j$; if the latter one holds then

$\bar{h}_i(\boldsymbol{p},\boldsymbol{p}')/\bar{h}_j(\boldsymbol{p},\boldsymbol{p}') = \bar{p}_i'/\bar{p}_j'$. Thus, since $\bar{h}_{n+1}(\boldsymbol{p},\boldsymbol{p}') = \bar{p}_{n+1} = \bar{p}_{n+1}' = 1$, we get that either $\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}') = (\boldsymbol{p},1)$ or $\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}') = (\boldsymbol{p}',1)$ hold.

If $\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}') = (\boldsymbol{p},1)$ (the arguments in the case of $\bar{\boldsymbol{h}}(\boldsymbol{p},\boldsymbol{p}') = (\boldsymbol{p}',1)$ are similar), then (32) reduces to

$$\frac{\bar{\boldsymbol{p}}' \cdot \boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}')}{\bar{\boldsymbol{p}}' \cdot \boldsymbol{x}'} = \frac{\bar{\boldsymbol{p}} \cdot \boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}')}{\bar{\boldsymbol{p}} \cdot \boldsymbol{x}'}. \tag{35}$$

The identity (35) holds for any $\boldsymbol{p}$ and $\boldsymbol{p}'$ if and only if $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}') = \alpha(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{x}'$ for some function $\alpha : \mathrm{X}^2 \to \mathrm{R}_{++}$.

The converse implication is trivial. ■

Proof of Proposition 5.

If G is a cone, then the map $g \mapsto \ln g$ defines a one-to-one correspondence between $\mathrm{G} \cap \mathrm{C}(f,\boldsymbol{p},M)$ and $\partial_{\ln \mathrm{G}} \ln f(\boldsymbol{p})$ modulo "differ by a constant" relation (two supergradients are considered equivalent if they are differ by a constant). If G is a cone closed under vertical shifts, then the map $g \mapsto M^{-1} f(\boldsymbol{p}) g$ is a bijection between $\mathrm{G} \cap \mathrm{C}(f,\boldsymbol{p},M)$ and $\partial_{\mathrm{G}} f(\boldsymbol{p})$ modulo "differ by a constant" relation.

To establish a one-to-one correspondence between $\mathrm{G} \cap \mathrm{C}(f,\boldsymbol{p},M)$ and $\partial_{\bar{\mathrm{G}}} \bar{f}(\bar{\boldsymbol{p}})$ we prove that $\bar{g}$ is a $\bar{\mathrm{G}}$-supergradient of a function $\bar{f}$ at a point $\bar{\boldsymbol{p}}$ if and only if $g(\boldsymbol{p}) = f(\boldsymbol{p})$ and $g(\boldsymbol{p}') \geq f(\boldsymbol{p}')$ for all $\boldsymbol{p}' \in \mathrm{P}$. Indeed, if $\bar{g}$ is a $\bar{\mathrm{G}}$-supergradient of $\bar{f}$ at $\bar{\boldsymbol{p}}$, then $p_{n+1}' g(\boldsymbol{p}') - g(\boldsymbol{p}) \geq p_{n+1}' f(\boldsymbol{p}') - f(\boldsymbol{p})$ holds for all $\boldsymbol{p}' \in \mathrm{P}$ and $p_{n+1}' \in \mathrm{R}_{++}$. Passing to the limit as $p_{n+1}' \to 0+$, we get $g(\boldsymbol{p}) \leq f(\boldsymbol{p})$. On the other hand, $g(\boldsymbol{p}') - f(\boldsymbol{p}') \geq (g(\boldsymbol{p}) - f(\boldsymbol{p}))/p_{n+1}'$. Passing to the limit as $p_{n+1}' \to +\infty$, we obtain $g(\boldsymbol{p}') \geq f(\boldsymbol{p}')$ for all $\boldsymbol{p}' \in \mathrm{P}$. The converse implication is obvious. ■

Proof of Proposition 6.

(a)$\Rightarrow$(b). Let $\boldsymbol{x}^* \in \mathrm{X}$ be the structure of a constant portfolio conservative with respect to I_f, $\boldsymbol{p}$, M and let $\tilde{v}(\boldsymbol{p},M) = M/f(\boldsymbol{p})$ be the indirect utility function generated by $\tilde{u}$ (15). Then $\boldsymbol{x}^* \in \mathrm{B}(\boldsymbol{p},M)$ and

$$\tilde{v}(\boldsymbol{p},\bar{\boldsymbol{p}} \cdot \boldsymbol{x}^*) \leq \tilde{v}(\boldsymbol{p}',\bar{\boldsymbol{p}}' \cdot \boldsymbol{x}^*) \text{ for all } \boldsymbol{p}' \in \mathrm{P}.$$

From (28) it follows that

$$\tilde{v}(\boldsymbol{p},M) = \tilde{v}(\boldsymbol{p},\bar{\boldsymbol{p}} \cdot \boldsymbol{x}^*) = \inf_{\boldsymbol{p}' \in \mathrm{P}} \tilde{v}(\boldsymbol{p}',\bar{\boldsymbol{p}}' \cdot \boldsymbol{x}^*) = \tilde{u}(\boldsymbol{x}^*);$$

thus $\boldsymbol{x}^*$ is a solution of the utility maximization problem $\langle \tilde{u}, \boldsymbol{p}, M \rangle$.

(b)$\Rightarrow$(a). Let $\boldsymbol{x}^*$ be a solution of the problem $\langle \tilde{u}, \boldsymbol{p}, M \rangle$. Since $\boldsymbol{x}^* \in \mathrm{B}(\boldsymbol{p}', \bar{\boldsymbol{p}}' \cdot \boldsymbol{x}^*)$ for any $\boldsymbol{p}' \in \mathrm{P}$, then

$$\tilde{v}(\boldsymbol{p}, \bar{\boldsymbol{p}} \cdot \boldsymbol{x}^*) = \tilde{v}(\boldsymbol{p}, M) = \tilde{u}(\boldsymbol{x}^*) \le \tilde{v}(\boldsymbol{p}', \bar{\boldsymbol{p}}' \cdot \boldsymbol{x}^*) \text{ for all } \boldsymbol{p}' \in \mathrm{P}.$$

(c)$\Leftrightarrow$(a), (a)$\Leftrightarrow$(d). The assertions follow from Proposition 4 and Proposition 5.

Since the sets $\partial f(\boldsymbol{p})$ and $\mathrm{L} \cap \mathrm{C}(f, \boldsymbol{p}, M)$ are equivalent (the map (19) is a bijection), $\mathrm{L} \cap \mathrm{C}(f, \boldsymbol{p}, M)$ is a singleton if and only if f is differentiable at $\boldsymbol{p}$ (Niculescu and Persson, 2006, theorem 3.8.2, p. 136). Equality (20) now follows from part (d). ■

Proof of Proposition 7.

All the assertions are the direct consequences of the equality

$$\sup_{g \in \mathrm{G}} \frac{g(\boldsymbol{p}')}{g(\boldsymbol{p})} = \max\left(\frac{\bar{\boldsymbol{p}}'}{\bar{\boldsymbol{p}}}\right),$$

which follows from part (e) of Proposition 1. ■

References

Abel A.B. (1990). Asset prices under habit formation and catching up with the Joneses // The American Economic Review. Vol. 80(2). P. 38–42.

Aczél J. (1966). Lectures on functional equations and their applications. New York: Academic Press.

Aczél J. (1990). Determining merged relative scores // Journal of Mathematical Analysis and Applications. Vol. 150. P. 205–243.

Admati A.R., Pfleiderer P. (1997). Does it all add up? Benchmarks and the compensation of active portfolio managers // The Journal of Business. Vol. 70(3). P. 323–350.

Alexeev A.G., Sokolov M.V. (2014). A theory of average growth rate indices // Mathematical Social Sciences. Vol. 71. P. 101–115.

Balk B.M. (2008). Price and quantity index numbers: models for measuring aggregate change and difference. New York: Cambridge University Press.

Barberà S., Bossert W., Pattanaik P.K. (2004). Ranking sets of objects. In S. Barberà, P.J. Hammond, Ch. Seidl (eds.). Handbook of utility theory: Volume 2 Extensions. P. 893–977. Boston: Kluwer Academic Publisher.

Barro D., Canestrelli E. (2009). Tracking error: a multistage portfolio model // Annals of Operations Research. Vol. 165(1). P. 47–66.

Basak S. (1995). A general equilibrium model of portfolio insurance // Review of Financial studies. Vol. 8 (4), 1059–1090.

Basak S., Pavlova A., Shapiro A. (2007). Optimal asset allocation and risk shifting in money management // Review of Financial Studies. Vol. 20(5). P. 1583–1621.

Basak S., Shapiro A., Teplá L. (2006). Risk management with benchmarking // Management Science. Vol. 52(4). P. 542–557.

Beasley J.E., Meade N., Chang T-J. (2003). An evolutionary heuristic for the index tracking problem // European Journal of Operational Research. Vol. 148(3). P. 621–643.

Björk T. (2009). Arbitrage theory in continuous time. Oxford: Oxford University Press.

Browne S. (1999). Beating a moving target: Optimal portfolio strategies for outperforming a stochastic benchmark // Finance and Stochastics. Vol. 3(3). P. 275–294.

Cox J.C., Huang C.-f. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process // Journal of economic theory. Vol. 49(1), 33–83.

Davis M., Lleo S. (2008). Risk-sensitive benchmarked asset management // Quantitative Finance. Vol. 8(4). P. 415–426.

Diewert W.E. (1993). The economic theory of index numbers: a survey // In W.E. Diewert and A.O. Nakamura (eds.). Essays in Index Number Theory. Vol. I. Chapter 7. Amsterdam: North-Holland Publishing Company. P. 177–221.

Ebert U. (1984). Exact welfare measures and economic index numbers // Journal of Economics. Vol. 44(1). P. 27–38.

Gali J. (1994). Keeping up with the Joneses: consumption externalities, portfolio choice, and asset prices // Journal of Money, Credit and Banking. Vol. 26(1). P. 1–8.

Gilli M., Këllezi E. (2002). The threshold accepting heuristic for index tracking // In P.M. Pardalos and V.K. Tsitsiringos (eds.). Financial Engineering, E-Commerce and Supply Chain. Norwell: Kluwer Academic Publishers. P. 1–18.

Gómez J.-P., Priestley R., Zapatero F. (2009). Implications of keeping-up-with-the-Joneses behavior for the equilibrium cross section of stock returns: international evidence // The Journal of Finance. Vol. 64(6). P. 2703–2737.

Gray Jr.K.B., Dewar R.B.K. (1971). Axiomatic characterization of the time-weighted rate of return // Management Science. Vol. 18(2). P. B32–B35.

Grossman S.J., Zhou Z. (1996). Equilibrium analysis of portfolio insurance // The Journal of Finance. Vol. 51(4). P. 1379–1403.

Guidolin M., Rinaldi F. (2013). Ambiguity in asset pricing and portfolio choice: a review of the literature // Theory and Decision. Vol. 74(2). P. 183–217.

Hammond P.J. (1994). Money metric measures of individual and social welfare allowing for environmental externalities. In W. Eichhorn (ed.). Models and measurement of welfare and inequality. P. 694–724. Berlin: Springer Verlag.

Krantz D.H., Luce R.D., Suppes P., Tversky A. (1971). Foundations of measurement. Vol. I: Additive and polynomial representations. New York: Academic Press.

Kumagai S. (1980). An implicit function theorem: comment // Journal of Optimization Theory and Applications. Vol. 31(2). P. 285–288.

Marinacci M., Montrucchio L. (2008). On concavity and supermodularity // Journal of Mathematical Analysis and Applications. Vol. 344. P. 642–654.

Martínez-Legaz J.E., Rubinov A.M., Schaible S. (2005). Increasing quasiconcave co-radiant functions with applications in mathematical economics // Mathematical Methods of Operations Research. Vol. 61. P. 261–280.

Niculescu C.P., Persson L.-E. (2006). Convex functions and their applications. A contemporary approach. New York: Springer.

Platen E., Heath D. (2006). A benchmark approach to quantitative finance. Berlin: Springer.

Promislow S.D., Spring D. (1996). Postulates for the internal rate of return of an investment project // Journal of Mathematical Economics. Vol. 26. P. 325–361.

Rader T. (1963). The existence of a utility function to represent preferences // Review of Economic Studies. Vol. 30. P. 229–232.

Roll R. (1992). A mean/variance analysis of tracking error // The Journal of Portfolio Management. Vol. 18(4). P. 13–22.

Rothstein M. (1972). On geometric and arithmetic portfolio performance indexes // Journal of Financial and Quantitative Analysis. Vol. 7(4). P. 1983–1992.

Rubinov A. (2000). Abstract convexity and global optimization. Dordrecht: Kluwer Academic Publishers.

Shah S.A. (2007). Dual representations of strongly monotonic utility functions // Centre for Development Economics. Working papers No. 161.

Shiller R.J. (1993). Stock prices and social dynamics // In R. Thaler (ed.) Advances in Behavioral Finance. New York: Russell Sage Foundation. P. 167–218.

Siegel L.B. (2003). Benchmarks and investment management. Charlottesville: The Research Foundation of AIRM.

Sion M. (1958). On general minimax theorems // Pacific Journal of Mathematics. Vol. 8(1). P. 171–176.

Teplá L. (2001). Optimal investment with minimum performance constraints // Journal of Economic Dynamics and Control. Vol. 25(10). P. 1629–1645.

Vilenskii P.L., Smolyak S.A. (1998). Internal rate of return and its modifications. Preprint # WP/98/060 CEMI RAS (in Russian).