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# Economic Growth and Property Rights on Natural Resources

Working Paper Ec-02/16
**Department of Economics**

St. Petersburg
2016

**УДК 330.35**
**ББК 65.012.2**
**В78**

Европейский Университет в Санкт-Петербурге

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Экономический рост и права собственности на природные ресурсы

*На английском языке*

**Borissov K., Pakhnin M.** *Economic Growth and Property Rights on Natural Resources.* — European University at St. Petersburg, Department of Economics. Working Paper Ec-02/16, 100 p.

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**Keywords:** economic growth, exhaustible resources, heterogeneous agents, voting
**JEL Classification:** D91, E13, O40, Q32

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# Economic Growth and Property Rights on Natural Resources *

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June 2016

**Abstract**

We consider two models of economic growth with exhaustible natural resources and agents heterogeneous in their time preferences. In the first model, we assume private ownership of natural resources and show that every competitive equilibrium converges to a balanced-growth equilibrium with the long-run rate of growth being determined by the discount factor of the most patient agents. In the second model, natural resources are public property and the resource extraction rate is determined by majority voting. For this model we define an intertemporal voting equilibrium and prove that it also converges to a balanced-growth equilibrium. In this scenario the long-run rate of growth is determined by the median discount factor. Our results suggest that if the most patient agents do not constitute a majority of the population, private ownership of natural resources results in a higher rate of growth than public ownership. At the same time, private ownership leads to higher inequality than public ownership, and if inequality impedes growth, then the public property regime is likely to result in a higher long-run rate of growth. However, an appropriate redistributive policy can eliminate the negative impact of inequality on growth.

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## 1 Introduction

The question of property rights[1] is one of the most controversial and complicated issues concerning the regulation of natural resources. Who should own natural resources, and in what form? Which individual, group or institution will best manage the resource stock in

*We are grateful to ExxonMobil for financial support. We also thank Thierry Bréchet, Stéphane Lambrecht and two anonymous referees for their helpful comments and suggestions.

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[1]As many other scholars, we do not make any difference between property rights and ownership throughout the paper. On our level of abstraction these two constructs are essentially the same. Therefore we use the terms "property regime" and "ownership" interchangeably.

the short and long run? How do different forms and extents of property rights on natural resources affect both present and future generations? These are important and inherently complex problems.

The vast amount of literature on property regimes over natural resources (see, e.g., Ostrom, 1990; Barnes, 2009; Cole and Ostrom, 2011) usually places great emphasis on the market failure that occurs when property rights are not properly specified. The relative advantages of private and common property rights in terms of efficiency, equity, and sustainability of natural resource use patterns have been widely discussed and studied.

However, even if property rights are clearly defined and assigned, the optimal choice from a wide array of diverse property regimes is not so obvious. Especially significant in this connection is the choice between private and public property. There has been much debate on the economic and political merits of private versus public ownership in general (see, e.g., a survey by Shleifer, 1998). It is believed that private firms are more efficient, mainly because of strong incentives to invest in improving the ways of using the assets. At the same time, state firms are usually considered inefficient for a number of reasons, e.g., weak incentives to reduce costs (agency problem), and government subsidies to the state-owned sector (soft budget constraints).[2] Moreover, it is widely recognized that public enterprises pursue political rather than economic goals (see Shleifer and Vishny, 1994). However, in this paper we do not take into account any political considerations and other possible sources of efficiency losses. We assume a government that acts in the interest of the people. Since we assume that extraction costs are zero, it is irrelevant for us, which kinds of enterprises (private or state-owned) extract and sell resources held in public ownership. There is no such an enterprise in our model. Our goal is to compare private and public property regimes over natural resources in terms of economic growth, and not in terms of profitability or efficiency.

Interestingly enough, economists have only recently begun to pay attention to the comparison of private and public ownership in the particular case of natural resources like crude oil or gold. This is even more surprising considering the ambiguity of this issue and its consequences for societies in resource-rich countries. There are many countries in the world that maintain full state ownership of their natural resources. In such countries private firms, especially foreign firms, have little or no operational and managerial control. Examples include Uzbekistan, Turkmenistan, Nigeria, and modern Venezuela. At the same time, there are countries like Kazakhstan or the Russian Federation, where leaders chose to privatize their energy sector (see Jones Luong and Weinthal, 2001). One should also mention the USA and Japan, where private firms own and control much of the countries' subsurface minerals.

There are certain rationales for such cleavage. On the one hand, exhaustible resource stocks (oil and gas fields, coal and ore mines) are universally regarded by the public at large as public property. It is felt that natural resources should belong to local peoples, who claim sovereign rights on their territorial habitat. It is often argued that "resources found in the territory of a state belong to the population of that state (...) The right to natural resources is a right of peoples or communities to determine how their natural resources should be protected, managed and explored" (Blanco and Razzaque, 2011). As Joseph Stiglitz put it, "a country's natural resources should belong to all of its people" (quoted after Kramer, 2014). Moreover, for many countries around the world, especially

[2]Bajona and Chu (2010) show that reduction in government subsidies leads to an increase in economic efficiency, and Gupta (2005) reports that even partial privatization increase productivity and profitability of state-owned enterprises.

developing countries, natural resources represent a significant share of income and are too important to be left to the market. It is believed that direct state control over resources is an indispensable feature of national sovereignty and political decision-making (see Mommer, 2002).

On the other hand, most economists are convinced that private ownership of natural resources leads to higher efficiency than public ownership. Empirical evidence shows that private natural resource companies are more efficient and profitable than nationalized firms, though the effects of privatization on employment and income distribution are not as desirable (see, e.g., Chong and de Silanes, 2005; Schmitz and Teixeira, 2008).

One may conjecture that this divergence between the positions of the public at large and economists partly explains the fact that privatization–nationalization cycles tend to occur more often in the natural resource sector (see, e.g., Kobrin, 1984; Chua, 1995; Hogan et al., 2010). This tendency provides an additional incentive to study the impact of property regime over natural resources on macroeconomic performance.

The existing empirical literature on ownership of the primary sectors (e.g., Megginson, 2005; Wolf, 2009) concentrates mostly on the productive efficiency and profitability of firms. The effects of different property regimes on aggregate income are studied by Brunnschweiler and Valente (2013), though they use a slightly different classification of ownership instead of the usual dichotomy between the categories of "private" and "public".

In this paper, we study private and public property regimes over exhaustible natural resources from the standpoint of economic growth theory. We do not compare private and public ownership in terms of efficiency or optimality. Thus we can abstract from any political considerations and focus only on the following question: which of the two property regimes does lead to a higher long-run rate of economics growth?

Developing the ideas of Borissov and Surkov (2010), we consider two models of economic growth with heterogeneous agents and exhaustible resources. These models are modifications to a well-known Ramsey-type model of economic growth with exhaustible resources (see, e.g., Dasgupta and Heal, 1979). Technical progress is exogenous. Under this assumption the long-run rate of growth is fully determined by the extraction rate.

The two models differ in the property regimes over natural resources. The first model assumes private ownership of natural resources. The resource stock is an asset. Agents can invest their savings in natural resources as well as in physical capital. This implies that the resource income belongs to the owners of natural resources. The extraction rate is determined by market forces. In the second model we assume that the resource stock is held in trust by the government for the common benefit. The resource income is equally distributed among all agents, and it is up to the agents to determine the extraction rate.

Following Becker (1980, 2006), we assume that agents are heterogeneous in their time preferences. The rates of time preference, or the degrees of impatience, are represented by agents' discount factors. The discount factors are higher for more patient agents and lower for less patient ones.

In the private property regime, only the most patient agents obtain income from the capital and resource stocks in the long run. We show that the discount factor of the most patient agents determines both the long-run extraction rate and the rate of growth. The extraction rate is decreasing and the growth rate is increasing in the discount factor of the most patient agents.

In the public property regime, the heterogeneity of agents results in different preferences over the resource extraction rate. Relatively impatient agents care less about the future and prefer to extract resources faster than relatively patient agents. Thus there naturally arises a problem of aggregating heterogeneous preferences. We use a conventional collective choice mechanism and suppose that the resource extraction rate is chosen by majority voting.

The performance of majoritarian institutions in dynamic settings has attracted growing interest and attention in recent years (see, e.g., Krusell et al., 1997; Rangel, 2003; Bernheim and Slavov, 2009). This body of literature studies appropriate dynamic generalizations of the standard solution concepts. One of these generalizations is presented by Borissov and Surkov (2010), who consider voting on extraction rates. The same voting mechanism is considered also in Borissov et al. (2014a), where heterogeneous agents vote for a tax aimed at environmental maintenance. In both cases the outcome of voting is the optimal policy for the agent with the median discount factor. However, this voting mechanism is oversimplified; it does not imply perfect rationality of agents, and allows one to analyze voting outcomes only in a balanced-growth equilibrium.

In this paper, we apply the approach to voting in a dynamic general equilibrium framework proposed by Borissov et al. (2014b).[3] We use the intertemporal setting of the model, and ask agents to vote on the extraction rate at each point in time under given expectations about future extraction rates. The sequence of winners in these one-dimensional votes under perfect foresight determines an intertemporal voting equilibrium. We show that in the long run an intertemporal voting equilibrium converges to a balanced-growth voting equilibrium. The long-run extraction rate and the rate of growth are determined by the median discount factor. The extraction rate is decreasing and the growth rate is increasing in the median discount factor.

Our results suggest that the long-run growth rate in the case of private ownership is equal to that of public ownership if the most patient agents constitute the majority of the population, and is higher otherwise. It seems reasonable to conclude that the private property regime is more favorable for promoting long-run economic growth than the public property regime, but this conclusion is somewhat hasty. The private property regime over natural resources, all other things being equal, results in higher income inequality. This can have detrimental effects on economic growth.

High inequality increases sociopolitical instability, the probability of revolutions and mass violence, and the risk of expropriation, thus creating uncertainty in the politico-economic environment. These factors reduce investment incentives and affect the security of property rights. Increased social tension can lead either to a higher extraction rate,[4] or to unproductive costs and losses in output. In both cases the growth rate of the economy is adversely affected.

We consider two different channels through which sociopolitical instability caused by inequality affects economic growth. The first channel assumes that the discount factors of agents are formed endogenously, and the rise in income inequality increases the impatience of agents (see Borissov and Lambrecht, 2009).[5] The second channel assumes that a certain

[3]See also Borissov et al. (2015), where the fundamentals of the proposed intertemporal majority voting approach are discussed in a general case, though in a somewhat different framework.

[4]Since Long (1975), the common wisdom has been that ownership risk induces a firm to overuse the stock of a resource, though the empirical evidence is ambiguous. For instance, Jacoby et al. (2002) support this point of view by reporting that a higher risk of expropriation reduces private investments and raises the current extraction. At the same time, Bohn and Deacon (2000) show that insecure ownership reduces present extraction for resources with capital-intensive extraction technology.

[5]The reasoning behind this assumption is as follows. All risks emerging from high inequality can be reduced to the threat of total political and economic breakdown. When making their decisions, agents do not take into account a new economic order which will be established after breakdown of the current

share of output, depending on inequality, is unproductively thrown away. This share might be used to increase military expenditures, to support and expand various social programs to pacify the population, etc. Under both assumptions, if inequality in the society is sufficiently high, then the public property regime over natural resources is likely to result in a higher long-run rate of growth compared with the private property regime.

In the most of the paper we compare two different institutional frameworks, private and public ownership of natural resources, in a purely positive manner. However, it may be interesting to ask whether differences in growth rate and inequality between private and public ownership could be undone by a social planner implementing certain economic policies. We show that the private property regime with an appropriately chosen capital income tax can have less inequality and a higher long-run growth rate than the public property regime.

The paper is organized as follows. In the main body of the paper we focus on the description of the models and on the general statement of the results. All technical details and proofs appear in Appendices. Wherever we formulate our results, there is a reference to the corresponding proposition or theorem in Appendix. The main body of the paper consists of the following sections. Section 2 presents the basic building blocks of the model and the descriptions of property regimes. In Section 3 we study the model with private ownership of natural resources. We define competitive and balanced-growth equilibria, and present the explicit expression for the equilibrium rate of growth. In Section 4 the model with public ownership of natural resources is considered. We define a competitive equilibrium under given extraction rates, characterize a temporary voting equilibrium, and study an intertemporal voting equilibrium, deriving the expression for the long-run rate of growth. Section 5 compares the long-run consequences of the two different property regimes. In Section 6 we modify the two models by taking into account the impact of sociopolitical instability caused by inequality on the growth rates. In Section 7 a model with private ownership is modified to include capital taxation. Section 8 concludes. Appendix A contains mathematical details and proofs of the statements related to the private property regime. Appendix B provides a thorough formulation of the public property regime. Appendix C is devoted to the private property regime with capital taxation.

# 2 The model

We consider a discrete time dynamic general equilibrium model of the economy endowed with the stock of exhaustible resources. The economy is populated with $L$ heterogeneous in their time preferences agents.

## 2.1 Production and resource extraction

Firms use physical capital, labor and natural resources to produce a homogeneous good, which is a numeraire in the model. Extraction is costless, and all markets are competitive.

economic order. This new order will be better for some agents, and worse for others, but agents can behave rationally only within the current economic order, and cannot extend their rationality beyond its end. Therefore, an increase in the probability of breakdown increases the impatience of all agents.

Output is given by the Cobb–Douglas production function:

$$Y_t = A_t K_t^{\alpha_1} L^{\alpha_2} E_t^{\alpha_3}, \quad \alpha_i > 0 \ (i = 1, 2, 3), \quad \sum_{i=1}^{3} \alpha_i = 1,$$

where $A_t$ is total factor productivity, $K_t$ is the physical capital stock, $E_t$ is the amount of resources extracted in period $t$ (which is identified with the amount of resources utilized in production), and $L$ is the constant over time labor supply. Capital fully depreciates during one time period. We assume that total factor productivity grows at an exogenously given constant rate $\lambda > 0$: $A_t = (1 + \lambda) A_{t-1}$.

The production function in intensive form is given by

$$y_t = \frac{Y_t}{L} = A_t k_t^{\alpha_1} e_t^{\alpha_3},$$

where $k_t = K_t/L$ and $e_t = E_t/L$.

The amount of resources extracted for production decreases the available stock: $R_t = R_{t-1} - E_t$. We denote the resource extraction rate by

$$\varepsilon_t = \frac{E_t}{R_{t-1}},$$

so that the per capita volume of extraction $e_t$ and the dynamics of the resource stock $R_t$ are given by

$$e_t = \frac{\varepsilon_t R_{t-1}}{L}, \qquad R_t = (1 - \varepsilon_t) R_{t-1}.$$

Since all markets are competitive, the interest rate $r_t$, the wage rate $w_t$, and the price of natural resources $q_t$ coincide with the respective marginal products:

$$1 + r_t = \alpha_1 A_t (k_t)^{\alpha_1 - 1} (e_t)^{\alpha_3}, \quad w_t = \alpha_2 A_t (k_t)^{\alpha_1} (e_t)^{\alpha_3}, \quad q_t = \alpha_3 A_t (k_t)^{\alpha_1} (e_t)^{\alpha_3 - 1}.$$

## 2.2 Households

There is an odd number $L$ of agents, indexed by $j = 1, \ldots, L$. Each agent is endowed with one unit of labor. Agent $j$ discounts future utilities by the factor $\beta_j$. We assume that

$$1 > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_L > 0,$$

i.e., agents are ordered by decreasing patience, from more to less patient. We denote by $J = \{j \mid \beta_j = \beta_1\}$ the set of agents with the highest discount factor. These agents appreciate the future higher than the others, and we refer to them as the most patient agents.

Agents obtain utility from their consumption over an infinite time horizon. Preferences of agent $j$ over consumption stream $\{c_t^j\}_{t=0}^{\infty}$ are given by the log-linear utility function:

$$U^j = \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j.$$

## 2.3 Property regimes

In almost every country of the world the state is the *de jure* owner of domestic natural resources. Thus the primary questions are: who has control over the rights to exploit the resource stock? And who has the right to obtain the resource income?

Following Borissov and Surkov (2010), we consider two different property rights regimes over exhaustible resources: private and public. We suppose that the stock of natural resources (e.g., oil or gas fields, coal mines, diamond mine with kimberlite pipes) is divisible, and we do not consider common-pool resources (e.g., oil in the common underground reservoir). It is, in principle, possible to divide the stock into individual parcels and to assign property rights over each parcel.

If proprietary rights over these parcels are established, we refer to this situation as the private property regime. The privately owned resource stock *in situ* is an asset to its owner. Agents can invest in natural resources as well as in physical capital. By the Hotelling (1931) rule, the equilibrium rate of return on the resource stock as an asset is equal to the return on capital. The resource income goes to the resource owner.

There can be other property rights regimes over natural resources. It is reasonable to consider the situation in which the exhaustible resource stock is controlled by a government that acts in the interest of the agents. We refer to this situation as the public property regime. In this case the resource income is equally distributed among agents, who choose the resource extraction rate by majority voting.

Note that in the public property regime there is no reason to expect that the Hotelling rule holds. There are incentives for arbitrage operations, as the rate of change of the natural resource price can differ from the interest rate. However, we assume that private storages are forbidden. There is no possibility to store resources; they are utilized immediately after extraction. Thus no arbitrage opportunities can be exploited.

# 3 Private property regime

Consider first the case in which the exhaustible resource stock is privately owned. In this section our exposition follows Borissov and Surkov (2010). We introduce the model and specify its main properties. Formal definitions and proofs can be found in Appendix A.

## 3.1 Competitive equilibrium

Suppose we start at time 0. Agent $j$ is endowed with some amount of physical capital $\hat{k}^j \geq 0$ and natural resources $\hat{R}^j \geq 0$. Given the equilibrium price of natural resources at $t = -1$, $q_{-1}$, the initial savings of agent $j$ are determined as $s^j_{-1} = q_{-1}\hat{R}^j + \hat{k}^j \geq 0$.[6]

Agent $j$ chooses her consumption plan by solving the problem of maximizing lifetime

[6]Note that the price of natural resources at time $t = -1$ is determined endogenously. Therefore, the initial savings of agent $j$ are not given exogenously. To ensure that the initial savings are non-negative, we impose the non-negativity constraints on initial holdings of physical capital and resources.

utility:

$$\max \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j,$$

$$\text{s. t.} \quad c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t,$$

$$s_t^j \geq 0,$$

$$s_{-1}^j = q_{-1}\hat{R}^j + \hat{k}^j.$$

Here $c_t^j$ and $s_t^j$ are consumption and savings of agent $j$, $r_t$ is the interest rate, and $w_t$ is the wage rate at time $t$.

Agents are prohibited from borrowing against their future earnings. Thus their savings must be non-negative. They can be invested in both physical capital and natural resources.[7] From the agents' point of view, the two assets are perfect substitutes, so we do not distinguish between physical capital and natural resource in the structure of agents' savings. The return to investment into physical capital and into natural resources must be equal (the Hotelling rule), hence the price of natural resources $q_t$ grows at the rate $r_t$:

$$q_t = (1 + r_t)\, q_{t-1}.$$

We define a *competitive equilibrium* in the private property regime (starting from some initial distribution of physical capital and natural resources),

$$\mathcal{E}^* = \left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\dots},$$

in a standard way by the following conditions:

- agents maximize their utilities subject to budget constraints;
- capital, labor and natural resources are paid their marginal products;
- the Hotelling rule holds;
- aggregate agents' savings are equal to the investment into physical capital and natural resources.

A competitive equilibrium exists (see Theorem A.1 in Appendix A), and *if initially the stocks of physical capital and natural resources belong to the most patient agents, then the competitive equilibrium starting from this state is unique* (Proposition A.1). Also, *in each competitive equilibrium from some time onward only the most patient agents can make positive savings, and from this time resources are extracted at a constant rate* $\varepsilon^* = 1 - \beta_1$ (Proposition A.2).

[7] Formally speaking, the non-negativity constraint on savings does not rule out the possibility that some agents have positive holdings of physical capital and negative holdings of resources, or vice versa. However, in an equilibrium only agents' savings are of interest for us, and it is irrelevant in which form they are held.

## 3.2 Balanced-growth equilibrium

A *balanced-growth equilibrium* is a competitive equilibrium in which output, consumption, savings, the capital stock and the wage rate grow at a constant rate $\gamma^*$, while the interest rate $r^*$ is constant over time. The price of natural resources grows at a constant rate equal to $r^*$ (by the Hotelling rule), and resources are depleted at a constant extraction rate $\varepsilon^*$.

In our model, *there exists a balanced-growth equilibrium* (see Proposition A.3). In such an equilibrium only the most patient agents make positive savings, while relatively impatient agents make no savings and consume their wages. This important property is known in the literature as the Ramsey (1928) conjecture.[8]

It can be checked that *the interest rate $r^*$, the extraction rate $\varepsilon^*$, and the growth rate $\gamma^*$ are the same for every balanced-growth equilibrium* (Proposition A.4). Moreover, *every competitive equilibrium converges to a balanced-growth equilibrium* (for the precise meaning of this statement see Proposition A.5). Thus we can concentrate on the characterization of balanced-growth equilibria when exploring the long-run perspective.

The equilibrium extraction rate is determined by the discount factor of the most patient agents:

$$\varepsilon^* = 1 - \beta_1.$$

The equilibrium rate of balanced growth depends on the extraction rate, and hence is given by

$$1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1-\varepsilon^*)^{\frac{\alpha_3}{1-\alpha_1}} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}.$$

Unlike Ramsey-type models without natural resources, in which the long-run growth rate is determined by the exogenously given rate of technical progress, in economies with natural resources the growth rate typically depends on the discount factor (see, e.g., Dasgupta and Heal, 1979). It is well-known that in representative agent models with exhaustible natural resources the extraction rate affects the growth rate (see the discussion in Stiglitz, 1974), and thus the discount factor plays a crucial role in determining the rate of balanced growth.

The equilibrium extraction rate is decreasing and the rate of balanced growth is increasing in the discount factor of the most patient agents. An increase in patience of the resource stock owners means that they put more weight on additional future consumption compared to additional present consumption. Thus they prefer to extract less amount of resources today, and the rate of natural resource utilization becomes lower. In turn, lower extraction rate (lower ratio of resource use per unit of time to stock) leads to a higher growth rate in the future.

# 4 Public property regime

Consider now the case in which the stock of exhaustible resources is held in trust by the government. Resources are extracted and sold to the private production sector. Income from the sale of natural resources is equally distributed among agents who choose the resource extraction rate by majority voting.[9] In this section we focus on the description

[8]For the history and discussion of this conjecture, see Becker (2006).

[9]This regime is also discussed by Borissov and Surkov (2010). However, their voting approach is not free from certain major drawbacks. The voting mechanism in their model allows one to define voting only in a balanced-growth equilibrium, and agents do not take into account the fact that policy changes have

of the model and on the general statement of the results. See Appendix B for formal definitions and proofs.

## 4.1 Competitive equilibrium under given extraction rates

Let us first define a competitive equilibrium under given extraction rates starting from an arbitrary point in time. Suppose that instead of time 0, we start at time $\tau$. Each agent $j$ has savings $\hat{s}^j_{\tau-1} \geq 0$ such that the corresponding stock of physical capital is positive, $k_\tau = \frac{1}{L}\sum_{j=1}^{L} \hat{s}^j_{\tau-1} > 0$. The stock of natural resources is also positive, $\hat{R}_{\tau-1} > 0$.

Suppose that at time $\tau$ agents have some non-degenerate expectations about future extraction rates, $\{\varepsilon^e_t\}^\infty_{t=\tau+1}$.[10] For any $\varepsilon_\tau \in (0,1)$ denote

$$\mathbb{E}_\tau(\varepsilon_\tau) = \{\varepsilon_\tau, \varepsilon^e_{\tau+1}, \varepsilon^e_{\tau+2}, \ldots\},$$

and notice that the sequence of extraction rates $\mathbb{E}_\tau(\varepsilon_\tau)$ is in fact arbitrary.

Clearly, given the sequence of extraction rates, the per capita volumes of extraction $e_t$ and the dynamics of the exhaustible resource stock $R_t$ are predetermined as follows:

$$R_\tau(\varepsilon_\tau) = (1-\varepsilon_\tau)\hat{R}_{\tau-1}, \quad R_t(\varepsilon_\tau) = (1-\varepsilon^e_t)R_{t-1}(\varepsilon_\tau),\ t = \tau+1, \tau+2, \ldots;$$
$$e_\tau(\varepsilon_\tau) = \frac{\varepsilon_\tau \hat{R}_{\tau-1}}{L}, \quad e_t(\varepsilon_\tau) = \frac{\varepsilon^e_t R_{t-1}(\varepsilon_\tau)}{L},\ t = \tau+1, \tau+2, \ldots.$$

Our notation emphasizes the fact that the future volumes of extraction and the dynamics of the resource stock depend on the time $\tau$ extraction rate $\varepsilon_\tau$.

Given the future volumes of extraction, agent $j$ solves the following maximization problem:

$$\max \sum_{t=\tau}^{\infty} \beta^t_j \ln c^j_t,$$
$$\text{s. t.} \quad c^j_t + s^j_t \leq (1+r_t)\, s^j_{t-1} + w_t + v_t,$$
$$s^j_t \geq 0,$$
$$s^j_{\tau-1} = \hat{s}^j_{\tau-1}.$$

Here $c^j_t$ and $s^j_t$ are consumption and savings of agent $j$, $r_t$ is the interest rate, $w_t$ is the wage rate, and $v_t$ is the per capita resource income at time $t$. The latter is the income from the sale of the extracted resource to the production sector, equally distributed among all agents.

In this model, agents are also prohibited from borrowing against their future income, and their savings must be non-negative. Savings can be invested only in physical capital, but not in natural resources. This is an important difference between the public property regime and the private property regime considered in Section 3.

A *competitive equilibrium*

$$\mathcal{E}^{**}_\tau(\varepsilon_\tau) = \left\{(c^{j**}_t(\varepsilon_\tau))^L_{j=1}, (s^{j**}_t(\varepsilon_\tau))^L_{j=1}, k^{**}_t(\varepsilon_\tau), r^{**}_t(\varepsilon_\tau), w^{**}_t(\varepsilon_\tau), q^{**}_t(\varepsilon_\tau), v^{**}_t(\varepsilon_\tau)\right\}_{t=\tau,\tau+1\ldots}$$

is defined in a standard way by the following conditions:

general equilibrium effects.

[10] We call a sequence of extraction rates $\{\varepsilon_t\}^\infty_{t=\tau}$ non-degenerate if $0 < \varepsilon_t < 1$ for all $t$, $\liminf_{t\to\infty} \varepsilon_t > 0$, and $\limsup_{t\to\infty} \varepsilon_t < 1$.

- agents maximize their utilities subject to the budget constraints, perfectly anticipating the profile of factor returns and resource incomes, $\{r_t^{**}(\varepsilon_\tau), w_t^{**}(\varepsilon_\tau), v_t^{**}(\varepsilon_\tau)\}_{t=\tau}^{\infty}$;
- capital, labor and natural resources are paid their marginal products;
- the resource income is determined by the marginal product of natural resources;
- aggregate agents' savings are equal to investment into physical capital.

Let us clarify our notation $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$. We use $**$ to denote the equilibrium values in the public property regime. The equilibrium starts at time $\tau$, hence the subscript. The equilibrium also depends on agents' expectations about future extraction rates, and on the parameters of the model. However, we are interested in the dependence of equilibrium variables on the current extraction rate $\varepsilon_\tau$. For instance, the notation $\{c_t^{j**}(\varepsilon_\tau)\}_{t=\tau}^{\infty}$ emphasizes the dependence of the equilibrium consumption stream for agent $j$ (and thus her utility) on $\varepsilon_\tau$.

In the above definition we do not suppose that the Hotelling rule holds. Indeed, the Hotelling rule corresponds to the equilibrium on the asset market, i.e., to the private property regime, where the stock of natural resources is an asset in which agents can invest. In the public property regime, the natural resource stock is not an asset, so the Hotelling rule can be violated.[11] Under some circumstances the rate of change of the resource price may not be equal to the interest rate. However, since we assume that resources cannot be stored privately, all arbitrage opportunities that arise from the violation of the Hotelling rule are forbidden.

There exists a competitive equilibrium under given non-degenerate sequence of extraction rates (see Theorem B.1 in Appendix B), and, similarly to the private property regime, *if initially the stock of physical capital is owned by the most patient agents, then the competitive equilibrium is unique* (Proposition B.1). Also, *in every competitive equilibrium all but the most patient agents run their capital to zero* (Proposition B.2). Eventually the whole capital stock belongs to the most patient agents. Thus in the public property regime the Ramsey conjecture also holds true.

## 4.2 Balanced-growth equilibrium under given extraction rate

Suppose that a sequence of extraction rates is constant over time. A *balanced-growth equilibrium* under given extraction rate $\varepsilon$ is a competitive equilibrium in which output, consumption, savings, the capital stock, the wage rate and the resource income grow at a constant rate $\gamma^{**}$, while the rate of change of the resource price and the interest rate are constant over time.

We show that *for any $\varepsilon$ there exists a balanced-growth equilibrium* (its characterization is given in Proposition B.3). In any balanced-growth equilibrium only the most patient agents make positive savings and thus own the whole capital stock. It follows that *the rate of balanced growth, the interest rate, and the rate of change of the resource price depend on the parameters of the model and on the given extraction rate $\varepsilon$* (see Proposition B.4). Moreover, *every competitive equilibrium under a constant sequence of extraction rates converges to a balanced-growth equilibrium* (Proposition B.5).

[11]See Chermak and Patrick (2002) for a discussion of the Hotelling rule applicability to the observable price dynamics.

Thus, we can give a qualitative description of competitive equilibria under given extraction rates. In every competitive equilibrium from some time onward only the most patient agents own the whole capital stock. They obtain not only wages and the resource income, but also the capital income. The incomes of all other agents consist only of wages and the resource income. If, in addition, from some time onward the sequence of extraction rates is constant, then a competitive equilibrium converges to a balanced-growth equilibrium.

### 4.3 Time $\tau$ voting equilibrium

Now we make extraction rates endogenous and introduce a voting procedure into our model. Our approach to voting in a dynamic general equilibrium framework follows Borissov et al. (2014b).[12] In our model agents vote on the current extraction rate at each point in time.

First we define a voting equilibrium under the assumption that a competitive equilibrium under given extraction rates is unique. Recall that this is true, in particular, when the stock of physical capital is initially owned only by the most patient agents. Further, in Subsection 4.6, we generalize our voting procedure to the case in which this assumption may not hold.

Under the uniqueness assumption a voting equilibrium is defined as follows. At each point in time agents choose the today's extraction rate by majority voting under given expectations about future extraction rates. We show that the agents' preferences are single-peaked, and hence the median voter theorem applies: at each point in time there exists an instantaneous Condorcet winner, i.e., an extraction rate which is preferred by a majority of agents to any other extraction rate. A *time $\tau$ (temporary) equilibrium* is determined by this Condorcet winner. We obtain the closed-form solution for the preferred extraction rate for each agent, and show that it depends only on the discount factor of the agent and is independent of expectations.[13] Hence the instantaneous Condorcet winner is the preferred extraction rate for the agent with the median discount factor. Since the outcome of voting at each point in time does not depend on expectations, an *intertemporal voting equilibrium* is defined in a natural way.

Formally, suppose at time $\tau$ agents are asked to vote on the time $\tau$ extraction rate. *Suppose that for given expectations about future extraction rates, $\{\varepsilon_t^e\}_{t=\tau+1}^{\infty}$, and for any $\varepsilon_\tau \in (0,1)$ a competitive equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ is unique.* Then we can unambiguously define agents' preferences over the time $\tau$ extraction rate by the indirect utility functions:

$$\mathcal{U}_\tau^j(\varepsilon_\tau) = \ln c_\tau^{j**}(\varepsilon_\tau) + \beta_j \ln c_{\tau+1}^{j**}(\varepsilon_\tau) + \dots, \quad j = 1, \dots, L, \tag{1}$$

where $\{c_t^{j**}(\varepsilon_\tau)\}_{t=\tau}^{\infty}$ is the equilibrium consumption stream for agent $j$. Consumption stream and objective functions depend on expectations and on the parameters of the model as well. However, we use a notation that emphasizes the dependence on the time $\tau$ extraction rate $\varepsilon_\tau$ on which agents vote.

[12] Borissov et al. (2014b) study voting on the shares of public goods in the GDP. Here we apply their approach to voting on extraction rates.

[13] This is due to the log-linear utility functions and the Cobb–Douglas production function. Only in this particular case we can apply our approach to voting in a dynamic general equilibrium framework. A model with general utility and production functions in a dynamic optimization context is proposed by Borissov et al. (2015).

The voting method is majority rule. We define a *time $\tau$ (temporal) voting equilibrium* as a couple $\{\varepsilon_\tau^{**}, \mathcal{E}_\tau^{**}\}$ such that the equilibrium extraction rate $\varepsilon_\tau^{**}$ is a Condorcet winner in voting on the time $\tau$ extraction rate, and $\mathcal{E}_\tau^{**} = \mathcal{E}_\tau^{**}(\varepsilon_\tau^{**})$ is the corresponding competitive equilibrium.

In order to determine the equilibrium extraction rate, we have to articulate how agents vote on the time $\tau$ extraction rate. When voting, agents maximize their indirect utility functions given by (1). Therefore, it is crucial to know how the competitive equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ changes when $\varepsilon_\tau$ changes. Suppose that the time $\tau$ extraction rate increases, while the initial resource stock and expectations about future extraction rates remain intact. In other words, let $\varepsilon_\tau$ be replaced by some other extraction rate, $\tilde{\varepsilon}_\tau > \varepsilon_\tau$.

The time $\tau$ volume of extraction increases by the factor $\tilde{\varepsilon}_\tau/\varepsilon_\tau$. Hence the output at time $\tau$ increases by the factor $(\tilde{\varepsilon}_\tau/\varepsilon_\tau)^{\alpha_3}$. Some simple but tedious calculations (see Lemma B.9) show that consumption and savings of all agents, the wage rate, the interest rate, the resource income, and the time $\tau + 1$ capital stock also increase by the factor $(\tilde{\varepsilon}_\tau/\varepsilon_\tau)^{\alpha_3}$, proportionally to the changed output.

Further, for all future periods of time the available resource stock decreases: it is multiplied by the factor $(1-\tilde{\varepsilon}_\tau)/(1-\varepsilon_\tau) < 1$. Since expectations about future extraction rates do not change, this leads to the proportional decline in the volumes of extraction. Thus there is a trade-off between the future and today's extraction, which leads to a trade-off between the future and today's consumption. The agent's decision on the time $\tau$ extraction rate explicitly affects her future consumption by changing the available resource stock.

Under our assumptions, *agents' preferences in voting on the time $\tau$ extraction rate, defined by (1), are single-peaked.* We show that *for each agent $j$ there exists a unique preferred time $\tau$ extraction rate, i.e., the value $\varepsilon_\tau^j$ that maximizes her indirect utility function $\mathcal{U}^j(\varepsilon_\tau)$*. It turns out that *the preferred time $\tau$ extraction rate for each agent $j$ is given by*

$$\varepsilon_\tau^j = 1 - \beta_j, \tag{2}$$

(see Proposition B.6). The preferred time $\tau$ extraction rate for each agent is constant over time, depends only on this agent's discount factor, and does not depend on expectations or on the current state of the economy.

By the median voter theorem, there exists a Condorcet winner in majority voting on the time $\tau$ extraction rate, which is the preferred extraction rate for the agent with the median discount factor. It follows that *there is a unique time $\tau$ voting equilibrium, with the extraction rate $\varepsilon_\tau^{**}$ given by*

$$\varepsilon_\tau^{**} = 1 - \beta_{med}, \tag{3}$$

*where $\beta_{med}$ is the median discount factor* (see Theorem B.2).

Note that the equilibrium time $\tau$ extraction rate actually depends neither on the current state of the economy nor on the expectations of agents. The equilibrium extraction rate is constant over time and depends only on the distribution of the discount factors across the population. This result in particular eliminates a strategic motive to influence the outcomes of future votes. This is the reason why they can really be taken as given.

## 4.4 Intertemporal voting equilibrium

Now let us assume perfect foresight about future extraction rates and define an intertemporal voting equilibrium.

Suppose we are given a sequence of extraction rates

$$\mathbb{E}^{**} = \mathbb{E}_0^{**} = \{\varepsilon_t^{**}\}_{t=0}^{\infty}.$$

Denote by

$$\mathcal{E}^{**} = \mathcal{E}_0^{**} = \big\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\big\}_{t=0,1,\ldots}$$

the corresponding competitive equilibrium starting at $t = 0$ under the sequence of extraction rates $\mathbb{E}_0^{**}$.

Let also for $\tau = 1, 2, \ldots$

$$\mathcal{E}_\tau^{**} = \big\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\big\}_{t=\tau,\tau+1,\ldots}$$

be the corresponding tail of $\mathcal{E}_0^{**}$. Clearly, it is a competitive equilibrium starting at $t = \tau$ under the sequence of extraction rates $\mathbb{E}_\tau^{**} = \{\varepsilon_t^{**}\}_{t=\tau}^{\infty}$.

An *intertemporal voting equilibrium* is a couple which consists of the sequence of extraction rates $\mathbb{E}^{**}$ and the corresponding competitive equilibrium $\mathcal{E}^{**}$ such that for every $\tau = 0, 1, \ldots$, the time $\tau$ extraction rate $\varepsilon_\tau^{**}$ and the competitive equilibrium $\mathcal{E}_\tau^{**}$ constitute a time $\tau$ voting equilibrium under perfect foresight about future extraction rates ($\varepsilon_t^e = \varepsilon_t^{**},\ t = \tau+1, \tau+2, \ldots$).

It is clear that *in every intertemporal voting equilibrium the sequence of extraction rates* $\mathbb{E}^{**}$ *is constant*:

$$\mathbb{E}^{**} = \{\varepsilon^{**}, \varepsilon^{**}, \ldots\}, \tag{4}$$

where $\varepsilon^{**} = 1 - \beta_{med}$ (see Theorem B.3).

The existence and uniqueness of an intertemporal voting equilibrium are related to the corresponding properties of an underlying competitive equilibria. In particular, *if initially the whole capital stock belongs to the most patient agents, then an intertemporal equilibrium exists and is unique* (Theorem B.4).

## 4.5 Balanced-growth voting equilibrium

A balanced-growth equilibrium for which the sequence of extraction rates is given by (4) is called a *balanced-growth voting equilibrium.* Put differently, a balanced-growth voting equilibrium is an intertemporal voting equilibrium in which output, consumption, savings, the capital stock, the wage rate, and the resource income grow at a constant rate $\gamma^{**}$, while the rate of change of the resource price and the interest rate are constant over time.

It can be checked that *if initially the whole capital stock is owned by the most patient agents, then the intertemporal voting equilibrium converges to a balanced-growth voting equilibrium* (Theorem B.5).

The most important conclusion to emerge from the characterization of a balanced-growth voting equilibrium is that the voting equilibrium extraction rate is fully determined by the median discount factor:

$$\varepsilon^{**} = 1 - \beta_{med}.$$

Consequently, the rate of balanced growth depends on the median discount factor:

$$1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1 - \varepsilon^{**})^{\frac{\alpha_3}{1-\alpha_1}} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_{med}^{\frac{\alpha_3}{1-\alpha_1}}.$$

It is clear that the more patient the population as a whole is (the higher $\beta_{med}$ is), the lower the voting equilibrium extraction rate is and the higher the long-run growth rate is. These results capture the intuition that the more patient agents value the future more highly and tend to smooth their consumption. They vote for the lower extraction rate today in order to maintain a higher resource stock level in the future, which leads to a higher growth rate.

As we have mentioned, the Hotelling rule can be violated. The rate of change of the resource price $\pi^{**}$ is not necessarily equal to the interest rate $r^{**}$. Namely, we have

$$\frac{1+\pi^{**}}{1+r^{**}} = \frac{\beta_1}{\beta_{med}}.$$

It follows that if $\beta_{med} < \beta_1$, then $\pi^{**}$ is larger than $r^{**}$. However, this should not be a great surprise. In the public property regime agents cannot invest in natural resources, so there is no reason for the Hotelling rule to hold. Indeed, in this model the return on capital is related to the discount factor of the most patient agents, and the resource extraction rate is determined by the agents with the median discount factor. Unless these types of agents coincide (i.e., unless $\beta_{med} = \beta_1$), the Hotelling rule is violated.

## 4.6 Generalized intertemporal voting equilibria

Our definition of an intertemporal voting equilibrium is given under the assumption of the uniqueness of the competitive equilibrium under given extraction rates. In particular, this assumption ensures the uniqueness of the time $\tau$ voting equilibrium and the existence of an intertemporal voting equilibrium.

Let us discuss the general case in which the uniqueness of a competitive equilibrium $\mathcal{E}_{\tau}^{**}(\varepsilon_{\tau})$ is not guaranteed. The difficulty here is that we cannot unambiguously define agents' indirect utility functions and obtain agents' preferred values of extraction rates. However, if we apply the technique proposed by Borissov et al. (2014b), we can get around this difficulty. To do this, we impose a certain additional assumption on the beliefs of agents.

Let us assume that *agents do not take into account the multiplicity of equilibria and believe that a competitive equilibrium after the change of the time $\tau$ extraction rate is associated with a competitive equilibrium before the change in the way discussed in Subsection 4.3 (and described in Lemma B.9).* Our assumption implies that agents simply act as if the competitive equilibrium $\mathcal{E}_{\tau}^{**}(\varepsilon_{\tau})$ is unique for any given extraction rate $\varepsilon_{\tau}$.

We define a *generalized intertemporal voting equilibrium* in essentially the same way as an intertemporal voting equilibrium. The only difference is that we do not assume the uniqueness of the competitive equilibrium; it is replaced with the additional assumption about agents' beliefs.

Clearly, any intertemporal voting equilibrium is a generalized intertemporal voting equilibrium. Moreover, if initially the whole capital stock belongs to the most patient agents, then any generalized intertemporal voting equilibrium is an intertemporal voting equilibrium.

Under our additional assumption all the results concerning voting equilibria remain the same as in the case of the unique competitive equilibrium. Namely, the preferred value of the time $\tau$ extraction rate for agent $j$ is given by (2), and the equilibrium time $\tau$ extraction rate is constant over time and given by (3). What is important, *there always exists a generalized intertemporal voting equilibrium, and the sequence of extraction rates in every generalized intertemporal equilibrium is constant over time and given by (4)* (this is the statement of Theorem B.6). Furthermore, *every generalized intertemporal equilibrium converges to a balanced-growth voting equilibrium* (see Theorem B.7).

## 5 Comparison of the balanced-growth equilibria

Now we can analyze the long-run consequences of different property regimes in terms of economic growth. In the private property regime, every competitive equilibrium converges to a balanced-growth equilibrium. In the long run only the most patient agents obtain income from the capital and resource stocks. Therefore, the equilibrium extraction rate and the rate of balanced growth are determined by the discount factor of the most patient agents and given by

$$\varepsilon^* = 1 - \beta_1,$$
$$1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}.$$

In the public property regime, the sequence of the resource extraction rates is chosen by majority voting. Every generalized intertemporal voting equilibrium converges to a balanced-growth voting equilibrium. In the long run the economy is characterized by the voting equilibrium extraction rate and the rate of balanced growth. They are given by structurally similar equations as in the private property regime, but depend on the median discount factor:

$$\varepsilon^{**} = 1 - \beta_{med},$$
$$1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_{med}^{\frac{\alpha_3}{1-\alpha_1}}.$$

It follows that in both regimes the more patient the decision makers are, the lower the extraction rate is and the higher the long-run growth rate is. This is reasonable since patient agents decide to extract less today in order to provide a higher standard of consumption in the future. Therefore it is possible to sustain a higher growth rate in the long run.

The question of which property regime leads to a higher long-run growth rate reduces then to the question of the relationship between the median discount factor and the highest discount factor. In other words, this is the question of whether the most patient agents constitute the majority of the population. If the most patient agents do not constitute the majority of the population ($\beta_{med} < \beta_1$), then the equilibrium extraction rate in the private property regime is lower than in the public property regime. This means that the long-run growth rate is higher in the private property regime. If the most patient agents constitute the majority of the population ($\beta_{med} = \beta_1$), then there is no difference in the growth rates between the two regimes in the long run.

It should be emphasized that we do not argue in favor of private or public ownership, and do not claim that a higher growth rate is better than a lower one. To answer the question, what is better for the society, a social welfare function should be used, and the optimal growth rate is determined by the discount factor of the social planner. However, aggregation of individual preferences and the choice of the discount factor of the social planner is not an easy task when agents are heterogeneous in their time preferences. For instance, Zuber (2011) shows that if agents have different discount factors, then no Paretian social welfare function satisfying natural requirements (history independence, time consistency and stationary) exists.

# 6 Property regimes and income inequality

Recall that in the case of public ownership, the resource income is equally distributed among agents, while the capital stock in the long run belongs only to the most patient agents. In the case of private ownership, the most patient agents obtain both capital and resource incomes in the long run. Hence private ownership, all other things being equal, results in higher income inequality than public ownership. Our results suggest that unless we take into account inequality, the private property regime over exhaustible resources is more favorable for promoting long-run economic growth than the public property regime. However, since inequality can have detrimental effects on economic growth, this conclusion is somewhat hasty.

There is conflicting evidence concerning the relationship between income inequality and economic growth. Various studies provide evidence for positive, negative or non-linear relationships (see, e.g., Henderson et al., 2015). However, there are some convincing theoretical and empirical arguments that inequality has a negative and statistically significant long-lasting impact on economic development (see, e.g., Keefer and Knack, 2002; Borissov and Lambrecht, 2009; Herzer and Vollmer, 2012; Cingano, 2014).

In particular, Alesina and Perotti (1996) emphasize the role of social conflict as a link between inequality and growth. Inequality increases sociopolitical instability and causes social tension. It creates uncertainty in the politico-economic environment, which in turn reduces investment incentives and affects the security of property rights. Uncertainty and social tension induce fear of losing the sources of income. All these can lead either to a faster depletion of resources, or to unproductive costs and losses in output. Both channels reduce the growth rate of the economy. Thus higher inequality in the private property regime may result in a lower long-run rate of growth compared with the public property regime.

Let us model two channels through which uncertainty and social tension caused by inequality affect economic growth. Suppose that there is a value $p$ which reflects a detrimental effect of inequality on the economy. Let us assume that $p = \psi(G)$, where $G$ is the Gini coefficient, and the function $\psi : [0, 1] \to [0, 1]$ satisfies the following properties:

- $\psi(G)$ is continuous;
- $\psi(G) = 0$ for $G$ smaller than some $\bar{G}$;
- $\psi(G)$ is increasing for $G > \bar{G}$.

An exemplary form of this function is shown in Figure 1. Below we will embed the function $\psi(G)$ in our model and clarify its role in affecting the long-run variables.

Clearly, income inequality changes over time. However, for a balanced-growth equilibrium it is constant over time. Let us compare balanced-growth equilibria in the private and public property regimes, taking into account inequality effects.

In a balanced-growth equilibrium the income distribution depends on two characteristics. The first characteristic is the fraction of stock owners, which we denote by $\sigma$. Stock owners are the agents who obtain income from both capital and resource stocks in the private property regime, or the agents who own the capital stock in the public property regime. It is clear that the set of stock owners is a subset of the set of the most patient agents, $J$. If every agent from the set $J$ makes positive savings in a balanced-growth equilibrium, then $\sigma = |J|/N$. Otherwise, $\sigma < |J|/N$.

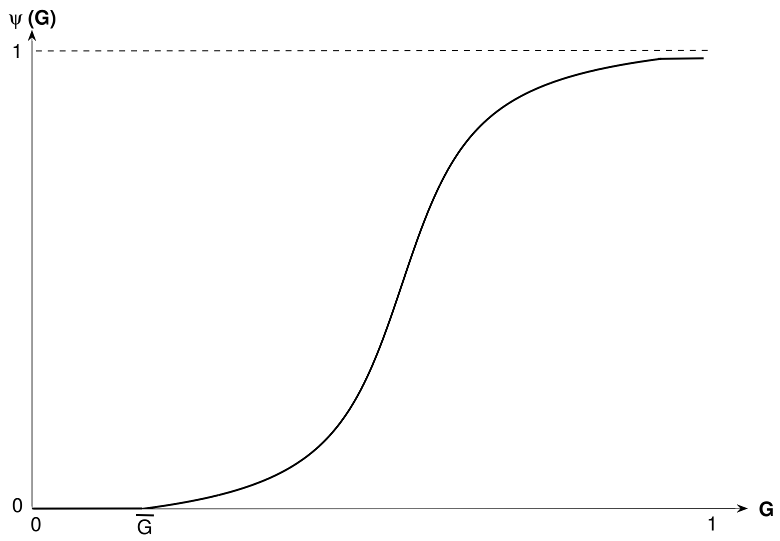


**Figure 1:** An exemplary form of the function $\psi(G)$

The second characteristic is the distribution of savings across the set of the stock owners. For simplicity we consider the case in which savings are distributed evenly (i.e., initial holdings of capital and resources are equal) across the stock owners. Given $\sigma$, any other pattern of the savings distribution in a balanced-growth equilibrium results in a higher Gini coefficient. Therefore, for a given $\sigma$, the even distribution of savings across the stock owners provides a lower bound on inequality in the society.[14]

It is not difficult to calculate the Gini coefficient based on the income distribution of agents in a balanced-growth equilibrium for both property regimes. We have

$$G = \alpha(1 - \sigma),$$

where $\alpha = \alpha_1$ for the public property regime, and $\alpha = \alpha_1 + \alpha_3$ for the private property regime.

Let us explore two channels through which inequality affects economic growth. Consider first how uncertainty and social tension caused by inequality lead to a faster resource depletion. Following Borissov and Lambrecht (2009), we model this possibility by assuming that the discount factors of agents are formed endogenously. Insecure property rights reduce confidence about the future and decrease the discount factors of agents. Agents are not sure of the future and are not able to put the high weight on their future utility. Namely, let us assume that the objective function of agent $j$ is given by

$$U^j = \sum_{t=0}^{\infty} \left((1 - p)\beta_j\right)^t \ln c_t^j,$$

where $p = \psi(G)$, as discussed above. In this case $p$ reflects the insecurity of property

[14]We assume that agents are negligible, and their decisions have no effect on inequality, which they take as given. However, if patient agents can by their actions affect the general equilibrium (e.g., in the case when they own shares in a single firm), then they should be treated as strategic actors who would anticipate that high income inequality causes instability, and thus inequality would be endogenous. The analysis of these issues typically adopts the Markov voting equilibria framework (see, e.g., Acemoglu et al., 2012, 2015), which is beyond the scope of our paper.

rights that lowers the discount factors of all agents.[15] The value $(1-p)\beta_j$ may be called the effective discount factor of agent $j$.

The long-run equilibrium extraction rates are now given by

$$\varepsilon^* = 1 - (1-\psi[(\alpha_1+\alpha_3)(1-\sigma)])\,\beta_1,$$
$$\varepsilon^{**} = 1 - (1-\psi[\alpha_1(1-\sigma)])\,\beta_{med}.$$

Thus the rates of balanced growth are given by

$$1+\gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \left((1-\psi[(\alpha_1+\alpha_3)(1-\sigma)])\beta_1\right)^{\frac{\alpha_3}{1-\alpha_1}},$$
$$1+\gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \left((1-\psi[\alpha_1(1-\sigma)])\beta_{med}\right)^{\frac{\alpha_3}{1-\alpha_1}}.$$

It follows that high inequality can lead to the increase in the resource extraction rate and to a decrease in the rate of balanced growth. Indeed, suppose that $\sigma$ satisfies the following condition:

$$(\alpha_1+\alpha_3)(1-\sigma) > \bar{G},$$

or, equivalently,

$$\sigma < 1 - \frac{\bar{G}}{\alpha_1+\alpha_3}. \tag{5}$$

Clearly, this may happen when inequality in the society is sufficiently high, i.e., when there are only a few stock owners. Then, the extraction rate $\varepsilon^*$ in the private property regime becomes higher, and the corresponding rate of growth $1+\gamma^*$ becomes lower, compared with the case in which we do not take into account the impact of the insecure property rights.

If inequality is so high that

$$\sigma < 1 - \frac{\bar{G}}{\alpha_1},$$

then the equilibrium extraction rates under both property regimes become higher, and both long-run growth rates become lower, compared with the case without the inequality impact.

Let us compare the growth rates between the two property regimes. It is clear that for $1+\gamma^{**} > 1+\gamma^*$, the following condition must hold:

$$\frac{1-\psi[\alpha_1(1-\sigma)]}{1-\psi[(\alpha_1+\alpha_3)(1-\sigma)]} > \frac{\beta_1}{\beta_{med}}. \tag{6}$$

Notice that when $\beta_{med} = \beta_1$, condition (6) is equivalent to condition (5). Thus if the most patient agents constitute the majority of the population, and inequality is just sufficient to increase the extraction rate in the private property regime, then our previous conclusion about the same growth rates in the two regimes is no longer true. Taking into account the impact of insecure property rights, it is the public property regime that results in a higher rate of growth compared with the private property regime.

If the most patient agents do not constitute the majority of the population, then condition (6) is more likely to hold when inequality is high, i.e., for the low values of $\sigma$. In other words, when there are only a few stock owners, it is more likely that the public property regime leads to a lower equilibrium extraction rate ($\varepsilon^{**} < \varepsilon^*$) and a higher rate of growth ($\gamma^{**} > \gamma^*$) than the private property regime.

[15] Similar reasoning is used by Gaddy and Ickes (2005).

Consider now how uncertainty and social tension caused by inequality lead to the high social costs and losses in output. Assume that in each period a certain share of output which depends on inequality is wasted. If inequality is low, then the wasted fraction is zero. If inequality is high, then instability is also high. In order to pacify the population, a certain share of output will be unproductively thrown away. This wasted share may represent military, police or other special forces expenditure, social spending, etc.

Assume that output per capita is given by

$$y_t = (1-p)A_t k_t^{\alpha_1} e_t^{\alpha_3},$$

where $p = \psi(G)$ now reflects the share of GDP which is drawn away to maintain public order and to prevent possible dissatisfaction of the population about inequality in the society.

Here, the impact of uncertainty and social tension caused by inequality does not influence the equilibrium extraction rates. However, it changes the rates of balanced growth. The growth rate in the case of private property is given by

$$1+\gamma^* = (1-\psi[(\alpha_1+\alpha_3(1-\sigma)]) \left((1+\lambda)\beta_1^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}}.$$

The growth rate in the case of public property is given by

$$1+\gamma^{**} = (1-\psi[\alpha_1(1-\sigma)]) \left((1+\lambda)\beta_{med}^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}}.$$

It follows that $1+\gamma^{**} > 1+\gamma^*$ if

$$\frac{1-\psi[\alpha_1(1-\sigma)]}{1-\psi[(\alpha_1+\alpha_3)(1-\sigma)]} > \left(\frac{\beta_1}{\beta_{med}}\right)^{\frac{\alpha_3}{1-\alpha_1}}. \tag{7}$$

Again note that when $\beta_{med} = \beta_1$, condition (7) is equivalent to condition (5). This means that if the most patient agents constitute the majority of the population, then as soon as inequality matters, the public property regime results in a higher long-run rate of growth than the private property regime. If the most patient agents do not constitute the majority of the population, then condition (7) is more likely to hold for the low values of $\sigma$.

Therefore, the two considered channels through which uncertainty and social tension caused by inequality affect economic growth lead to similar results. When inequality in the society is sufficiently high, the public property regime may lead to a higher long-run rate of growth than the private property regime.

# 7 Income inequality and capital taxation

Up to this point our analysis was purely positive, and our concern was in comparing two different institutional frameworks, private and public ownership. As we have noticed above, it is difficult to make normative judgements and discuss economic policy within models where agents are heterogeneous in their time preferences. The very existence of a social welfare function satisfying certain natural requirements is questionable (see Zuber, 2011), and, in particular, it is unclear, what discount factor a social planner should have.

One might conjecture that a social planner can achieve her goals by implementing an appropriate economic policy in the form of taxes (e.g., introducing capital income or resource income tax). Following the logic of our models, it is natural to assume that the tax rate is also chosen by majority voting. This seems to be a difficult task[16] which may be a fruitful topic for further study, as we have no developed theory of voting for this case.

Having said that, suppose that a social planner seeks to maximize the long-run growth rate of the economy, and simultaneously tries to reach tolerable income inequality. Then it is reasonable to consider the question of whether differences in growth rate and inequality between private and public ownership could be undone by capital income taxation.

Our initial model with private ownership of natural resources is easily modified to include capital taxation (see Appendix C for details). Suppose that capital income is taxed at some fixed rate $\theta$, and tax revenues are lump-sum distributed to the agents. Then *in the private property regime with capital tax the long-run growth rate and the long-run extraction rate are exactly the same as in the private property regime without capital tax:*

$$1 + \gamma^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}},$$
$$\varepsilon^* = 1 - \beta_1,$$

(see Proposition C.1 in Appendix C). The introduction of a capital income tax does not affect the long-run growth rate, and, at the same time, increases the share of wages with transfers in the total income (from $\alpha_2$ to $\alpha_2 + \theta\alpha_1$), thus decreasing inequality.[17]

Therefore, if a social planner seeks to maximize the long-run growth rate and explicitly takes into account that inequality adversely affects the growth rate, she may effectively implement a tax on capital income. Private ownership with capital tax is equivalent to private ownership without tax in terms of growth rates, while an appropriately chosen capital tax ($\theta = \alpha_3/\alpha_1$) generates the same income inequality as public ownership. Moreover, it is in principle possible to achieve less income inequality and obtain a higher long-run growth rate in the private property regime with capital taxation than in the public property regime.

# 8 Conclusion

In this paper, we consider two Ramsey-type models of economic growth with exhaustible natural resources and agents heterogeneous in their time preferences. The important difference between the two models lies in the property regime over natural resources. The first model assumes that the resource stock is privately owned. The second model assumes that the resource stock is controlled by a government for the common benefit.

In the private property regime, the resource income belongs to the owners of natural resources. The extraction rate is determined by the market forces of supply and demand. Eventually only the most patient agents obtain income from both capital and resource stocks. We show that every competitive equilibrium in this model converges to

[16] It should be noted that the impact of capital taxation on inequality depends on the capital distribution among the most patient agents (cf. Alesina and Rodrik, 1994), while the distribution of capital among the most patient agents on a balanced-growth path in our models is in principle indeterminate.

[17] Recall that in a balanced-growth equilibrium in the private property regime impatient agents consume only their wages. In a balanced-growth equilibrium with capital taxation, each impatient agent receives her wage, $w_t = \alpha_2 y_t$, and a lump-sum transfer payment, $\theta\alpha_1 y_t$. In both cases the capital stock belongs only to the most patient agents.

a balanced-growth equilibrium. In the long run the discount factor of the most patient agents determines the long-run extraction rate and the growth rate.

In the public property regime, the resource income is equally distributed among all agents, who choose the resource extraction rate by majority voting. We define an intertemporal voting equilibrium and establish its convergence to a balanced-growth voting equilibrium. It turns out that the preferences of the agent with the median discount factor determine all voting decisions. In particular, the long-run extraction rate and the rate of growth are determined by the median discount factor.

When comparing the long-run effects of the two property regimes in terms of economic growth, we have to distinguish between two cases. The first is the case in which the most patient agents constitute the majority of the population. Here the rates of balanced growth under the two regimes are equal. The second is the case in which the most patient agents do not constitute the majority of the population. It follows that the equilibrium extraction rate in the private property regime is lower than that in the public property regime. Correspondingly, the growth rate is higher in the private property regime. The intuition behind this result is as follows. More patient agents prefer to extract less amount of resource today in order to provide a higher standard of consumption in the future. Therefore, the private property regime, in which the extraction rate in the long run is determined by the discount factor of the most patient agents, sustains a higher growth rate.

The conclusion that the private property regime is better for economic growth than the public property regime is not necessarily true if we take into account the detrimental impact of inequality on economic development. In the case of private ownership, only the most patient agents obtain the resource income in the long run, while in the case of public ownership, the resource income is equally distributed among all agents. Hence the private property regime, all other things being equal, results in higher income inequality than the public property regime.

We explore two different channels through which uncertainty and social tension caused by inequality affect economic growth. The first channel assumes that the discount factors of agents are formed endogenously. When inequality is high, agents' confidence about the future reduces, their discount factors decrease, and thus inequality explicitly affects the long-run extraction rates. The second channel assumes that a certain share of output, depending on inequality, is unproductively wasted. This wasted share may be thought of as military expenditures, or social spending to pacify the population. In both cases if inequality is sufficiently high, then the public property regime will likely result in a higher long-run rate of growth.

These results suggest that in the absence of redistributive policy, there is no unambiguous answer to the question, which of the two property regimes does lead to a higher long-run rate of economics growth. In societies with moderate inequality, private ownership of natural resources is likely to provide a higher long-run growth rate than public ownership. In societies with high inequality, the public property regime may result in a higher long-run growth rate compared to the private property regime.

It turns out, however, that if a social planner seeks to maximize the growth rate of the economy taking into account the detrimental effect of inequality on the economy, then she may effectively implement a capital income tax and lump-sum transfers. The private property regime with such a redistributive policy will lead to less income inequality and a higher long-run growth rate compared with the public property regime.

# A Appendix 1. Private property regime

## A.1 Competitive equilibrium

Suppose we are given an initial state of the economy, $\mathcal{I}_0 = \{(\hat{k}_0^j)_{j=1}^L, (\hat{R}_{-1}^j)_{j=1}^L\}$, where $(\hat{k}_0^j)_{j=1}^L$ and $(\hat{R}_{-1}^j)_{j=1}^L$ are initial distributions of physical capital and natural resources among agents.

We assume that $\mathcal{I}_0$ is a non-degenerate initial state,[18] i.e.,

$$\hat{k}_0^j \geq 0, \quad \hat{R}_{-1}^j \geq 0, \quad j = 1, \ldots, L,$$
$$\frac{1}{L}\sum_{j=1}^{L} \hat{k}_0^j > 0, \qquad \sum_{j=1}^{L} \hat{R}_{-1}^j > 0.$$

**Definition A.1.** *A competitive equilibrium starting from the initial state $\mathcal{I}_0$ is a sequence*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\ldots}$$

*such that*

1. *For each $j = 1, \ldots, L$, the sequence $\{c_t^{j*}, s_t^{j*}\}_{t=0}^{\infty}$ is a solution to the following utility maximization problem:*

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j, \\ \text{s. t.} \quad & c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t, \quad t = 0, 1, \ldots, \\ & s_t^j \geq 0, \quad t = 0, 1, \ldots \end{aligned} \tag{A.1}$$

   *at $r_t = r_t^*$, $w_t = w_t^*$, and $s_{-1}^j = \frac{q_0^*}{1+r_0^*}\hat{R}_{-1}^j + \hat{k}_0^j$;*

2. *Capital is paid its marginal product:*

$$1 + r_t^* = \alpha_1 A_t (k_t^*)^{\alpha_1 - 1}(e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots,$$

   *where $k_0^* = \frac{1}{L}\sum_{j=1}^{L} \hat{k}_0^j$;*

3. *Labor is paid its marginal product:*

$$w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots;$$

4. *The price of natural resources is equal to the marginal product:*

$$q_t^* = \alpha_3 A_t (k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3 - 1}, \quad t = 0, 1, \ldots;$$

5. *The Hotelling rule holds:*

$$q_{t+1}^* = (1 + r_{t+1}^*) q_t^*, \quad t = 0, 1, \ldots;$$

[18] We impose the non-negativity constraints on initial distributions of physical capital and natural resources only for technical convenience. The individual holdings of capital and resources are indeterminate on the equilibrium path, they may be positive as well as negative. However, they do not appear in the definition of equilibrium; it is irrelevant, in which proportion agents invest their savings in different assets. Since the two assets are perfect substitutes, only agents' savings are important in an equilibrium.

6. *The natural balance of exhaustible resources is fulfilled:*[19]

$$R_t^* = R_{t-1}^* - Le_t^*, \quad t = 0, 1, \dots,$$

*where* $R_{-1}^* = \sum_{j=1}^{L} \hat{R}_{-1}^j$*;*

7. *Total agents' savings are equal to the investment into physical capital and natural resources:*

$$\sum_{j=1}^{L} s_t^{j*} = \frac{q_{t+1}^*}{1 + r_{t+1}^*} R_t^* + Lk_{t+1}^*, \quad t = 0, 1, \dots.$$

The existence of a competitive equilibrium is established in the following theorem.[20]

**Theorem A.1.** *For any initial state* $\mathcal{I}_0$ *there exists a competitive equilibrium starting from* $\mathcal{I}_0$*.*

*Proof.* We divide the proof of the theorem into two steps. First we show the existence of a competitive equilibrium in the finite horizon model. We prove that for any $T > 0$ there exists a finite $T$-period competitive equilibrium. Second, we construct a candidate for a competitive equilibrium in the infinite horizon model by applying some kind of diagonalization procedure to the sequence of finite $T$-period equilibrium paths, and then prove that this candidate is indeed a competitive equilibrium in the infinite horizon model.

**Step I. Competitive equilibrium in the finite horizon model**.

Let us define a finite $T$-period competitive equilibrium along the lines of the above definition.

**Definition A.2.** *A finite* $T$*-period competitive equilibrium starting from the initial state* $\mathcal{I}_0$ *is a sequence*

$$\mathcal{E}_T^* = \left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\dots,T}$$

*such that*

1. *For each* $j = 1, \dots, L$*, the sequence* $\{c_t^{j*}, s_t^{j*}\}_{t=0}^T$ *is a solution to the following utility maximization problem:*

$$\begin{gathered} \max \sum_{t=0}^{T} \beta_j^t \ln c_t^j, \\ \text{s. t.} \quad c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t, \quad t = 0, 1, \dots, T, \\ s_t^j \geq 0, \quad t = 0, 1, \dots, T, \end{gathered} \tag{A.2}$$

*at* $r_t = r_t^*$*,* $w_t = w_t^*$*, and*

$$s_{-1}^j = \frac{q_0^*}{1 + r_0^*} \hat{R}_{-1}^j + \hat{k}_0^j;$$

[19]Note that extraction rates are determined here in an equilibrium by the market forces of supply and demand.

[20]The proof of Theorem A.1 is based on the ideas presented in Becker et al. (2015). The existence of equilibrium in the considered Ramsey-type model can be also proved along the lines of Becker et al. (1991).

2. *Capital is paid its marginal product:*

$$1 + r_t^* = \alpha_1 A_t (k_t^*)^{\alpha_1 - 1} (e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots, T,$$

*where* $k_0^* = \frac{1}{L} \sum_{j=1}^{L} \hat{k}_0^j$*;*

3. *Labor is paid its marginal product:*

$$w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots, T;$$

4. *The price of natural resources is equal to the marginal product:*

$$q_t^* = \alpha_3 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3 - 1}, \quad t = 0, 1, \ldots, T;$$

5. *The Hotelling rule holds:*

$$q_{t+1}^* = (1 + r_{t+1}^*) q_t^*, \quad t = 0, 1, \ldots, T - 1;$$

6. *The natural balance of exhaustible resources is fulfilled:*

$$R_t^* = R_{t-1}^* - L e_t^*, \quad t = 0, 1, \ldots T,$$

*where* $R_{-1}^* = \sum_{j=1}^{L} \hat{R}_{-1}^j$*, and* $R_t \geq 0$*,* $e_t \geq 0$*, for all* $t = 0, 1, \ldots, T$*;*

7. *Total agents' savings are equal to the investment into physical capital and natural resources:*

$$\sum_{j=1}^{L} s_t^{j*} = \frac{q_{t+1}^*}{1 + r_{t+1}^*} R_t^* + L k_{t+1}^*, \quad t = 0, 1, \ldots, T - 1.$$

It is clear that $\{c_t^{j*}, s_t^{j*}\}_{t=0}^{T}$ is a solution to (A.2) if and only if it satisfies the following conditions:

$$c_t^{j*} + s_t^{j*} = (1 + r_t) s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \ldots, T, \tag{A.3}$$

$$c_{t+1}^{j*} \geq \beta_j (1 + r_{t+1}^*) c_t^{j*} \text{ (= if } s_t^{j*} > 0), \quad t = 0, 1, \ldots, T - 1, \tag{A.4}$$

$$s_T^{j*} = 0.$$

Let $\varepsilon_t^*$ be the extraction rate at time $t$:

$$\varepsilon_t^* = \frac{L e_t^*}{R_{t-1}^*}.$$

The existence of a competitive equilibrium in the finite horizon model is shown via the following steps. First we present some preliminary definitions and results that will be useful in what follows. Second, we reduce our finite horizon model to a game, and show that there exists a Nash equilibrium in this game. Third, we prove that a Nash equilibrium in the game that represents our model determines a competitive equilibrium in the finite horizon model.

**Step I.1. Preliminaries**

We use the notation

$$e = e(\varepsilon, R) := \frac{\varepsilon R}{L},$$

for the volume of extraction as depending on the extraction rate $\varepsilon$ and resource stock $R$, and the notation

$$\begin{aligned} f(k,e,A) &:= Ak^{\alpha_1}e^{\alpha_3}, \\ 1 + r(k,e,A) &:= \alpha_1 Ak^{\alpha_1-1}e^{\alpha_3}, \\ w(k,e,A) &:= \alpha_2 Ak^{\alpha_1}e^{\alpha_3}, \\ q(k,e,A) &:= \alpha_3 Ak^{\alpha_1}e^{\alpha_3-1}, \end{aligned}$$

for the output (production function), interest rate, wage rate, and resource price as depending on the capital stock $k$, the volume of extraction $e$ and total factor productivity $A$. It is clear that for all $k, e, A$,

$$(1 + r(k,e,A))k + w(k,e,A) + q(k,e,A)e = f(k,e,A), \tag{A.5}$$

$$f(k,e,A) = \frac{(1 + r(k,e,A))k}{\alpha_1} = \frac{w(k,e,A)}{\alpha_2} = \frac{q(k,e,A)e}{\alpha_3}. \tag{A.6}$$

In particular,

$$\frac{q(k,e,A)}{1 + r(k,e,A)} = \frac{\alpha_3}{\alpha_1}\frac{k}{e} = \frac{\alpha_3}{\alpha_1}\frac{k}{\varepsilon}\frac{L}{R}. \tag{A.7}$$

Denote

$$\begin{aligned} \tilde{\varepsilon} &:= \frac{\alpha_3(1-\beta_1)}{1 - (\alpha_1+\alpha_2)(1-\beta_1)}, \\ \bar{\varepsilon} &:= \frac{1}{1 + \beta_L(1-\beta_1)^2}. \end{aligned}$$

Let

$$\begin{aligned} \tilde{R}_{-1} &:= \hat{R}_{-1}, \\ \tilde{R}_t &:= (1-\bar{\varepsilon})\tilde{R}_{t-1}, \quad t = 0, 1, \ldots, \\ \tilde{e}_t &:= \frac{\bar{\varepsilon}\tilde{R}_{t-1}}{L}, \quad t = 0, 1, \ldots, \\ \bar{e} &:= \frac{\hat{R}_{-1}}{L}. \end{aligned} \tag{A.8}$$

Denote

$$1 + \bar{g} = \max\left\{(1+\lambda)^{\frac{1}{1-\alpha_1}}, (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}}\right\},$$

where $\lambda$ is the growth rate of the total factor productivity:

$$A_t = (1+\lambda)A_{t-1} = (1+\lambda)^t A_0. \tag{A.9}$$

Let also

$$1 + \tilde{g} = \min\left\{(1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \frac{A_0(\tilde{e}_0)^{\alpha_3}}{(\hat{k}_0)^{1-\alpha_1}}\right\},$$

and

$$1 + \underline{g} = \beta_L\alpha_1(1+\tilde{g}).$$

It is clear that

$$
\begin{aligned}
1+\bar{g} &\geq (1+\lambda)^{\frac{1}{1-\alpha_1}}, \\
1+\bar{g} &\geq (1+\lambda)^{\frac{1}{1-\alpha_1}} \left(\frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}},
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
1+\tilde{g} &\leq \frac{A_0(\tilde{e}_0)^{\alpha_3}}{(\hat{k}_0)^{1-\alpha_1}}, \\
1+\tilde{g} &\leq (1+\lambda)^{\frac{1}{1-\alpha_1}} \left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}},
\end{aligned} \tag{A.11}
$$

and

$$1+\bar{g} > 1+\tilde{g} > 1+\underline{g}. \tag{A.12}$$

Suppose that $\bar{\kappa} > 0$ is given by

$$(1+\tilde{g})\bar{\kappa} = (\bar{\kappa})^{\alpha_1}, \tag{A.13}$$

Let the sequence $\{\bar{k}_t\}$ be given by

$$\bar{k}_{t+1} = (1+\bar{g})\bar{k}_t,$$

where

$$\bar{k}_0 = \bar{\kappa}(A_0 \bar{e}^{\alpha_3})^{\frac{1}{1-\alpha_1}}.$$

We show that

$$\bar{\kappa}(A_t(\bar{e})^{\alpha_3})^{\frac{1}{1-\alpha_1}} \leq \bar{k}_t. \tag{A.14}$$

Due to (A.9), (A.10), and the choice of $\bar{k}_0$, we get

$$\bar{\kappa}(A_t(\bar{e})^{\alpha_3})^{\frac{1}{1-\alpha_1}} = \bar{\kappa}\left((1+\lambda)^t A_0(\bar{e})^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} \leq \bar{\kappa}\left(A_0(\bar{e})^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} (1+\bar{g})^t = (1+\bar{g})^t \bar{k}_0 = \bar{k}_t.$$

Moreover,

$$f(\bar{k}_t, \bar{e}, A_t) < \bar{k}_{t+1}, \quad t = 0, 1, \ldots. \tag{A.15}$$

Indeed, using (A.14), (A.13) and (A.12), we get

$$
\begin{aligned}
f(\bar{k}_t, \bar{e}, A_t) - \bar{k}_{t+1} &= (\bar{k}_t)^{\alpha_1} A_t(\bar{e})^{\alpha_3} - (1+\bar{g})\bar{k}_t \\
&= \bar{k}_t \left(\frac{A_t(\bar{e})^{\alpha_3}}{(\bar{k}_t)^{1-\alpha_1}} - (1+\bar{g})\right) \leq \bar{k}_t \left(\frac{A_t(\bar{e})^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t(\bar{e})^{\alpha_3}} - (1+\bar{g})\right) \\
&= \bar{k}_t \left(\frac{\bar{\kappa}^{\alpha_1}}{\bar{\kappa}} - (1+\bar{g})\right) = \bar{k}_t\left((1+\tilde{g}) - (1+\bar{g})\right) < 0.
\end{aligned}
$$

Denote

$$\bar{c}_t := L\bar{k}_{t+1}. \tag{A.16}$$

Clearly,

$$\bar{c}_{t+1} = (1+\bar{g})\bar{c}_t. \tag{A.17}$$

Let the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ be defined recursively as follows. We take $\tilde{k}_0$ such that $0 < \tilde{k}_0 < \hat{k}_0$. Suppose we are given $\tilde{k}_t > 0$. Consider the following equation in $k$:

$$\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right) k + \frac{\bar{c}_{t+1}}{\beta_L(1 + r(k, \tilde{e}_{t+1}, A_{t+1}))} = f(\tilde{k}_t, \tilde{e}_t, A_t).$$

The left-hand side of the above equation is increasing in $k$, and equals to 0 when $k = 0$. Thus there is a unique positive solution to this equation. We take $\tilde{k}_{t+1} > 0$ as this solution. Clearly, the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ satisfies the following equation:

$$\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right)\tilde{k}_{t+1} + \frac{\bar{c}_{t+1}}{\beta_L(1 + r(\tilde{k}_{t+1}, \tilde{e}_{t+1}, A_{t+1}))} = f(\tilde{k}_t, \tilde{e}_t, A_t), \quad t = 0, 1, \ldots. \tag{A.18}$$

**Step I.2. A game.**

We reduce our finite horizon model to a game $\Gamma = (X_k, G_k)_{k \in I}$. Recall that to specify a game, we need to describe a set of players, $I$, and for each player $k \in I$ define the strategy set $X_k$ and the loss function

$$G_k : \prod_{i \in I} X_i \to \mathbb{R}.$$

Elements of $\prod_{i \in I} X_i$ are called multistrategies. The equilibrium of the game $\Gamma$ is defined as follows.

**Definition.** *A multistrategy $\left(x_1^*, \ldots, x_{|I|}^*\right)$ is called a Nash equilibrium of the game $\Gamma$ if for each $k \in I$, $x_k^*$ is a solution to*

$$\begin{gathered}\min_{x_k} G_k\left(x_1^*, \ldots, x_{k-1}^*, x_k, x_{k+1}^*, \ldots, x_{|I|}^*\right), \\ \text{s. t.} \quad x_k \in X_k.\end{gathered}$$

The sufficient conditions for the existence of a Nash equilibrium of this game are well-known (see, e.g., Ichiishi, 2014): for each $k \in I$ the set $X_k$ is a convex and compact subset of a finite dimensional space, and the function $G_k(x_1, \ldots, x_k, \ldots, x_{|I|})$ is continuous in all variables and quasi-convex in $x_k$.

Consider the following game $\Gamma_T$ with $3T + (2T + 1)L$ players where

1. for each agent $j = 1, \ldots, L$,

   (a) $T$ players determine $s_t^j$, $t = 0, 1, \ldots, T - 1$, by solving

$$\begin{gathered}\min_s s\left(c_{t+1}^j - \beta_j(1 + r(k_{t+1}, e(\varepsilon_{t+1}, R_t), A_{t+1}))c_t^j\right), \\ \text{s. t.} \quad 0 \le s \le \tfrac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}.\end{gathered} \tag{A.19}$$

   (b) $T + 1$ players determine $c_t^j$, $t = 0, 1, \ldots, T$, by solving

$$\begin{gathered}\min_c \left|c - \left((1 + r(k_t, e(\varepsilon_t, R_{t-1}), A_t))s_{t-1}^j + w(k_t, e(\varepsilon_t, R_{t-1}), A_t) - s_t^j\right)\right|, \\ \text{s. t.} \quad 0 \le c \le \tfrac{\bar{c}_t}{\tilde{\varepsilon}},\end{gathered} \tag{A.20}$$

   where $s_T^j = 0$ and $R_{-1} = \hat{R}_{-1}$.

2. $T$ players determine $k_t$, $t = 1, 2, \ldots, T$, by solving

$$\begin{gathered}\min_k \left|k - \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^j \frac{\alpha_1\varepsilon_t}{\alpha_3 + \alpha_1\varepsilon_t}\right|, \\ \text{s. t.} \quad \tilde{k}_t \le k \le \tfrac{\bar{k}_{t+1}}{\tilde{\varepsilon}}\end{gathered} \tag{A.21}$$

3. $T$ players determine $R_t$, $t = 0, 1, \ldots, T-1$, by solving

$$\min_R |R - (1-\varepsilon_t) R_{t-1}|, \quad \text{s. t.} \quad \tilde{R}_t \le R \le \hat{R}_{-1}, \tag{A.22}$$

where $R_{-1} = \hat{R}_{-1}$.

4. $T$ players determine $\varepsilon_t$, $t = 0, 1, \ldots, T-1$, by solving

$$\min_\varepsilon \left| \frac{(e(\varepsilon, R_{t-1}))^{1-\alpha_3}}{A_t k_t^{\alpha_1}} - \alpha_1 \frac{e(\varepsilon_{t+1}, R_t)}{k_{t+1}} \right|, \quad \text{s. t.} \quad \tilde{\varepsilon} \le \varepsilon \le \bar{\varepsilon}, \tag{A.23}$$

where $R_{-1} = \hat{R}_{-1}$, and $\varepsilon_T = 1$.

**Lemma.** *There exists a Nash equilibrium in the game $\Gamma_T$ with $3T + (2T+1)L$ players having the strategy sets and loss functions described by (A.19)–(A.23).*

*Proof.* All the strategy sets are closed intervals, and for each player the loss function is continuous in all variables and quasi-convex in the player's own strategy variable. □

**Step I.3. Nash equilibrium vs. competitive equilibrium**.

The following lemma maintains that the Nash equilibrium of the game $\Gamma_T$ determines a finite $T$-period competitive $\mathbb{E}_0$-equilibrium.

**Lemma A.1.** *Let*

$$\left\{ (c_t^{j*})_{t=0,1,\ldots,T}^{j=1,\ldots,L}, (s_t^{j*})_{t=0,1,\ldots,T-1}^{j=1,\ldots,L}, (k_t^*)_{t=1,2,\ldots,T}, (R_t^*)_{t=0,1,\ldots,T-1}, (\varepsilon_t^*)_{t=0,1,\ldots,T-1} \right\}$$

*be a Nash equilibrium of the game $\Gamma_T$. Let $k_0^* = \hat{k}_0$, $R_{-1}^* = \hat{R}_{-1}$, $R_T^* = 0$, $\varepsilon_T^* = 1$, and $s_T^{j*} = 0$ for all $j$. Let also*

$$\begin{aligned} e_t^* &= e(\varepsilon_t^*, R_{t-1}^*), \quad t = 0, 1, \ldots, T, \\ 1 + r_t^* &= 1 + r(k_t^*, e_t^*, A_t), \quad t = 0, 1, \ldots, T, \\ w_t^* &= w(k_t^*, e_t^*, A_t), \quad t = 0, 1, \ldots, T, \\ q_t^* &= q(k_t^*, e_t^*, A_t), \quad t = 0, 1, \ldots, T. \end{aligned}$$

*Then*

$$\left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\ldots,T}$$

*is a finite $T$-period competitive equilibrium starting from the initial state $\mathcal{I}_0$.*

*Proof.* First, observe that

- if $c_{t+1}^j > \beta_j(1 + r(k_{t+1}, e(\varepsilon_{t+1}, R_t), A_{t+1}))c_t^j$, then the only solution to the problem (A.19) is $s = 0$;
- if $c_{t+1}^j = \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then any $s$ from the interval $[0, \frac{L\bar{k}_{t+1}}{\hat{\varepsilon}}]$ is a solution to the problem (A.19);

- if $c_{t+1}^j < \beta_j(1+r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then the only solution to the problem (A.19) is $s = \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}$.

Second, notice that minimization problems (A.20)–(A.22) are of the form

$$\min_x |x - \hat{x}|,$$
$$\text{s. t.} \quad a_1 \le x \le a_2.$$

The unique solution to this problem, $x^*$, is given by

$$x^* = \begin{cases} a_1, & \text{if} \quad \hat{x} < a_1; \\ a_2, & \text{if} \quad \hat{x} > a_2; \\ \hat{x}, & \text{if} \quad a_1 \le \hat{x} \le a_2. \end{cases}$$

**Remark A.1.** *When $\hat{x} \ge a_1$, we have $\hat{x} \ge x^*$.*

**Remark A.2.** *When $\hat{x} \le a_2$, we have $\hat{x} \le x^*$.*

Third, note that minimization problems (A.23) are of the form

$$\min_x |g(x) - \hat{x}|,$$
$$\text{s. t.} \quad a_1 \le x \le a_2,$$

where the function $g(x)$ is increasing in $x$. The unique solution to this problem, $x^*$, is given by

$$x^* = \begin{cases} a_1, & \text{if} \quad \hat{x} < g(a_1); \\ a_2, & \text{if} \quad \hat{x} > g(a_2); \\ g^{-1}(\hat{x}), & \text{if} \quad g(a_2) \le \hat{x} \le g(a_1). \end{cases}$$

Let

$$\left\{ (c_t^{j*})_{t=0,1,\dots,T}^{j=1,\dots,L}, (s_t^{j*})_{t=-1,0,\dots,T-1}^{j=1,\dots,L}, (k_t^*)_{t=1,2,\dots,T}, (R_t^*)_{t=0,1,\dots,T-1}, (\varepsilon_t^*)_{t=0,1,\dots,T-1} \right\}$$

be a Nash equilibrium of the game $\Gamma_T$. Denote $s_{-1}^{j*} = \frac{q_0^*}{1+r_0^*}\hat{R}_{-1}^j + \hat{k}_0^j$. Notice that that for all $t = 0, 1, \dots, T$, $k_t^* \ge \tilde{k}_t > 0$, and

$$0 < \tilde{\varepsilon} \le \varepsilon_t^* \le \bar{\varepsilon} < 1. \tag{A.24}$$

Therefore, for all $t = 0, 1, \dots, T$, $e_t^* > 0$, $w_t^* > 0$, $0 < 1 + r_t^* < \infty$, and $0 < q_t^* < \infty$.

The proof of Lemma A.1 is divided into several claims.

**Claim A.1.** *For each $j = 1, \dots, L$,*

$$0 < c_t^{j*} \le (1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*}, \quad t = 0, 1, \dots, T, \tag{A.25}$$

*and hence*

$$c_t^{j*} + s_t^{j*} \le (1 + r_t^*)s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \dots, T, \tag{A.26}$$

*Proof.* Assume the converse. Then, by the structure of the problem (A.20), there are $j$ and $0 \le \tau \le T$ such that

$$0 < c_t^{j*} \le (1 + r_t^*) s_{t-1}^{j*} + w_t^* - s_t^{j*}, \quad t = 0, 1, \ldots, \tau - 1,$$

and

$$0 = c_\tau^{j*} \ge (1 + r_\tau^*) s_{\tau-1}^{j*} + w_\tau^* - s_\tau^{j*}. \tag{A.27}$$

Consider two cases. First, let $\tau \le T - 1$. By (A.27),

$$s_\tau^{j*} \ge (1 + r_\tau^*) s_{\tau-1}^{j*} + w_\tau^* \ge w_\tau^* > 0.$$

Then, by the structure of the problem (A.19),

$$c_{\tau+1}^{j*} \le \beta_j (1 + r_{\tau+1}^*) c_\tau^{j*} = 0,$$

because otherwise we would have $s_\tau^{j*} = 0$. Therefore, using Remark A.1, we conclude that

$$0 = c_{\tau+1}^{j*} \ge (1 + r_{\tau+1}^*) s_\tau^{j*} + w_{\tau+1}^* - s_{\tau+1}^{j*}.$$

Repeating the argument, and using the structure of the problem (A.19), we obtain for $t = \tau, \tau + 1, \ldots, T - 1$,

$$s_t^{j*} > 0,$$
$$c_{t+1}^{j*} = 0.$$

However, $c_T^{j*} = 0$ is impossible, because $s_T^{j*} = 0$, and by the structure of the problem (A.20) we have

$$0 = c_T^{j*} = c_T^{j*} + s_T^{j*} \ge (1 + r_T^*) s_{T-1}^{j*} + w_T^* > 0,$$

a contradiction.

Second, let $\tau = T$. Since $c_{T-1}^{j*} > 0$, and $c_T^{j*} = 0$, we have

$$c_T^{j*} - \beta_j (1 + r_T^*) c_{T-1}^{j*} = -\beta_j (1 + r_T^*) c_{T-1}^{j*} < 0,$$

and, by the structure of the problem (A.19), $s_{T-1}^{j*} = \frac{L\bar{k}_T}{\tilde{\varepsilon}}$. Using the fact that $s_T^{j*} = 0$, by the structure of the problem (A.20) we have

$$0 = c_T^{j*} + s_T^{j*} \ge (1 + r_T^*) s_{T-1}^{j*} + w_T^* > 0,$$

a contradiction. □

**Claim A.2.** *For each* $j = 1, \ldots, L$,

$$(1 + r_t^*) s_{t-1}^{j*} + w_t^* < \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}, \quad t = 0, 1, \ldots, T, \tag{A.28}$$

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j*} \frac{\alpha_1 \varepsilon_t^*}{\alpha_3 + \alpha_1 \varepsilon_t^*} < \frac{\bar{k}_t}{\tilde{\varepsilon}}, \quad t = 0, 1, \ldots, T, \tag{A.29}$$

*and*

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j*} \le k_t^* + \frac{q_t^*}{1 + r_t^*} \frac{R_{t-1}^*}{L}, \quad t = 0, 1, \ldots, T. \tag{A.30}$$

*Proof.* Using the definition of $s_{-1}^{j*}$, we obtain

$$\frac{1}{L}\sum_{j=1}^{L} s_{-1}^{j*} = \frac{q_0^*}{1+r_0^*}\frac{1}{L}\sum_{j=1}^{L}\hat{R}_{-1}^j + \frac{1}{L}\sum_{j=1}^{L}\hat{k}_0^j = \frac{q_0^*}{1+r_0^*}\frac{R_{-1}^*}{L} + k_0^*. \tag{A.31}$$

Therefore, by (A.31), (A.24), and (A.15)

$$\begin{aligned}
(1+r_0^*)s_{-1}^{j*} + w_0^* &\le \sum_{j=1}^{L}\left((1+r_0^*)s_{-1}^{j*} + w_0^*\right)\\
&= (1+r_0^*)\frac{q_0^*}{1+r_0^*}R_{-1}^* + L(1+r_0^*)k_0^* + Lw_0^* = L(1+r_0^*)k_0^* + Lw_0^* + q_0^*R_{-1}^*\\
&= \alpha_1 Lf(k_0^*, e_0^*, A_0) + \alpha_2 Lf(k_0^*, e_0^*, A_0) + \alpha_3 f(k_0^*, e_0^*, A_0)\frac{R_{-1}^*}{e_0^*}\\
&= (\alpha_1+\alpha_2)Lf(k_0^*, e_0^*, A_0) + \alpha_3 Lf(k_0^*, e_0^*, A_0)\frac{1}{\varepsilon_0^*}\\
&= Lf(k_0^*, e_0^*, A_0) + \alpha_3 Lf(k_0^*, e_0^*, A_0)\frac{1-\varepsilon_0^*}{\varepsilon_0^*}\\
&\le Lf(\bar{k}_0, \bar{e}, A_0) + \alpha_3 Lf(\bar{k}_0, \bar{e}, A_0)\frac{1-\tilde{\varepsilon}}{\tilde{\varepsilon}} < L\bar{k}_1 + L\bar{k}_1\frac{1-\tilde{\varepsilon}}{\tilde{\varepsilon}} = \frac{L\bar{k}_1}{\tilde{\varepsilon}},
\end{aligned}$$

which proves (A.28) for $t=0$.

Moreover, by (A.26),

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j*} \le \frac{1}{L}\sum_{j=1}^{L}\left(c_0^{j*} + s_0^{j*}\right) < \frac{1}{L}\sum_{j=1}^{L}\left((1+r_0^*)s_{-1}^{j*} + w_0^*\right) \le \frac{\bar{k}_1}{\tilde{\varepsilon}}.$$

Since $\varepsilon_1^* > 0$,

$$\frac{\alpha_1\varepsilon_1^*}{\alpha_3 + \alpha_1\varepsilon_1^*} < 1,$$

and therefore

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j*}\frac{\alpha_1\varepsilon_1^*}{\alpha_3 + \alpha_1\varepsilon_1^*} < \frac{\bar{k}_1}{\tilde{\varepsilon}},$$

which proves (A.29) for $t=0$.

It follows from Remark A.2 that

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j*}\frac{\alpha_1\varepsilon_1^*}{\alpha_3 + \alpha_1\varepsilon_1^*} \le k_1^*.$$

Using (A.7),

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j*} \le k_1^* + \frac{\alpha_3}{\alpha_1}\frac{k_1^*}{\varepsilon_1^*} = k_1^* + \frac{q_1^*}{1+r_1^*}\frac{R_0^*}{L}.$$

Thus, (A.30) holds for $t=0$.

To obtain inequalities (A.28)–(A.30) for all $t \le T$, it is sufficient to repeat the argument. ∎

**Claim A.3.** *For each agent $j = 1, \dots, L$,*

$$c_t^{j*} + s_t^{j*} = (1 + r_t^*)s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \dots, T. \tag{A.32}$$

*Proof.* Using (A.28), (A.16) and the fact that $s_t^{j*} \geq 0$ for all $t = 0, 1, \dots, T$, we get

$$(1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*} < \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}} = \frac{\bar{c}_t}{\tilde{\varepsilon}}.$$

Therefore, by the structure of the problem (A.20), for each $j = 1, \dots, L$,

$$c_t^{j*} \geq (1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*}, \quad t = 0, 1, \dots, T.$$

Combining this inequality with (A.25), we obtain (A.32). □

**Claim A.4.** *For each $j = 1, \dots, L$,*

$$c_{t+1}^{j*} \geq \beta_j(1 + r_{t+1}^*)c_t^{j*} \ (= \textit{if } s_t^{j*} > 0), \quad t = 0, 1, \dots, T. \tag{A.33}$$

*Proof.* Assume that for some $j$ and $t < T$,

$$c_{t+1}^{j*} < \beta_j(1 + r_{t+1}^*)c_t^{j*}.$$

Then, by the structure of the problem (A.19), $s_t^{j*} = \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}$. By (A.28),

$$(1 + r_t^*)s_{t-1}^{j*} + w_t^* < \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}} = s_t^{j*},$$

and hence

$$(1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*} < 0,$$

which contradicts (A.25). Thus we have proved that

$$c_{t+1}^{j*} \geq \beta_j(1 + r_{t+1}^*)c_t^{j*}.$$

Moreover, if

$$c_{t+1}^{j*} > \beta_j(1 + r_{t+1}^*)c_t^{j*},$$

then by the structure of the problem (A.19), $s_t^{j*} = 0$. □

**Claim A.5.** *For $t = 0, 1, \dots, T - 1$,*

$$R_t^* = (1 - \varepsilon_t^*)R_{t-1}^*. \tag{A.34}$$

*Proof.* Due to the (A.8), (A.24) and the bounds for $R$ in (A.22), for all $t = 0, 1, \dots, T-1$, we have

$$\tilde{R}_t = (1 - \bar{\varepsilon})\tilde{R}_{t-1} \leq (1 - \varepsilon_t^*)R_{t-1}^* \leq (1 - \varepsilon_t^*)\hat{R}_{-1} < \hat{R}_{-1}.$$

Now (A.34) follows from the structure of the problem (A.22). □

It follows from (A.8), (A.24), the bounds for $R$ in (A.22) and the definition of $e_t^*$ that

$$\tilde{e}_t \leq e_t^* \leq \bar{e}, \quad t = 0, 1, \dots, T. \tag{A.35}$$

**Claim A.6.** *For all* $t = 0, 1, \ldots, T$,

$$k_t^* > \tilde{k}_t, \tag{A.36}$$

*and*

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} = \frac{q_t^*}{1+r_t^*}\frac{R_{t-1}^*}{L} + k_t^*. \tag{A.37}$$

*Proof.* Note that by the choice of $\tilde{k}_0$,

$$k_0^* > \tilde{k}_0,$$

and it follows from (A.31) that (A.37) holds for $t = 0$.

Assume that for some $t = 1, 2, \ldots T$,

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-2}^{j*}\frac{\alpha_1\varepsilon_{t-1}^*}{\alpha_3 + \alpha_1\varepsilon_{t-1}^*} = \left(\frac{q_{t-1}^*}{1+r_{t-1}^*}R_{t-2}^* + k_{t-1}^*\right)\frac{\alpha_1\varepsilon_{t-1}^*}{\alpha_3 + \alpha_1\varepsilon_{t-1}^*} = k_{t-1}^* > \tilde{k}_{t-1},$$

and

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}\frac{\alpha_1\varepsilon_t^*}{\alpha_3 + \alpha_1\varepsilon_t^*} \le \left(\frac{q_t^*}{1+r_t^*}R_{t-1}^* + k_t^*\right)\frac{\alpha_1\varepsilon_t^*}{\alpha_3 + \alpha_1\varepsilon_t^*} = k_t^* = \tilde{k}_t.$$

By (A.32), (A.24), (A.6) and (A.35),

$$\begin{aligned}
\frac{1}{L}\sum_{j=1}^{L}\left(c_{t-1}^{j*} + s_{t-1}^{j*}\right) &= \frac{1}{L}\sum_{j=1}^{L}\left((1+r_{t-1}^*)s_{t-2}^{j*} + w_{t-1}^*\right) \\
&= (1+r_{t-1}^*)k_{t-1}^* + q_{t-1}^*\frac{R_{t-2}^*}{L} + w_{t-1}^* = (1+r_{t-1}^*)k_{t-1}^* + w_{t-1}^* + q_{t-1}^*e_{t-1}^*\frac{1}{\varepsilon_{t-1}^*} \\
&> (1+r_{t-1}^*)k_{t-1}^* + w_{t-1}^* + q_{t-1}^*e_{t-1}^* = f(k_{t-1}^*, e_{t-1}^*, A_{t-1}) > f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}).
\end{aligned}$$

Hence, due to (A.24),

$$\begin{aligned}
\frac{1}{L}\sum_{j=1}^{L} c_{t-1}^{j*} &> f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} \\
&\ge f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t - \frac{\alpha_3}{\alpha_1}\frac{\tilde{k}_t}{\varepsilon_t^*} \ge f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right).
\end{aligned}$$

Therefore, there is $j$ such that

$$c_{t-1}^{j*} > f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right) > 0. \tag{A.38}$$

Using (A.18), the bounds for $c$ in (A.20), and taking into account that $\tilde{k}_t = k_t^*$, we get

$$\begin{aligned}
c_{t-1}^{j*} &> f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right) \\
&= \frac{\bar{c}_t}{\beta_L(1 + r(\tilde{k}_t, \tilde{e}_t, A_t))} \ge \frac{\bar{c}_t}{\beta_j(1 + r(\tilde{k}_t, e_t^*, A_t))} \ge \frac{c_t^{j*}}{\beta_j(1+r_t^*)},
\end{aligned}$$

and hence

$$c_t^{j*} < \beta_j(1+r_t^*)c_{t-1}^{j*}.$$

It follows from the structure of the problem (A.19) that

$$s_{t-1}^{j*} = \frac{L\bar{k}_t}{\tilde{\varepsilon}}.$$

By (A.32) and (A.28), we have

$$c_{t-1}^{j*} = (1+r_{t-1}^*)s_{t-2}^{j*} + w_{t-1}^* - s_{t-1}^{j*} < \frac{L\bar{k}_t}{\tilde{\varepsilon}} - \frac{L\bar{k}_t}{\tilde{\varepsilon}} = 0,$$

a contradiction of (A.38). This proves (A.36).

Now it follows from (A.29) and (A.36) that

$$\tilde{k}_t < \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}\frac{\alpha_1\varepsilon_t^*}{\alpha_3+\alpha_1\varepsilon_t^*} < \frac{\bar{k}_t}{\tilde{\varepsilon}}, \quad t = 0,1,\ldots,T.$$

By the structure of the problem (A.21),

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}\frac{\alpha_1\varepsilon_t^*}{\alpha_3+\alpha_1\varepsilon_t^*} = k_t^*, \quad t = 0,1,\ldots,T.$$

Therefore, using (A.7), for all $t = 0,1,\ldots,T$, we obtain

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} = k_t^* + \frac{\alpha_3}{\alpha_1}\frac{k_t^*}{\varepsilon_t^*} = k_t^* + \frac{q_t^*}{1+r_t^*}\frac{R_{t-1}^*}{L}.$$

□

**Claim A.7.** *For all $t = 0,1,\ldots,T$ and for all $j = 1,2,\ldots,L$,*

$$c_t^{j*} \geq (1-\beta_1)\left((1+r_t^*)s_{t-1}^{j*} + w_t^*\right). \tag{A.39}$$

*Proof.* Let us prove (A.39) for $t = 0$. It is clear that there is $0 \leq \tau \leq T$ such that $s_t > 0$ for all $t < \tau$ and $s_\tau = 0$. If $\tau = 0$, then it is sufficient to note that

$$c_0^{j*} = (1+r_0^*)s_{-1}^{j*} + w_0^* \geq (1-\beta_1)\left((1+r_0^*)s_{-1}^{j*} + w_0^*\right).$$

If $\tau > 0$, then, by (A.33),

$$c_1^{j*} = \beta_j(1+r_1^*)c_0^{j*}, \quad \ldots, \quad c_\tau^{j*} = \beta_j^\tau(1+r_1^*)\cdots(1+r_\tau^*)c_0^{j*},$$

and, by (A.32),

$$c_0^{j*} + \frac{1}{1+r_1^*}c_1^{j*} + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}c_\tau^{j*}$$
$$= (1+r_0^*)s_{-1}^{j*} + w_0^* + \frac{1}{1+r_1^*}w_1^* + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}w_\tau^*.$$

Therefore,

$$\frac{1}{1-\beta_1}c_0^{j*} > c_0^{j*} + \beta_j c_0^{j*} + \ldots + \beta_j^{\tau} c_0^{j*}$$
$$= c_0^{j*} + \frac{1}{1+r_1^*}c_1^{j*} + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}c_\tau^{j*}$$
$$= (1+r_0^*)s_{-1}^{j*} + w_0^* + \frac{1}{1+r_1^*}w_1^* + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}w_\tau^* \geq (1+r_0^*)s_{-1}^{j*} + w_0^*,$$

which proves (A.39) for $t = 0$. To prove it for $t > 0$ it is sufficient to repeat the argument.
□

**Claim A.8.**

$$k_{t+1}^* > \alpha_1\beta_L(1-\beta_1)^2 f(k_t^*, e_t^*, A_t), \quad t = 0, 1, \ldots, T-1. \tag{A.40}$$

*Proof.* Note that

$$\alpha_1 + \alpha_2 + \frac{\alpha_3}{\tilde{\varepsilon}} = \alpha_1 + \alpha_2 + \frac{1-(\alpha_1+\alpha_2)(1-\beta_1)}{1-\beta_1} = \frac{1}{1-\beta_1}. \tag{A.41}$$

Due to (A.32), (A.31), (A.5), (A.24), and (A.41), we get for $t = 0, 1, \ldots, T-1$,

$$\frac{1}{L}\sum_{j=1}^{L} c_t^{j*} \leq \left((1+r_t^*)\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} + w_t^*\right)$$
$$= \left(q_t^*\frac{R_{t-1}^*}{L} + (1+r_t^*)k_t^* + w_t^*\right) = \left(\frac{\alpha_3}{\varepsilon_t^*} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t)$$
$$\leq \left(\frac{\alpha_3}{\tilde{\varepsilon}} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) = \frac{f(k_t^*, e_t^*, A_t)}{1-\beta_1},$$

or

$$f(k_t^*, e_t^*, A_t) \geq (1-\beta_1)\frac{1}{L}\sum_{j=1}^{L} c_t^{j*}. \tag{A.42}$$

By (A.39), (A.31), (A.5), (A.6), and (A.24), we get

$$\frac{1}{L}\sum_{j=1}^{L} c_0^{j*} \geq (1-\beta_1)\left((1+r_0^*)\frac{1}{L}\sum_{j=1}^{L} s_{-1}^{j*} + w_0^*\right) = (1-\beta_1)\left(q_0^*\frac{R_{-1}^*}{L} + (1+r_0^*)k_0^* + w_0^*\right) =$$
$$(1-\beta_1)\left(\frac{\alpha_3}{\varepsilon_0^*} + \alpha_1 + \alpha_2\right) f(k_0^*, e_0^*, A_0) > (1-\beta_1) f(k_0^*, e_0^*, A_0),$$

and therefore, taking into account (A.42) and (A.33),

$$f(k_1^*, e_1^*, A_1) \geq (1-\beta_1)\frac{1}{L}\sum_{j=1}^{L} c_1^{j*} \geq (1-\beta_1)\frac{1}{L}\sum_{j=1}^{L} \beta_j(1+r_1^*)c_0^{j*}$$
$$\geq (1-\beta_1)\beta_L(1+r_1^*)\frac{1}{L}\sum_{j=1}^{L} c_0^{j*} > \beta_L(1-\beta_1)^2(1+r_1^*) f(k_0^*, e_0^*, A_0).$$

Repeating the argument, we obtain for $t = 0, 1, \ldots, T-1$,

$$f(k^*_{t+1}, e^*_{t+1}, A_{t+1}) > \beta_L(1-\beta_1)^2(1+r^*_{t+1})f(k^*_t, e^*_t, A_t). \tag{A.43}$$

It follows from (A.6) and (A.43) that

$$\frac{(1+r^*_{t+1})k^*_{t+1}}{\alpha_1} = f(k^*_{t+1}, e^*_{t+1}, A_{t+1}) > \beta_L(1-\beta_1)^2(1+r^*_{t+1})f(k^*_t, e^*_t, A_t),$$

and hence (A.40) holds. □

**Claim A.9.** *For $t = 0, 1, \ldots, T-1$,*

$$\frac{\big(e(\varepsilon^*_t, R^*_{t-1})\big)^{1-\alpha_3}}{A_t(k^*_t)^{\alpha_1}} = \alpha_1 \frac{e(\varepsilon^*_{t+1}, R^*_t)}{k^*_{t+1}}, \tag{A.44}$$

*or, equivalently,*

$$\frac{1}{q(k^*_t, e^*_t, A_t)} = \frac{1 + r(k^*_{t+1}, e^*_{t+1}, A_{t+1})}{q(k^*_{t+1}, e^*_{t+1}, A_{t+1})}. \tag{A.45}$$

*Proof.* Using (A.7) and the definition of $q^*_t$, it is easily seen that (A.44) is equivalent to (A.45).

Suppose that for some $t = 0, 1, \ldots, T-1$ equality (A.44) does not hold. Then either

$$\frac{\big(e(\varepsilon^*_t, R^*_{t-1})\big)^{1-\alpha_3}}{A_t(k^*_t)^{\alpha_1}} > \alpha_1 \frac{e(\varepsilon^*_{t+1}, R^*_t)}{k^*_{t+1}}, \tag{A.46}$$

or

$$\frac{\big(e(\varepsilon^*_t, R^*_{t-1})\big)^{1-\alpha_3}}{A_t(k^*_t)^{\alpha_1}} < \alpha_1 \frac{e(\varepsilon^*_{t+1}, R^*_t)}{k^*_{t+1}}. \tag{A.47}$$

Consider the first case. It follows from the structure of the problem (A.23) that

$$\varepsilon^*_t = \tilde{\varepsilon} = \frac{\alpha_3(1-\beta_1)}{1-(\alpha_1+\alpha_2)(1-\beta_1)}.$$

By (A.32),

$$\frac{1}{L}\sum_{j=1}^{L}\big(c^{j*}_t + s^{j*}_t\big) = (1+r^*_t)\frac{1}{L}\sum_{j=1}^{L} s^{j*}_{t-1} + w^*_t, \quad t = 0, 1, \ldots, T-1,$$

and, using (A.37) and (A.34), we get for $t = 0, 1, \ldots, T-1$,

$$\begin{aligned} \frac{1}{L}\sum_{j=1}^{L} c^{j*}_t + k^*_{t+1} &= q^*_t \frac{R^*_{t-1}}{L} - \frac{q^*_{t+1}}{1+r^*_{t+1}}\frac{R^*_t}{L} + (1+r^*_t)k^*_t + w^*_t \\ &= (1+r^*_t)k^*_t + w^*_t + q^*_t e^*_t + q^*_t\frac{R^*_t}{L} - \frac{q^*_{t+1}}{1+r^*_{t+1}}\frac{R^*_t}{L} \\ &= f(k^*_t, e^*_t, A_t) + \frac{R^*_t}{L}\left(q^*_t - \frac{q^*_{t+1}}{1+r^*_{t+1}}\right). \end{aligned} \tag{A.48}$$

Using (A.45), it is easily seen that (A.46) is equivalent to

$$\frac{q^*_{t+1}}{1+r^*_{t+1}} > q^*_t,$$

and thus from (A.48) we get

$$f(k_t^*, e_t^*, A_t) > \frac{1}{L}\sum_{j=1}^{L} c_t^{j*}, \quad t = 0, 1, \ldots, T-1. \tag{A.49}$$

By (A.49), (A.39), (A.31), (A.5), (A.6), and (A.41), we get

$$\begin{aligned} f(k_t^*, e_t^*, A_t) &> \frac{1}{L}\sum_{j=1}^{L} c_t^{j*} \geq (1-\beta_1)\left((1+r_t^*)\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} + w_t^*\right) \\ &= (1-\beta_1)\left(q_t^*\frac{R_{t-1}^*}{L} + (1+r_t^*)k_t^* + w_t^*\right) = (1-\beta_1)\left(\frac{\alpha_3}{\varepsilon_t^*} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) \\ &= (1-\beta_1)\left(\frac{\alpha_3}{\tilde{\varepsilon}} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) = (1-\beta_1)\frac{1}{1-\beta_1} f(k_t^*, e_t^*, A_t) = f(k_t^*, e_t^*, A_t), \end{aligned}$$

a contradiction.

Consider the second case. It follows from the structure of the problem (A.23) that $\varepsilon_t^* = \bar{\varepsilon}$. By (A.24),

$$\varepsilon_t^* = \bar{\varepsilon} = \frac{1}{1+\beta_L(1-\beta_1)^2} \geq \frac{\varepsilon_{t+1}^*}{\varepsilon_{t+1}^* + \beta_L(1-\beta_1)^2},$$

or

$$\varepsilon_{t+1}^*(1-\varepsilon_t^*) \leq \beta_L(1-\beta_1)^2\varepsilon_t^*.$$

By (A.34) and the definition of $e_t^*$,

$$e_{t+1}^* \leq \beta_L(1-\beta_1)^2 e_t^*.$$

At the same time, it follows from (A.47) that

$$\frac{e_{t+1}^*}{e_t^*} > \frac{k_{t+1}^*}{\alpha_1 f(k_t^*, e_t^*, A_t)}.$$

Hence, by (A.40),

$$e_{t+1}^* > \beta_L(1-\beta_1)^2 e_t^*,$$

a contradiction.

Thus equality (A.44) holds for all $t = 0, 1, \ldots, T-1$. □

Claims A.3 and A.4 show that condition 1 of Definition A.2 holds. Due to the choice of $e_t^*$, $r_t^*$, $q_t^*$ and $w_t^*$, conditions 2–4 of Definition A.2 are satisfied. Claims A.9, A.5 and A.6 show that conditions 5, 6 and 7 of Definition A.2 are valid. Thus the proof of Lemma A.1 is complete. □

**Step II. Competitive equilibrium in the infinite horizon model**.

**Step II.1. A candidate equilibrium path**.
Let for $T = 1, 2, \ldots$,

$$\mathcal{E}_T^* = \left\{(c_t^{j*}(T))_{j=1}^L, (s_t^{j*}(T))_{j=1}^L, k_t^*(T), r_t^*(T), w_t^*(T), q_t^*(T), e_t^*(T), R_t^*(T)\right\}_{t=0,1,\ldots,T}$$

be a finite $T$-period equilibrium path. Let us apply the following procedure to the sequence $\{\mathcal{E}_T^*\}_{T=1,2,\dots}$.

At the first step of the process we take a cluster point of the sequence

$$\left\{(c_0^{j*}(T))_{j=1}^L,(s_0^{j*}(T))_{j=1}^L,k_0^*(T),r_0^*(T),w_0^*(T),q_0^*(T),e_0^*(T),R_0^*(T)\right\}_{T=1,2,\dots},$$

denote it as

$$\left\{(c_0^{j*})_{j=1}^L,(s_0^{j*})_{j=1}^L,k_0^*,r_0^*,w_0^*,q_0^*,e_0^*,R_0^*\right\},$$

and extract a subsequence $\{T_{0n}\}_{n=1}^\infty$ from $\{T\}_{T=1,2,\dots}$ such that

$$\left\{(c_0^{j*}(T_{0n}))_{j=1}^L,(s_0^{j*}(T_{0n}))_{j=1}^L,k_0^*(T_{0n}),r_0^*(T_{0n}),w_0^*(T_{0n}),q_0^*(T_{0n}),e_0^*(T_{0n}),R_0^*(T_{0n})\right\}_{n=1}^\infty$$

converges to $\left\{(c_0^{j*})_{j=1}^L,(s_0^{j*})_{j=1}^L,k_0^*,r_0^*,w_0^*,q_0^*,e_0^*,R_0^*\right\}$.

At the second step we take a cluster point of the sequence

$$\left\{(c_1^{j*}(T_{0n}))_{j=1}^L,(s_1^{j*}(T_{0n}))_{j=1}^L,k_1^*(T_{0n}),r_1^*(T_{0n}),w_1^*(T_{0n}),q_1^*(T_{0n}),e_1^*(T_{0n}),R_1^*(T_{0n})\right\}_{n=1}^\infty,$$

denote it as

$$\left\{(c_1^{j*})_{j=1}^L,(s_1^{j*})_{j=1}^L,k_1^*,r_1^*,w_1^*,q_1^*,e_1^*,R_1^*\right\},$$

and extract a subsequence $\{T_{1n}\}_{n=1}^\infty$ from the sequence $\{T_{0n}\}_{n=1}^\infty$ such that $T_{11}>1$ and

$$\left\{(c_1^{j*}(T_{1n}))_{j=1}^L,(s_1^{j*}(T_{1n}))_{j=1}^L,k_1^*(T_{1n}),r_1^*(T_{1n}),w_1^*(T_{1n}),q_1^*(T_{1n}),e_1^*(T_{1n}),R_1^*(T_{1n})\right\}_{n=1}^\infty$$

converges to $\left\{(c_1^{j*})_{j=1}^L,(s_1^{j*})_{j=1}^L,k_1^*,r_1^*,w_1^*,q_1^*,e_1^*,R_1^*\right\}$. This procedure continues ad infinitum.

As a result, we obtain an infinite sequence

$$\mathcal{E}_\infty^*=\left\{(c_t^{j*})_{j=1}^L,(s_t^{j*})_{j=1}^L,k_t^*,r_t^*,w_t^*,q_t^*,e_t^*,R_t^*\right\}_{t=0,1,\dots}. \tag{A.50}$$

This sequence is a natural candidate to be a competitive equilibrium in our model.

**Step II.2. Bounds of the $T$-period equilibrium capital sequence.**

It follows from Lemma A.1 that in any $T$-period finite equilibrium $\varepsilon_t^*=\frac{Le_t^*}{R_{t-1}^*}$ satisfies (A.24), and $e_t^*$ satisfies (A.35). Moreover, by condition 6 of Definition A.2,

$$\frac{e_{t+1}^*}{e_t^*}=\frac{\varepsilon_{t+1}^*R_t^*}{L}\frac{L}{\varepsilon_t^*R_{t-1}^*}=\frac{\varepsilon_{t+1}^*(1-\varepsilon_t^*)}{\varepsilon_t^*},$$

and hence, for $t=0,1,\dots$,

$$\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}<\frac{e_{t+1}^*}{e_t^*}<\frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}. \tag{A.51}$$

We also know that $k_t^*$, is bounded from below by $\tilde{k}_t$. However, we need to establish a more precise lower bound of the capital sequence in a $T$-period finite equilibrium.

Let the value $1+r'$ be such that

$$\beta_L(1+r')>2(1+\bar{g}), \tag{A.52}$$

and $k'$ be given by

$$\alpha_1A_0(\tilde{e}_0)^{\alpha_3}(k')^{\alpha_1-1}=1+r'. \tag{A.53}$$

Let further the sequence $\{k_t'\}$ be given by

$$k_{t+1}'=(1+\underline{g})k_t', \tag{A.54}$$

where

$$0<k_0'<\min\{\hat{k}_0,k'\}.$$

**Claim A.10.** *For all $t$,*

$$(1+\underline{g}) < \frac{f(k'_{t+1}, e^*_{t+1}, A_{t+1})}{f(k'_t, e^*_t, A_t)} < (1+\bar{g}). \tag{A.55}$$

*Proof.* By (A.54), (A.9), (A.51), (A.11), and (A.12),

$$\begin{aligned}\frac{f(k'_{t+1}, e^*_{t+1}, A_{t+1})}{f(k'_t, e^*_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}} \frac{A_{t+1}}{A_t} \frac{(e^*_{t+1})^{\alpha_3}}{(e^*_t)^{\alpha_3}} \\ &> \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\alpha_3} \geq \left(1+\underline{g}\right)^{\alpha_1} (1+\tilde{g})^{1-\alpha_1} > (1+\underline{g}).\end{aligned}$$

Analogously, using (A.54), (A.9), (A.51), (A.10), and (A.12), we have

$$\begin{aligned}\frac{f(k'_{t+1}, e^*_{t+1}, A_{t+1})}{f(k'_t, e^*_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}} \frac{A_{t+1}}{A_t} \frac{(e^*_{t+1})^{\alpha_3}}{(e^*_t)^{\alpha_3}} \\ &< \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \left(\frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)^{\alpha_3} \leq (1+\bar{g})^{\alpha_1} (1+\bar{g})^{1-\alpha_1} = (1+\bar{g}).\end{aligned}$$

□

**Claim A.11.** *For all $t = 0, 1, \ldots, T$,*

$$1 + r(k'_t, e^*_t, A_t) > 1 + r'. \tag{A.56}$$

*Proof.* It follows from (A.6), (A.54), and (A.55) that

$$1 + r(k'_t, e^*_t, A_t) = \alpha_1 \frac{f(k'_t, e^*_t, A_t)}{k'_t} > \alpha_1 \frac{f(k'_{t-1}, e^*_{t-1}, A_{t-1})}{k'_{t-1}} = 1 + r(k'_{t-1}, e^*_{t-1}, A_{t-1}).$$

Repeating the argument, and using (A.35) along with (A.53), we get

$$\begin{aligned}1 + r(k'_t, e^*_t, A_t) > 1 + r(k'_t, e^*_0, A_t) &\geq 1 + r(k'_0, \tilde{e}_0, A_0) \\ &> 1 + r(k', \tilde{e}_0, A_0) = \alpha_1 A_0 (\tilde{e}_0)^{\alpha_3} (k')^{\alpha_1 - 1} = 1 + r'.\end{aligned}$$

□

Let

$$w'_{t+1} = (1+\underline{g}) w'_t, \quad t = 0, 1, \ldots,$$

where

$$w'_0 = \alpha_2 A_0 (k'_0)^{\alpha_1} (\tilde{e}_0)^{\alpha_3} > 0.$$

**Claim A.12.** *In any finite $T$-period competitive equilibrium*

$$\left\{(c^{j*}_t)^L_{j=1}, (s^{j*}_t)^L_{j=1}, k^*_t, r^*_t, w^*_t, q^*_t, e^*_t, R^*_t\right\}_{t=0,1,\ldots,T},$$

*for $t \leq T-1$,*

$$k^*_t > k'_t > 0, \tag{A.57}$$

*and*

$$w^*_t > w'_t > 0.$$

*Proof.* First let us prove (A.57). Assume the converse. Then there is $\tau > 0$ such that $k^*_\tau > k'_\tau$, and $k^*_{\tau+1} \leq k'_{\tau+1}$. It follows from (A.4), (A.35), and (A.56), that for all $j$,

$$c^{j*}_{\tau+1} \geq \beta_j(1+r^*_{\tau+1})c^{j*}_\tau \geq \beta_L(1+r^*_{\tau+1})c^{j*}_\tau = \beta_L(1+r(k^*_{\tau+1},e^*_{\tau+1},A_{\tau+1}))c^{j*}_\tau$$
$$\geq \beta_L(1+r(k'_{\tau+1},e^*_{\tau+1},A_{\tau+1}))c^{j*}_\tau > \beta_L(1+r')c^{j*}_\tau.$$

By (A.52),

$$c^{j*}_{\tau+1} > 2(1+\bar{g})c^{j*}_\tau. \tag{A.58}$$

Adding together the budget constraints of all agents at time $t$ in (A.3), and using conditions 5–7 of Definition A.2, we get

$$\frac{1}{L}\sum_{j=1}^{L} c^{j*}_t + k^*_{t+1} = (1+r^*_t)\,k^*_t + w^*_t = f(k^*_t,e^*_t,A_t), \quad t = 0,1,\ldots,T. \tag{A.59}$$

Applying (A.59) for $t = \tau+1$ and $t = \tau$, and using (A.58), we have

$$f(k^*_{\tau+1},e^*_{\tau+1},A_{\tau+1}) - k^*_{\tau+2} = \frac{1}{L}\sum_{j=1}^{L} c^{j*}_{\tau+1}$$
$$> 2(1+\bar{g})\frac{1}{L}\sum_{j=1}^{L} c^{j*}_\tau = 2(1+\bar{g})\left(f(k^*_\tau,e^*_\tau,A_\tau) - k^*_{\tau+1}\right).$$

Hence, by the choice of $\tau$ and (A.35),

$$\begin{aligned} k^*_{\tau+2} &< 2(1+\bar{g})k^*_{\tau+1} + f(k^*_{\tau+1},e^*_{\tau+1},A_{\tau+1}) - 2(1+\bar{g})f(k^*_\tau,e^*_\tau,A_\tau) \\ &\leq (1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau,e^*_\tau,A_\tau)\right) + f(k'_{\tau+1},e^*_{\tau+1},A_{\tau+1}) - (1+\bar{g})f(k'_\tau,e^*_\tau,A_\tau). \end{aligned} \tag{A.60}$$

It follows from (A.55) that

$$f(k'_{\tau+1},e^*_{\tau+1},A_{\tau+1}) < (1+\bar{g})f(k'_\tau,e^*_\tau,A_\tau). \tag{A.61}$$

Moreover, using (A.56) and (A.52), we get

$$\frac{f(k'_\tau,e^*_\tau,A_\tau)}{k'_\tau} = \frac{1+r(k'_\tau,e^*_\tau,A_\tau)}{\alpha_1} > \frac{1+r'}{\alpha_1} > \frac{2(1+\bar{g})}{\alpha_1\beta_L} > 2(1+\bar{g}),$$

and hence, by (A.54) and (A.12), we get

$$2k'_{\tau+1} = 2(1+\underline{g})k'_\tau < 2(1+\bar{g})k'_\tau < f(k'_\tau,e^*_\tau,A_\tau). \tag{A.62}$$

Combining (A.61) and (A.62), we have

$$(1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau,e^*_\tau,A_\tau)\right) + f(k'_{\tau+1},e^*_{\tau+1},A_{\tau+1}) - (1+\bar{g})f(k'_\tau,e^*_\tau,A_\tau) < 0.$$

Now it follows from (A.60) that $k^*_{\tau+2} < 0$, which is impossible. Hence $\tau + 2 > T$, and therefore the inequality (A.57) holds for $t \leq T-1$.

Using (A.57), (A.35), (A.54), (A.9), (A.51), (A.11) and (A.12), we obtain for all $t = 0, 1, \ldots, T-1$,

$$
\begin{aligned}
w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3} &> \alpha_2 A_t (k_t')^{\alpha_1} (e_t^*)^{\alpha_3} \\
&= \alpha_2 (1+\lambda)^t A_0 (1+\underline{g})^{t\alpha_1} (k_0')^{\alpha_1} \left( \frac{e_t^*}{e_0^*} \right)^{\alpha_3} (e_0^*)^{\alpha_3} \\
&> (1+\underline{g})^{t\alpha_1} (1+\lambda)^t \left( \frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}} \right)^{t\alpha_3} \alpha_2 A_0 (k_0')^{\alpha_1} (\tilde{e}_0)^{\alpha_3} \\
&\geq (1+\underline{g})^{t\alpha_1} (1+\tilde{g})^{t(1-\alpha_1)} w_0' > (1+\underline{g})^t w_0' = w_t'.
\end{aligned}
$$

□

**Step II.3. Existence of an equilibrium**.

Now we are ready to prove the following lemma which maintains that the sequence $\mathcal{E}_\infty^*$ defined by (A.50) is a competitive equilibrium under given extraction rates in our model.

**Lemma A.2.** *The sequence $\mathcal{E}_\infty^*$ defined by (A.50) is a competitive equilibrium starting from $\mathcal{I}_0$.*

*Proof.* It is clear that by construction $\mathcal{E}_\infty^*$ satisfies conditions 2–7 in Definition A.1. Thus to prove that $\mathcal{E}_\infty^*$ is a competitive equilibrium it is sufficient to show that $\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L\}_{t=0}^\infty$ is a solution to the problem (A.1) at $r_t = r_t^*$, $w_t = w_t^*$, and $s_{-1}^j = \frac{q_0^*}{1+r_0^*} \hat{R}_{-1}^j + \hat{k}_0^j$.

Let $c_t'$ be such that

$$
c_t' = \frac{w_t'}{2}, \quad t = 0, 1, \ldots. \tag{A.63}
$$

It is clear that

$$
c_{t+1}' = (1+\underline{g}) c_t',
$$

and hence

$$
\sum_{t=0}^\infty \beta^t \ln c_t' = \frac{\ln c_0'}{1-\beta} + \ln(1+\underline{g}) \sum_{t=0}^\infty t\beta^t = \frac{\ln c_0'}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g}).
$$

Consider the instantaneous utility function

$$
u_t(c) = \ln c - \ln c_t'.
$$

Clearly, the solution to the problem (A.1) will not change if we replace the instantaneous utility function $\ln c$ with the function $u_t(c)$. It is also clear that $u_t(c_t') = 0$.

Note that for all $t$,

$$
\bar{c}_t > Lf(\bar{k}_t, \bar{e}, A_t) > Lf(k_t', \tilde{e}_t, A_t) > c_t',
$$

and hence $u_t(\bar{c}_t) > 0$. Moreover, it follows from (A.17) that

$$
\begin{aligned}
\sum_{t=0}^\infty \beta^t u_t(\bar{c}_t) = \frac{\ln \bar{c}_0}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\bar{g}) - \frac{\ln c_0'}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g}) \\
= \frac{1}{1-\beta} \ln\left( \frac{\bar{c}_0}{c_0'} \right) + \frac{\beta}{(1-\beta)^2} \ln\left( \frac{1+\bar{g}}{1+\underline{g}} \right).
\end{aligned}
$$

Now assume that $\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L\}_{t=0}^\infty$ is not a solution to the problem (A.1). Then for some $j$ (we fix this $j$ and omit it in the remaining part of the proof for the simplicity of notation) there is a feasible sequence $\{\widehat{c}_t, \widehat{s}_t\}_{t=0}^\infty$ such that

$$\widehat{U} > U^*, \text{ where } \widehat{U} = \sum_{t=0}^{\infty} \beta^t u_t(\widehat{c}_t), \text{ and } U^* = \sum_{t=0}^{\infty} \beta^t u_t(c_t^*).$$

Let $0 < \Delta < \widehat{U} - U^*$, and let $\Theta$ be such that

$$\sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \min\left\{\frac{\Delta}{2}, \ln 2\right\}.$$

Further, let

$$U^{*\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*), \quad \widehat{U}^{\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t),$$

and

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^*(T)), \quad U^{*\Theta}(T) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*(T)),$$

for $T = \Theta + 1, \Theta + 2, \ldots$.

**Claim A.13.** *There is a sequence $\{T_{\Theta n}\}_{n=1}^\infty$ such that*

$$U^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} U^{*\Theta}.$$

*Proof.* It is sufficient to note that since $\mathcal{E}_\infty^*$ is obtained as a result of the application of the process described at Step II.1 to the sequence $\{\mathcal{E}_T^*\}_{T=1,2,\ldots}$, there is a sequence $\{T_{\Theta n}\}_{n=1}^\infty$ such that for $t = 0, 1, \ldots, \Theta$:

$$\lim_{n\to\infty} k_t^*(T_{\Theta n}) = k_t^*, \quad \lim_{n\to\infty} r_t^*(T_{\Theta n}) = r_t^*, \quad \lim_{n\to\infty} w_t^*(T_{\Theta n}) = w_t^*,$$
$$\lim_{n\to\infty} q_t^*(T_{\Theta n}) = q_t^*, \quad \lim_{n\to\infty} e_t^*(T_{\Theta n}) = e_t^*, \quad \lim_{n\to\infty} R_t^*(T_{\Theta n}) = R_t^*,$$
$$\lim_{n\to\infty} c_t^*(T_{\Theta n}) = c_t^*, \quad \lim_{n\to\infty} s_t^*(T_{\Theta n}) = s_t^*.$$

□

Let us formulate a claim that will be useful in what follows.

**Statement 1.** *Suppose that $F_r(x, y)$, $r = 1, \ldots, R$, are continuous and concave in $y$ functions defined on $X \times Y$, where $X$ and $Y$ are convex compact subsets of finite dimensional spaces. If there exists $\hat{y} \in Y$ such that $F_r(x, \hat{y}) > 0$ for all $x \in X$, $r = 1, \ldots, R$, then the correspondence*

$$x \to \bigcap_{r=1}^{R} \{y \in Y \mid F_r(x, y) \geq 0\}$$

*is upper and lower semi-continuous, and all sets*

$$\bigcap_{r=1}^{R} \{y \in Y \mid F_r(x, y) \geq 0\}$$

*are non-empty, convex and closed.*

*Proof.* It is trivial. □

Let $W^{*\Theta}$ be the maximum value of utility in the problem

$$\max \sum_{t=0}^{\Theta} \beta^t u_t(c_t),$$
$$\text{s. t.} \quad c_t + s_t \leq (1 + r_t^*)\, s_{t-1} + w_t^*,$$
$$s_t \geq 0, \quad t = 0, 1, \ldots, \Theta,$$

and $W^{*\Theta}(T)$ be the maximum value of utility in the problem

$$\begin{aligned} &\max \sum_{t=0}^{\Theta} \beta^t u_t(c_t), \\ \text{s. t.} \quad & c_t + s_t \leq (1 + r_t^*(T))\, s_{t-1} + w_t^*(T), \\ & s_t \geq 0, \quad t = 0, 1, \ldots, \Theta, \end{aligned} \tag{A.64}$$

for $T = \Theta + 1, \Theta + 2, \ldots$.

**Claim A.14.**

$$W^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} W^{*\Theta}.$$

*Proof.* Consider the correspondence that takes to each

$$\{(1 + r_0, w_0), \ldots, (1 + r_\Theta, w_\Theta)\}$$
$$\in \prod_{t=0}^{\Theta} \Big( [1 + r(\bar{k}_t, \tilde{e}_t, A_t), 1 + r(\tilde{k}_t, \bar{e}_t, A_t)] \times [w(\tilde{k}_t, \tilde{e}_t, A_t), w(\bar{k}_t, \bar{e}_t, A_t)] \Big)$$

the set

$$\{(c_0, s_0), \ldots, (c_\Theta, s_\Theta\} \in \mathbb{R}^{2(\Theta+1)}$$

which is such that, with $s_{-1} = \hat{s}_{-1}$ being given,

$$c_t + s_t \leq (1 + r_t^*(T))\, s_{t-1} + w_t^*(T), \text{ and } s_t \geq 0,$$

hold for all $t = 0, 1, \ldots, \Theta$.

By Statement 1, this correspondence is lower- and upper-semicontinuous, and it is sufficient to apply the Maximum Theorem. □

**Claim A.15.**

$$U^*(T) \geq W^{*\Theta}(T).$$

*Proof.* Let for some $T > \Theta + 1$, the sequence $\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_\Theta, \breve{s}_\Theta)\}$ be a solution to (A.64). Let further for $t = \Theta + 1, \ldots, T$, $\{(\breve{c}_t, \breve{s}_t)\}$ be defined recursively by

$$\breve{c}_t = c'_t, \quad \breve{s}_t = (1 + r_t^*(T))\, \breve{s}_{t-1} + w_t^*(T) - \breve{c}_t. \tag{A.65}$$

We show that given $s_{-1} = \hat{s}_{-1}$, the sequence

$$\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_\Theta, \breve{s}_\Theta), (\breve{c}_{\Theta+1}, \breve{s}_{\Theta+1}), \ldots, (\breve{c}_T, \breve{s}_T)\} \tag{A.66}$$

is feasible for the problem

$$\begin{aligned}
&\max \textstyle\sum_{t=0}^{T} \beta^t u_t(c_t),\\
\text{s. t.}\quad & c_t + s_t \le (1 + r_t^*(T))\, s_{t-1} + w_t^*(T),\\
& s_t \ge 0, \quad t = 0, 1, \ldots, T.
\end{aligned} \tag{A.67}$$

It is sufficient to check that $\breve{s}_t \ge 0$ for $t = \Theta + 1, \ldots, T$. By Claim A.12, we have for $\Theta + 1 \le t \le T - 1$,

$$\breve{c}_t = c'_t = \frac{w'_t}{2} < w_t^*(T).$$

We prove that $\breve{s}_t > 0$ for $t = \Theta + 1, \ldots, T - 1$ recursively. Clearly, $\breve{s}_\Theta = 0$. Suppose that $\breve{s}_{t-1} \ge 0$ for $\Theta + 1 \le t < T - 2$. Then

$$\breve{s}_t = (1 + r_t^*(T))\, \breve{s}_{t-1} + w_t^*(T) - \breve{c}_t \ge w_t^*(T) - c'_t > 0.$$

In particular, $\breve{s}_{T-2} > 0$. For $t = T - 1$ we have

$$\begin{aligned}
\breve{s}_{T-1} &= \big(1 + r_{T-1}^*(T)\big)\, \breve{s}_{T-2} + w_{T-1}^*(T) - \breve{c}_{T-1}\\
&\ge w_{T-1}^*(T) - c'_{T-1} > 2c'_{T-1} - c'_{T-1} = c'_{T-1}.
\end{aligned}$$

For $t = T$ we know from Claim A.12 that either

$$w_T^*(T) > c'_T,$$

or

$$1 + r_T^* \ge 1 + r'.$$

In the first case, we can apply the same reasoning as before:

$$\breve{s}_T = (1 + r_T^*(T))\, \breve{s}_{T-1} + w_T^*(T) - \breve{c}_T \ge w_T^*(T) - c'_T > 0.$$

In the second case, using (A.52) and (A.12), we have

$$\begin{aligned}
\breve{s}_T &= (1 + r_T^*(T))\, \breve{s}_{T-1} + w_T^*(T) - \breve{c}_T > (1 + r')c'_{T-1} - c'_T\\
&> \frac{2}{\beta_L}(1 + \bar{g})c'_{T-1} - c'_T > (1 + \bar{g})c'_{T-1} - c'_T > (1 + \underline{g})c'_{T-1} - c'_T = 0.
\end{aligned}$$

Thus we have proved that the sequence (A.66) is feasible for the problem (A.67). Since the sequence

$$\{(c_0^*(T), s_0^*(T)), \ldots, (c_T^*(T), s_T^*(T))\}$$

is the solution to this problem, we have

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^*(T)) \ge \sum_{t=0}^{\Theta} \beta^t u_t(\breve{c}_t) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c'_t) = \sum_{t=0}^{\Theta} \beta^t u_t(\breve{c}_t) = W^{*\Theta}(T).$$

□

Let us prove another useful claim.

**Claim A.16.** *For all* $t = 0, 1, \ldots, T$,

$$k_t^*(T) \le \bar{\kappa}(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}. \tag{A.68}$$

*Proof.* It is sufficient to show that

$$\kappa_t^* \leq \bar{\kappa}, \quad t = 0, 1, \ldots, T, \tag{A.69}$$

where

$$\kappa_t^* := \frac{k_t^*(T)}{(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}}.$$

By (A.13), (A.11), and (A.35),

$$\bar{\kappa} = \frac{1}{(1+\tilde{g})^{\frac{1}{1-\alpha_1}}} \geq \frac{\hat{k}_0}{(A_0\tilde{e}_0)^{\alpha_3})^{\frac{1}{1-\alpha_1}}} \geq \frac{\hat{k}_0}{(A_0(e_0^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_0^*,$$

which proves (A.69) for $t = 0$. We prove it for $t = 1, \ldots, T$ recursively. Suppose that $\kappa_t^* \leq \bar{\kappa}$. If follows from (A.59) that for all $t$,

$$k_{t+1}^*(T) \leq f(k_t^*(T), e_t^*(T), A_t) = (k_t^*(T))^{\alpha_1} \left( (A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}} \right)^{1-\alpha_1},$$

and hence, due to (A.9), (A.51), and (A.11),

$$\begin{aligned}(\kappa_t^*)^{\alpha_1} &\geq \frac{k_{t+1}^*(T)}{(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_{t+1}^* \frac{(A_{t+1}(e_{t+1}^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}}{(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}} \\ &= \kappa_{t+1}^* \left( (1+\lambda) \left( \frac{e_{t+1}^*(T)}{e_t^*(T)} \right)^{\alpha_3} \right)^{\frac{1}{1-\alpha_1}} > \kappa_{t+1}^*(1+\lambda)^{\frac{1}{1-\alpha_1}} \left( \frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}} \right)^{\frac{\alpha_3}{1-\alpha_1}} \geq (1+\tilde{g})\kappa_{t+1}^*.\end{aligned}$$

Therefore, by (A.13),

$$\kappa_{t+1}^* \leq \frac{(\kappa_t^*)^{\alpha_1}}{1+\tilde{g}} \leq \frac{(\bar{\kappa})^{\alpha_1}}{1+\tilde{g}} = \bar{\kappa}.$$

Thus (A.69) holds for all $t = 0, 1, \ldots, T$. □

Denote

$$1 + \bar{r} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}}.$$

By (A.13) and the choice of $1 + \underline{g}$,

$$\beta_L(1+\bar{r}) = \beta_L\alpha_1 \frac{(\bar{\kappa})^{\alpha_1}}{\bar{\kappa}} = \beta_L\alpha_1(1+\tilde{g}) = (1+\underline{g}), \tag{A.70}$$

and hence, by (A.68), for all $t = 0, 1, \ldots, T$, we have

$$1 + r_t^*(T) = \alpha_1 \frac{A_t(e_t^*(T))^{\alpha_3}}{(k_t^*(T))^{1-\alpha_1}} \geq \alpha_1 \frac{A_t(e_t^*(T))^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t(e_t^*(T))^{\alpha_3}} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}} = 1 + \bar{r}. \tag{A.71}$$

**Claim A.17.**

$$U^* \geq U^{*\Theta}.$$

*Proof.* Let us prove that for any $T > \Theta + 1$,

$$c_\Theta^*(T) \geq c'_\Theta. \tag{A.72}$$

Assume that $c^*_\Theta(T) < c'_\Theta$. We show that this inequality implies $c^*_t(T) < c'_t$ for all $t \le \Theta$. Indeed, if $c^*_t(T) < c'_t$ for some $t < \Theta$, then it follows from (A.4), (A.71) and (A.70) that

$$c^*_t(T) \ge \beta_j(1 + r^*_t(T))c^*_{t-1}(T) \ge \beta_L(1 + \bar{r})c^*_{t-1}(T) = (1 + \underline{g})c^*_{t-1}(T), \tag{A.73}$$

and thus

$$c^*_{t-1}(T) \le \frac{c^*_t(T)}{1+\underline{g}} < \frac{c'_t}{1+\underline{g}} = c'_{t-1}.$$

Hence

$$\sum_{t=0}^{\Theta} \beta^t u_t(c^*_t(T)) = \sum_{t=0}^{\Theta} \beta^t \left(\ln c^*_t(T) - \ln c'_t\right) < 0.$$

At the same time, by the choice of $\Theta$, we have

$$\sum_{t=\Theta+1}^{T} \beta^t u_t(c^*_t(T)) \le \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \le \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \ln 2.$$

Therefore

$$\sum_{t=0}^{T} \beta^t u_t(c^*_t(T)) = \sum_{t=0}^{\Theta} \beta^t u_t(c^*_t(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c^*_t(T)) < \ln 2. \tag{A.74}$$

Consider the sequence

$$\{(\check{c}_0, \check{s}_0), \dots, (\check{c}_T, \check{s}_T)\}, \tag{A.75}$$

defined as follows: for $t \le \Theta$

$$\check{c}_t = w^*_t(T), \quad \check{s}_t = 0,$$

and for $t = \Theta + 1, \dots, T$, $\{(\check{c}_t, \check{s}_t)\}$ is given by (A.65). It follows from Claim A.12 that for $t \le \Theta$,

$$\check{c}_t = w^*_t(T) > w'_t > c'_t. \tag{A.76}$$

Repeating the argument from the proof of Claim A.15, we obtain that the sequence (A.75) is feasible for the problem (A.67). At the same time, the sequence

$$\{(c^*_0(T), s^*_0(T)), \dots, (c^*_T(T), s^*_T(T))\}$$

is the solution to this problem. Hence, using (A.76), we get

$$\begin{aligned}\sum_{t=0}^{T} \beta^t u_t(c^*_t(T)) &\ge \sum_{t=0}^{T} \beta^t u_t(\check{c}_t) = \sum_{t=0}^{\Theta} \beta^t u_t(w^*_t(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c'_t) \\ &= u_0(w^*_0(T)) + \sum_{t=1}^{\Theta} \beta^t u_t(\check{c}_t) > u_0(w^*_0(T)) = \ln w^*_0(T) - \ln c'_0 \\ &> \ln w'_0 - \ln c'_0 = \ln\left(\frac{w'_0}{c'_0}\right) = \ln 2,\end{aligned}$$

a contradiction of (A.74).

Thus (A.72) holds, and using the fact that $c^*_\Theta$ is a limit of the sequence $\{c^*_\Theta(T_{\Theta n})\}_{n=1}^{\infty}$, we have

$$c^*_\Theta \ge c'_\Theta.$$

It immediately follows from (A.73) that for all $\Theta + 1 \le t \le T$,

$$c_t^*(T) \ge c_t'.$$

Since every $c_t^*$ is a cluster point of the sequence $\{c_t^*(T)\}_{T=1,2,\ldots}$, we get

$$c_t^* \ge c_t', \quad t = \Theta + 1, \Theta + 2, \ldots.$$

It follows that

$$U^* - U^{*\Theta} = \sum_{t=\Theta+1}^{\infty} \beta^t u_t(c_t^*) = \sum_{t=\Theta+1}^{\infty} \beta^t (\ln c_t^* - \ln c_t') \ge 0,$$

which completes the proof. □

**Claim A.18.**

$$U^{*\Theta}(T) > U^*(T) - \frac{\Delta}{2}, \quad T = \Theta + 1, \Theta + 2, \ldots; \tag{A.77}$$

$$W^{*\Theta} > \widehat{U} - \frac{\Delta}{2}. \tag{A.78}$$

*Proof.* Clearly, $\bar{c}_t > c_t^*(T)$ and $\bar{c}_t > \widehat{c}_t$ for all $t$. It follows from the choice of $\Theta$ that

$$U^*(T) - U^{*\Theta}(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^*(T)) - \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*(T)) = \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^*(T))$$
$$< \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \le \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \frac{\Delta}{2},$$

which proves (A.77).

Due to the definition of $W^{*\Theta}$, we have $W^{*\Theta} \ge \widehat{U}^\Theta$. Now it is easily seen that

$$W^{*\Theta} \ge \widehat{U}^\Theta = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t) = \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\widehat{c}_t) \ge \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) > \widehat{U} - \frac{\Delta}{2},$$

which proves (A.78). □

Now, combining Claims A.13–A.15 and A.17–A.18, we obtain

$$U^* \ge U^{*\Theta} = \lim_{n\to\infty} U^{*\Theta}(T_{\Theta n}) \ge \lim_{n\to\infty} U^*(T_{\Theta n}) - \frac{\Delta}{2}$$
$$\ge \lim_{n\to\infty} W^{*\Theta}(T_{\Theta n}) - \frac{\Delta}{2} = W^{*\Theta} - \frac{\Delta}{2} > \widehat{U} - \Delta,$$

which contradicts the choice of $\Delta$. This contradiction completes the proof of the lemma. □

Thus the proof of Theorem A.1 is finally complete, and there exists a competitive equilibrium. □

Let us prove two important results about a competitive equilibrium. The following proposition states that if at the initial instant the stocks of physical capital and natural resources are owned by the most patient agents, then the competitive equilibrium starting from this state is unique.

**Proposition A.1.** *Suppose that the initial state $\mathcal{I}_0$ is such that*

$$\hat{k}_0^j = 0, \quad \hat{R}_{-1}^j = 0 \quad (j \notin J).$$

*Then there exists a unique competitive equilibrium starting from $\mathcal{I}_0$,*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots},$$

*which is given for $t = 0, 1, \dots$ by*

$$\begin{gathered} c_t^{j*} = (1-\beta_1)(1+r_t^*)s_{t-1}^{j*} + w_t^*, \qquad s_t^{j*} = \beta_1(1+r_t^*)s_{t-1}^{j*} \quad (j \in J), \\ c_t^{j*} = w_t^*, \qquad s_t^{j*} = 0 \quad (j \notin J), \\ k_{t+1}^* = \beta_1(1+r_t^*)k_t^*, \qquad 1 + r_t^* = \alpha_1 A_t (k_t^*)^{\alpha_1 - 1}(e_t^*)^{\alpha_3}, \\ w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3}, \qquad q_t^* = \alpha_3 A_t (k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3 - 1}, \\ R_t^* = \beta_1 R_{t-1}^*, \qquad e_t^* = \frac{1-\beta_1}{L} R_{t-1}^*, \end{gathered}$$

*where $R_{-1}^* = \sum_{j=1}^L \hat{R}_{-1}^j$, $k_0^* = \frac{1}{L}\sum_{j=1}^L \hat{k}_0^j$, and $s_{-1}^{j*} = \frac{q_0^*}{1+r_0^*}\hat{R}_{-1}^j + \hat{k}_0^j$.*

The following proposition verifies that in every competitive equilibrium from some time onward only the most patient agents can make positive savings. From this time relatively less patient agents make no savings, and the extraction rate is constant over time and equals $1 - \beta_1$.

**Proposition A.2.** *Suppose that*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

*is a competitive equilibrium starting from an arbitrary initial state $\mathcal{I}_0$. Then there exists a point in time $T$ such that for all $t > T$,*

$$\begin{aligned} s_t^{j*} &= 0 \quad (j \notin J), \\ R_t^* &= \beta_1 R_{t-1}^*, \\ e_{t+1}^* &= \beta_1 e_t^*. \end{aligned}$$

***Proof of Propositions A.1 and A.2.***
Consider a competitive equilibrium

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

starting from a non-degenerate state $\mathcal{I}_0 = \{(\hat{k}_0^j)_{j=1}^L, (\hat{R}_{-1}^j)_{j=1}^L\}$. Since for each $j = 1, \dots, L$, the sequence $\{c_t^{j*}, s_t^{j*}\}_{t=0}^\infty$ is a solution to problem (A.1), it satisfies the first-order conditions,

$$c_{t+1}^{j*} \ge \beta_j(1 + r_{t+1}^*)c_t^{j*} \ (= \text{ if } s_t^{j*} > 0), \quad t = 0, 1, \dots, \tag{A.79}$$

and the transversality condition,

$$\lim_{t\to\infty} \frac{\beta_j^t s_t^{j*}}{c_t^{j*}} = 0. \tag{A.80}$$

**Lemma A.3.** *Let $\beta > 0$ be such that for some $T$*

$$k^*_{t+1} > \beta(1 + r^*_t)k^*_t = \beta\alpha_1 A_t(k^*_t)^{\alpha_1}(e^*_t)^{\alpha_3}, \quad t > T.$$

*If $\beta_j < \beta$, then $s^{j*}_t = 0$ for all sufficiently large $t$.*

*Proof.* First let us show that if $\beta_j < \beta$, then $s^{j*}_t = 0$ for some $t \geq T$.

Assume the converse. By (A.79), for all $t \geq T$,

$$c^{j*}_{t+1} = \beta_j(1 + r^*_{t+1})c^{j*}_t,$$

and hence

$$\frac{c^{j*}_t}{k^*_{t+1}} \leq \frac{\beta_j(1 + r^*_t)c^{j*}_{t-1}}{\beta(1 + r^*_t)k^*_t} \leq \frac{\beta_j}{\beta}\frac{c^{j*}_{t-1}}{k^*_t}.$$

By assumption, $\beta_j/\beta < 1$, and thus $c^{j*}_t/k^*_{t+1} \xrightarrow[t\to\infty]{} 0$. Furthermore, it is clear that $k^*_{t+1} \leq A_t(k^*_t)^{\alpha_1}(e^*_t)^{\alpha_3}$, and therefore

$$\frac{c^{j*}_t}{w^*_t} = \frac{c^{j*}_t}{\alpha_2 A_t(k^*_t)^{\alpha_1}(e^*_t)^{\alpha_3}} \leq \frac{c^{j*}_t}{\alpha_2 k^*_{t+1}} \xrightarrow[t\to\infty]{} 0.$$

Thus for all sufficiently large $t$, $c^{j*}_t < w^*_t$, which is not optimal for agent $j$.

Now we know that $s^{j*}_t = 0$ at least for some $t$. Let us show that $s^{j*}_t = 0$ for all $t \geq T$.

Indeed, assume the converse. Then there are only two possibilities. The first is that there exists $T' > T$ such that $s^{j*}_t > 0$ for all $t \geq T'$. However, applying the same argument as above, we obtain that $s^{j*}_t = 0$ for some $t \geq T'$.

The second possibility is that there are $t_1$ and $t_2$ such that $T \leq t_1 < t_2 - 1$, and

$$s^{j*}_{t_1} = 0, \qquad s^{j*}_{t_2} = 0, \qquad s^{j*}_t > 0, \quad t_1 < t < t_2.$$

It follows from the budget constraints of agent $j$ that

$$c^{j*}_{t_1+1} < w^*_{t_1+1}, \qquad c^{j*}_{t_2} > w^*_{t_2}. \tag{A.81}$$

However, for $t \geq T$,

$$\frac{\alpha_1}{\alpha_2}\frac{w^*_{t+1}}{1 + r^*_{t+1}} = k^*_{t+1} > \beta(1 + r^*_t)k^*_t = \frac{\alpha_1}{\alpha_2}\beta w^*_t.$$

Thus $w^*_{t+1} > \beta(1 + r^*_{t+1})w^*_t$. Using (A.79), we get

$$c^{j*}_{t_1+2} = \beta_j(1 + r^*_{t_1+2})c^{j*}_{t_1+1} < \beta_j(1 + r^*_{t_1+2})w^*_{t_1+1} < \beta(1 + r^*_{t_1+2})w^*_{t_1+1} < w^*_{t_1+2}.$$

Repeating this argument, we obtain

$$c^{j*}_{t+1} < w^*_{t+1}, \quad t_1 < t < t_2,$$

which implies $c^{j*}_{t_2} < w^*_{t_2}$, a contradiction of (A.81). □

**Lemma A.4.**

$$k^*_{t+1} \leq \beta_1(1 + r^*_t)k^*_t, \quad t = 0, 1, \ldots.$$

*Proof.* Assume the converse. Then there are $T$ and $\zeta > 1$ such that

$$k^*_{T+1} \geq \zeta\beta_1(1+r^*_T)k^*_T. \tag{A.82}$$

Let us show that (A.82) implies

$$k^*_{t+1} \geq \zeta\beta_1(1+r^*_t)k^*_t, \quad t \geq T. \tag{A.83}$$

Denote

$$J(T) = \left\{j \in \{1,2,\ldots,L\} \ \mid \ s^{j*}_T > 0\right\},$$

and recall that

$$\frac{(1+r^*_t)k^*_t}{\alpha_1} = \frac{w^*_t}{\alpha_2} = \frac{q^*_t e^*_t}{\alpha_3}. \tag{A.84}$$

We have

$$\begin{aligned}
&\sum_{j\in J(T)} \left(c^{j*}_{T+1} + s^{j*}_{T+1}\right) - \frac{q^*_{T+2}}{1+r^*_{T+2}} R^*_{T+1} \\
&= \sum_{j\in J(T)} \left((1+r^*_{T+1})s^{j*}_T + w^*_{T+1}\right) - \frac{q^*_{T+2}}{1+r^*_{T+2}} R^*_{T+1} \\
&= q^*_{T+1}R^*_T + (1+r^*_{T+1})Lk^*_{T+1} + |J(T)|w^*_{T+1} - q^*_{T+1}R^*_{T+1} \\
&= q^*_{T+1}Le^*_{T+1} + (1+r^*_{T+1})Lk^*_{T+1} + |J(T)|w^*_{T+1} \\
&= (1+r^*_{T+1})k^*_{T+1}\left(L + L\frac{\alpha_3}{\alpha_1} + |J(T)|\frac{\alpha_2}{\alpha_1}\right) \\
&\geq \zeta\beta_1(1+r^*_{T+1})(1+r^*_T)k^*_T\left(L + L\frac{\alpha_3}{\alpha_1} + |J(T)|\frac{\alpha_2}{\alpha_1}\right) \\
&= \zeta\beta_1(1+r^*_{T+1})\left(L(1+r^*_T)k^*_T + Lq^*_Te^*_T + |J(T)|w^*_T\right) \\
&= \zeta\beta_1(1+r^*_{T+1})\left((1+r^*_T)\left(\sum_{j=1}^{L} s^{j*}_{T-1} - \frac{q^*_T}{1+r^*_T}R_{T-1}\right) + Lq^*_Te^*_T + \sum_{j\in J(T)} w^*_T\right) \\
&\geq \zeta\beta_1(1+r^*_{T+1})\left((1+r^*_T)\sum_{j\in J(T)} s^{j*}_{T-1} + \sum_{j\in J(T)} w^*_T - q^*_TR^*_{T-1} + Lq^*_Te^*_T\right) \\
&= \zeta\beta_1(1+r^*_{T+1})\left(\sum_{j\in J(T)}(c^{j*}_T + s^{j*}_T) - \frac{q^*_{T+1}}{1+r^*_{T+1}}R^*_T\right),
\end{aligned}$$

or, finally,

$$\begin{aligned}
&\sum_{j\in J(T)} \left(c^{j*}_{T+1} + s^{j*}_{T+1}\right) - \frac{q^*_{T+2}}{1+r^*_{T+2}} R^*_{T+1} \\
&\geq \zeta\beta_1(1+r^*_{T+1})\left(\sum_{j\in J(T)}(c^{j*}_T + s^{j*}_T) - \frac{q^*_{T+1}}{1+r^*_{T+1}}R^*_T\right).
\end{aligned} \tag{A.85}$$

For $j \in J(T)$, (A.79) holds with equality, so

$$\begin{aligned}
&\sum_{j\in J(T)} c^{j*}_{T+1} = \sum_{j\in J(T)} \beta_j(1+r^*_{T+1})c^{j*}_T \\
&\leq \beta_1(1+r^*_{T+1})\sum_{j\in J(T)} c^{j*}_T < \zeta\beta_1(1+r^*_{T+1})\sum_{j\in J(T)} c^{j*}_T.
\end{aligned} \tag{A.86}$$

Now (A.85) is consistent with (A.86) only if

$$\sum_{j\in J(T)} s_{T+1}^{j*} - \frac{q_{T+2}^*}{1+r_{T+2}^*} R_{T+1}^* \geq \zeta\beta_1(1+r_{T+1}^*) \left( \sum_{j\in J(T)} s_T^{j*} - \frac{q_{T+1}^*}{1+r_{T+1}^*} R_T^* \right).$$

Therefore,

$$\begin{aligned} Lk_{T+2}^* &= \sum_{j=1}^{L} s_{T+1}^{j*} - \frac{q_{T+2}^*}{1+r_{T+2}^*} R_{T+1}^* \\ &\geq \sum_{j\in J(T)} s_{T+1}^{j*} - \frac{q_{T+2}^*}{1+r_{T+2}^*} R_{T+1}^* \geq \zeta\beta_1(1+r_{T+1}^*) \left( \sum_{j\in J(T)} s_T^{j*} - \frac{q_{T+1}^*}{1+r_{T+1}^*} R_T^* \right) \\ &\qquad = \zeta\beta_1(1+r_{T+1}^*) \left( \sum_{j=1}^{L} s_T^{j*} - \frac{q_{T+1}^*}{1+r_{T+1}^*} R_T^* \right) = \zeta\beta_1(1+r_{T+1}^*) Lk_{T+1}^*. \end{aligned}$$

Repeating the argument, we infer that (A.83) holds for all $t \geq T$.

However, from Lemma A.3 it follows that $s_t^{j*} = 0$ for all $j$ and for all sufficiently large $t$. This contradicts the evident positivity of $k_t^*$ for all $t = 0, 1, \ldots$. □

**Lemma A.5.**

$$\begin{aligned} w_{t+1}^* &\leq \beta_1(1+r_{t+1}^*) w_t^*, \quad t = 0, 1, \ldots, \\ e_{t+1}^* &\leq \beta_1 e_t^*, \quad t = 0, 1, \ldots. \end{aligned}$$

*Proof.* Both inequalities follow from (A.84) and Lemma A.4. Indeed, for all $t$

$$\frac{w_{t+1}^*}{1+r_{t+1}^*} = \frac{\alpha_2(1+r_{t+1}^*)k_{t+1}^*}{\alpha_1(1+r_{t+1}^*)} \leq \frac{\beta_1\alpha_2(1+r_t^*)k_t^*}{\alpha_1} = \beta_1 w_t^*.$$

Moreover, for all $t$

$$\frac{e_{t+1}^*}{e_t^*} = \frac{q_t^*}{q_{t+1}^*} \frac{(1+r_{t+1}^*)k_{t+1}^*}{(1+r_t^*)k_t^*} = \frac{(1+r_{t+1}^*)q_t^*}{q_{t+1}^*} \frac{k_{t+1}^*}{(1+r_t^*)k_t^*} \leq \beta_1,$$

since $q_{t+1}^* = (1+r_{t+1}^*)q_t^*$ by the Hotelling rule. □

**Lemma A.6.**

$$s_{t+1}^{j*} \geq \beta_1(1+r_{t+1}^*) s_t^{j*} \quad (j \in J), \quad t = -1, 0, \ldots.$$

*Proof.* Consider $j \in J$. Then by (A.79),

$$\beta_1^t c_0^{j*} \leq \frac{c_t^{j*}}{(1+r_1^*)\cdots(1+r_t^*)}, \quad t = 1, 2, \ldots,$$

and hence

$$c_0^{j*}(1+\beta_1+\beta_1^2+\ldots) \leq c_0^{j*} + \frac{c_1^{j*}}{(1+r_1^*)} + \frac{c_2^{j*}}{(1+r_1^*)(1+r_2^*)} + \ldots. \tag{A.87}$$

Adding together all budget constraints of agent $j$, we obtain

$$\begin{aligned} c_0^{j*} + \frac{c_1^{j*}}{(1+r_1^*)} + \frac{c_2^{j*}}{(1+r_1^*)(1+r_2^*)} + \ldots \\ \le (1+r_0^*)s_{-1}^j + w_0^* + \frac{w_1^*}{(1+r_1^*)} + \frac{w_2^*}{(1+r_1^*)(1+r_2^*)} + \ldots. \end{aligned} \tag{A.88}$$

Moreover, by Lemma A.5 for $t = 0, 1, \ldots$,

$$\frac{w_{t+1}^*}{(1+r_1^*)\cdots(1+r_{t+1}^*)} \le \frac{\beta_1 w_t^*}{(1+r_1^*)\cdots(1+r_t^*)} \le \ldots \le \beta_1^{t+1} w_0^*,$$

which implies

$$\begin{aligned} (1+r_0^*)s_{-1}^{j*} + w_0^* + \frac{w_1^*}{(1+r_1^*)} + \frac{w_2^*}{(1+r_1^*)(1+r_2^*)} + \ldots \\ \le (1+r_0^*)s_{-1}^{j*} + w_0^*(1 + \beta_1 + \beta_1^2 + \ldots). \end{aligned} \tag{A.89}$$

Combining (A.87)–(A.89), we finally get

$$c_0^{j*}(1 + \beta_1 + \beta_1^2 + \ldots) \le (1+r_0^*)s_{-1}^{j*} + w_0^*(1 + \beta_1 + \beta_1^2 + \ldots),$$

and therefore

$$c_0^{j*} \le (1+r_0^*)(1-\beta_1)s_{-1}^{j*} + w_0^*.$$

Thus,

$$s_0^{j*} = (1+r_0^*)s_{-1}^{j*} + w_0^* - c_0^{j*} \ge (1+r_0^*)s_{-1}^{j*} + w_0^* - (1+r_0^*)(1-\beta_1)s_{-1}^{j*} - w_0^* = \beta_1(1+r_0^*)s_{-1}^{j*}.$$

This proves the desired inequality for $t = -1$. To prove it for $t = 0, 1, \ldots$, it is sufficient to repeat the argument. □

**Lemma A.7.** *For all $\delta > 0$ there exists a point in time $T$ such that for all $t > T$,*

$$k_{t+1}^* > \beta_1(1-\delta)(1+r_t^*)k_t^*.$$

*Proof.* From (A.79) and Lemma A.5 it is clear that for $j \in J$

$$\frac{c_{t+1}^{j*}}{w_{t+1}^*} \ge \frac{\beta_1(1+r_{t+1}^*)c_t^{j*}}{\beta_1(1+r_{t+1}^*)w_t^*} = \frac{c_t^{j*}}{w_t^*}, \quad t = 0, 1, \ldots.$$

This means that the sequence $\left\{\frac{c_t^{j*}}{w_t^*}\right\}_{t=0}^{\infty}$ is non-decreasing. It is also bounded from above, as consumption cannot exceed total output:

$$c_t^{j*} \le L\frac{w_t^*}{\alpha_2}, \quad t = 0, 1, \ldots.$$

Therefore, the sequence $\left\{\frac{c_t^{j*}}{w_t^*}\right\}_{t=0}^{\infty}$ converges, so the sequence $\left\{\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}}\right\}_{t=0}^{\infty}$ converges to 1.

It follows from Lemma A.6 that if $s_{-1}^j > 0$, then $s_t^{j*} > 0$ for all $t \ge 0$ and $j \in J$. Thus,

$$\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}} = \frac{w_{t+1}^*}{\beta_1(1+r_{t+1}^*)w_t^*}, \quad t = 0, 1, \ldots,$$

and the sequence $\left\{\frac{w^*_{t+1}}{\beta_1(1+r^*_{t+1})w^*_t}\right\}_{t=0}^{\infty}$ converges to 1 as well.

Hence for all $\delta > 0$ there exists $T$ such that for $t > T$,

$$\frac{w^*_{t+1}}{\beta_1(1+r^*_{t+1})w^*_t} > (1-\delta),$$

which implies

$$k^*_{t+1} = \frac{\alpha_1}{\alpha_2}\frac{w^*_{t+1}}{1+r^*_{t+1}} > \frac{\alpha_1}{\alpha_2}\beta_1(1-\delta)w^*_t = \beta_1(1-\delta)(1+r^*_t)k^*_t.$$

□

Consider $\delta$ that satisfies $\beta_1(1-\delta) > \max_{j\notin J}\beta_j$. Applying Lemma A.3 with $\beta = \beta_1(1-\delta)$, we obtain that for any competitive equilibrium starting from the non-degenerate initial state there exists a point in time $T$ such that for all $t > T$, $s^{j*}_t = 0$ $(j \notin J)$.

**Lemma A.8.** *For all $t > T$,*

$$k^*_{t+1} = \beta_1(1+r^*_t)k^*_t,$$
$$c^{j*}_t = (1-\beta_1)(1+r^*_t)s^{j*}_{t-1} + w^*_t, \qquad s^{j*}_t = \beta_1(1+r^*_t)s^{j*}_{t-1} \quad (j \in J),$$
$$c^{j*}_t = w^*_t, \qquad s^{j*}_t = 0 \quad (j \notin J).$$

*Moreover,*

$$e^*_{t+1} = \beta_1 e^*_t,$$
$$R^*_t = \beta_1 R^*_{t-1}.$$

*Proof.* First let us show that

$$\lim_{t\to\infty} R^*_t = 0. \tag{A.90}$$

To prove this, note that for all $j$,

$$\lim_{t\to\infty}\frac{s^{j*}_t}{(1+r^*_1)\cdots(1+r^*_t)} = 0.$$

Indeed, it follows from (A.88) that this limit exists. Since savings are non-negative, this limit is also non-negative. Suppose that $\lim_{t\to\infty}\frac{s^{j*}_t}{(1+r^*_1)\cdots(1+r^*_t)} > 0$. Then $s^{j*}_t > 0$ for all $t$, and thus by (A.79),

$$c^{j*}_t = \beta_j(1+r^*_t)c^{j*}_{t-1} = \ldots = \beta^t_j(1+r^*_t)\cdots(1+r^*_1)c^{j*}_0.$$

Hence

$$\frac{\beta^t_j s^{j*}_t}{c^{j*}_t} = \frac{s^{j*}_t}{(1+r^*_t)\cdots(1+r^*_1)c^{j*}_0} = \frac{1}{c^{j*}_0}\frac{s^{j*}_t}{(1+r^*_1)\cdots(1+r^*_t)}.$$

It follows from (A.80) that $\lim_{t\to\infty}\frac{s^{j*}_t}{(1+r^*_1)\cdots(1+r^*_t)} = 0$, which is a contradiction. Therefore,

$$\lim_{t\to\infty}\frac{\sum_{j=1}^{L}s^{j*}_t}{(1+r^*_1)\cdots(1+r^*_t)} = 0.$$

Since $\sum_{j=1}^{L} s_t^{j*} = q_t^* R_t^* + L k_{t+1}^*$, and both terms are non-negative, we get

$$\lim_{t\to\infty} \frac{q_t^* R_t^*}{(1+r_1^*)\cdots(1+r_t^*)} = \lim_{t\to\infty} q_0^* R_t^* = 0.$$

As $q_0^* > 0$, (A.90) indeed holds.

Now suppose that $t > T$. By Lemma A.6,

$$\begin{aligned}\beta_1(1+r_t^*)\left(Lk_t^* + \frac{q_t^*}{1+r_t^*}R_{t-1}^*\right) &= \beta_1(1+r_t^*)\sum_{j\in J} s_{t-1}^{j*} \\ &\leq \sum_{j\in J} s_t^{j*} = Lk_{t+1}^* + \frac{q_{t+1}^*}{1+r_{t+1}^*}R_t^*.\end{aligned} \tag{A.91}$$

At the same time, by Lemma A.4,

$$k_{t+1}^* \leq \beta_1(1+r_t^*)k_t^*, \quad t = 0, 1, \ldots.$$

Therefore for $t > T$,

$$\frac{q_{t+1}^*}{1+r_{t+1}^*}R_t^* \geq \beta_1(1+r_t^*)\frac{q_t^*}{1+r_t^*}R_{t-1}^*,$$

or, equivalently,

$$R_t^* \geq \beta_1 R_{t-1}^*. \tag{A.92}$$

It follows from the natural balance of exhaustible resources and (A.90) that

$$\begin{aligned}R_T^* = R_{T+1}^* + Le_{T+1}^* = R_{T+2}^* + Le_{T+2}^* + Le_{T+1}^* = \ldots \\ = Le_{T+1}^*\left(1 + \frac{e_{T+2}^*}{e_{T+1}^*} + \frac{e_{T+3}^*}{e_{T+2}^*}\frac{e_{T+2}^*}{e_{T+1}^*} + \ldots\right).\end{aligned}$$

Hence, using Lemma A.5, we conclude that

$$R_T^* \leq Le_{T+1}^*\left(1 + \beta_1 + \beta_1^2 + \ldots\right) = Le_{T+1}^*\frac{1}{1-\beta_1}.$$

It follows that

$$(1-\beta_1)\left(R_{T+1}^* + Le_{T+1}^*\right) \leq Le_{T+1}^*,$$

or

$$R_{T+1}^* \leq \beta_1\left(R_{T+1}^* + Le_{T+1}^*\right) = \beta_1 R_T^*.$$

Thus, using (A.92) we get

$$R_{T+1}^* = \beta_1 R_T^*.$$

Repeating the argument, we obtain that

$$R_t^* = \beta_1 R_{t-1}^*, \quad t > T.$$

Therefore, for all $t > T$, $Le_t^* = R_{t-1}^* - R_t^* = (1-\beta_1)R_{t-1}^*$, and

$$\frac{e_{t+1}^*}{e_t^*} = \frac{R_t^*}{R_{t-1}^*} = \beta_1, \quad t > T.$$

We have proved that eventually the extraction rate becomes constant over time and equal to $\varepsilon_t = 1 - \beta_1$ ($t > T$).

Since for $t > T$

$$\frac{q_{t+1}^*}{1+r_{t+1}^*} R_t^* = q_t^* R_t^* = \beta_1 q_t^* R_{t-1}^*,$$

it follows from (A.91) that

$$\beta_1(1+r_t^*)k_t^* \le k_{t+1}^*, \quad t > T.$$

Using Lemma A.4, we obtain that

$$k_{t+1}^* = \beta_1(1+r_t^*)k_t^*, \quad t > T,$$

and hence for $t > T$, $s_t^{j*} = \beta_1(1+r_t^*)s_{t-1}^{j*}$ $(j \in J)$, while $s_t^{j*} = 0$ $(j \notin J)$. □

Proposition A.2 is a corollary of Lemma A.8.

Proposition A.1 also easily follows from Lemma A.8. If the initial state $\mathcal{I}_0$ is such that

$$\hat{k}_0^j = 0, \qquad \hat{R}_{-1}^j = 0 \quad (j \notin J),$$

then $s_{-1}^j = 0$ $(j \notin J)$, and we can take $T = -1$. The sequences $\{r_t^*\}$, $\{w_t^*\}$, and $\{q_t^*\}$ are derived from the known sequences $\{k_t^*\}$ and $\{e_t^*\}$, described in Lemma A.8. □

Thus, in every competitive equilibrium from some time onward only the most patient agents make positive savings, and from this time resources are extracted at the constant rate $\varepsilon^* = 1 - \beta_1$.

## A.2 Balanced-growth equilibrium

**Definition A.3.** *A competitive equilibrium*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

*starting from a non-degenerate initial state $\mathcal{I}_0$ is called a balanced-growth equilibrium if there exist an equilibrium rate of balanced growth $\gamma^*$ and an equilibrium extraction rate $\varepsilon^*$ such that for $t = 0, 1, \dots,$*

$$c_{t+1}^{j*} = (1+\gamma^*)c_t^{j*}, \qquad s_t^{j*} = (1+\gamma^*)s_{t-1}^{j*}, \quad j = 1, \dots, L, \tag{A.93}$$

$$k_{t+1}^* = (1+\gamma^*)k_t^*, \qquad w_{t+1}^* = (1+\gamma^*)w_t^*, \tag{A.94}$$

$$1 + r_t^* = 1 + r^*, \qquad q_{t+1}^* = (1+r^*)\, q_t^*, \tag{A.95}$$

$$e_{t+1}^* = (1-\varepsilon^*)\, e_t^*, \qquad R_t^* = (1-\varepsilon^*)\, R_{t-1}^*. \tag{A.96}$$

The following proposition proves the existence of a balanced-growth equilibrium, and provides its characterization. In particular, it maintains that in every balanced-growth equilibrium less patient agents make no savings.

**Proposition A.3.** *A balanced-growth equilibrium*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

*starting from a non-degenerate initial state $\mathcal{I}_0 = \{(\hat{k}_0^j)_{j=1}^L, (\hat{R}_{-1}^j)_{j=1}^L\}$ exists if and only if*

$$\hat{k}_0^j = 0, \quad \hat{R}_{-1}^j = 0 \quad (j \notin J), \tag{A.97}$$

$$\alpha_1 A_0 \left(\frac{1}{L}\sum_{j=1}^L \hat{k}_0^j\right)^{\alpha_1 - 1} \left(\frac{1-\beta_1}{L}\sum_{j=1}^L \hat{R}_{-1}^j\right)^{\alpha_3} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}}, \tag{A.98}$$

*and (A.93)–(A.96) hold.*

*Proof.* **Necessity**. Suppose that there exists a balanced-growth equilibrium

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

starting from a non-degenerate state $\mathcal{I}_0$. It is a competitive equilibrium which satisfies (A.93)–(A.96) for some $r^*$, $\varepsilon^*$ and $\gamma^*$.

Repeating a well-known argument by Becker (1980, 2006), we infer that every balanced-growth equilibrium is characterized by the following properties:

$$s_{t-1}^{j*} = 0 \quad (j \notin J), \quad t = 0, 1, \dots, \tag{A.99}$$

$$1 + \gamma^* = \beta_1(1 + r^*). \tag{A.100}$$

Moreover, comparing the definitions of competitive and balanced-growth equilibria, we obtain that for every balanced-growth equilibrium the following relationships hold:

$$(1 + \gamma^*)^{1-\alpha_1} = (1 + \lambda)\left(1 - \varepsilon^*\right)^{\alpha_3}, \tag{A.101}$$

$$1 + r^* = \frac{1 + \gamma^*}{1 - \varepsilon^*}. \tag{A.102}$$

Indeed, (A.101) follows from the fact that

$$1 = \frac{1 + r_{t+1}^*}{1 + r_t^*} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1 - 1}\left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3} = (1 + \lambda)\left(1 + \gamma^*\right)^{\alpha_1 - 1}\left(1 - \varepsilon^*\right)^{\alpha_3}.$$

We also have

$$1 + r^* = \frac{q_{t+1}^*}{q_t^*} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1}\left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3 - 1} = (1 + \lambda)\left(1 + \gamma^*\right)^{\alpha_1}\left(1 - \varepsilon^*\right)^{\alpha_3 - 1},$$

which is equivalent to (A.102).

Using (A.100)–(A.102), it is easily checked that

$$1 + r^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}}\beta_1^{\frac{\alpha_1 + \alpha_3 - 1}{1 - \alpha_1}}. \tag{A.103}$$

It follows from (A.99) that $s_{-1}^{j*} = 0$ for $j \notin J$. Since $\mathcal{I}_0$ is non-degenerate, $\hat{k}_0^j \geq 0$ and $\hat{R}_{-1}^j \geq 0$ for all $j$, and thus (A.97) holds.[21] Furthermore, a constant over time interest rate is consistent with the definition of a competitive equilibrium if and only if

$$1 + r^* = 1 + r_0^* = \alpha_1 A_0 (k_0^*)^{\alpha_1 - 1}(e_0^*)^{\alpha_3} = \alpha_1 A_0\left(\frac{1}{L}\sum_{j=1}^L \hat{k}_0^j\right)^{\alpha_1 - 1}\left(\frac{1 - \beta_1}{L}\sum_{j=1}^L \hat{R}_{-1}^j\right)^{\alpha_3}.$$

Taking into account (A.103), we obtain (A.98).

**Sufficiency**. Suppose that the initial state $\mathcal{I}_0$ is such that (A.97)–(A.98) hold. Consider the sequence

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

[21]Recall that a balanced-growth equilibrium is defined up to the distribution of physical capital and natural resources in the structure of each agent's savings. Individual holdings of capital and resources are indeterminate in an equilibrium. However, since we assumed for convenience that the initial state is non-degenerate, and initial savings of less patient agents must be zero, it follows that a balanced-growth equilibrium can start only from the state where individual holdings of capital and resources of less patient agents are zero.

starting from $\mathcal{I}_0$ and determined by (A.93)–(A.96).

It is easily checked that this sequence is a competitive equilibrium which is described in Proposition A.1, with the constant interest rate

$$1 + r_t^* = 1 + r_0^* = \alpha_1 A_0 \left( \frac{1}{L} \sum_{j=1}^{L} \hat{k}_0^j \right)^{\alpha_1 - 1} \left( \frac{1 - \beta_1}{L} \sum_{j=1}^{L} \hat{R}_{-1}^j \right)^{\alpha_3} .$$

Therefore, $\mathcal{E}^*$ is a competitive equilibrium which satisfies (A.93)–(A.96), i.e., a balanced-growth equilibrium. □

It follows that the interest rate $r^*$, the equilibrium extraction rate $\varepsilon^*$, and the equilibrium rate of balanced growth $\gamma^*$ are uniquely determined by the parameters of the model and are the same for every balanced-growth equilibrium.

**Proposition A.4.** *For every balanced-growth equilibrium,*

$$1 + \gamma^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}, \tag{A.104}$$

$$1 + r^* = \frac{1 + \gamma^*}{\beta_1}, \tag{A.105}$$

$$\varepsilon^* = 1 - \beta_1. \tag{A.106}$$

*Proof.* It is sufficient to repeat the argument used in the proof of Proposition A.3. Combining (A.100)–(A.102), we obtain (A.104)–(A.106). □

The following proposition maintains that every competitive equilibrium converges in some sense to a balanced-growth equilibrium.

**Proposition A.5.** *Every competitive equilibrium starting from an arbitrary non-degenerate initial state satisfies the following asymptotic properties:*

$$\lim_{t \to \infty} 1 + r_t^* = 1 + r^* = \frac{1 + \gamma^*}{\beta_1}, \tag{A.107}$$

$$\lim_{t \to \infty} \frac{k_{t+1}^*}{k_t^*} = \lim_{t \to \infty} \frac{w_{t+1}^*}{w_t^*} = 1 + \gamma^*, \tag{A.108}$$

$$\lim_{t \to \infty} \frac{s_{t+1}^{j*}}{s_t^{j*}} = 1 + \gamma^* \quad (j \in J), \tag{A.109}$$

$$\lim_{t \to \infty} \frac{c_{t+1}^{j*}}{c_t^{j*}} = 1 + \gamma^*, \quad j = 1, \ldots, L, \tag{A.110}$$

$$\lim_{t \to \infty} \frac{q_{t+1}^*}{q_t^*} = 1 + r^*, \tag{A.111}$$

*where* $1 + \gamma^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}$.

*Proof.* It is sufficient to consider $t > T$. It follows from Proposition A.2 that

$$\frac{1 + r_{t+1}^*}{1 + r_t^*} = \frac{A_{t+1}}{A_t} \left( \frac{k_{t+1}^*}{k_t^*} \right)^{\alpha_1 - 1} \left( \frac{e_{t+1}^*}{e_t^*} \right)^{\alpha_3} = (1 + \lambda) \left( \beta_1 (1 + r_t^*) \right)^{\alpha_1 - 1} (\beta_1)^{\alpha_3} ,$$

and thus

$$1 + r_{t+1}^* = (1 + \lambda) (\beta_1)^{\alpha_1 + \alpha_3 - 1} (1 + r_t^*)^{\alpha_1} .$$

Iterating, we get

$$1 + r^*_{t+1+n} = (1+\lambda)^{1+\alpha_1+\ldots+\alpha_1^n} (\beta_1)^{(\alpha_1+\alpha_3-1)(1+\alpha_1+\ldots+\alpha_1^n)} (1+r^*_t)^{\alpha_1^{n+1}},$$

and

$$\lim_{n\to\infty} 1 + r^*_{t+n+1} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}}.$$

From Lemma A.8 we know that $\frac{k^*_{t+1}}{k^*_t} = \beta_1(1+r^*_t)$. Moreover, $\frac{w^*_t}{k^*_t} = \frac{\alpha_2}{\alpha_1}(1+r^*_t)$. Now (A.108) is straightforward. It also follows from Lemma A.8 that $\frac{s^{j*}_{t+1}}{s^{j*}_t} = \beta_1(1+r^*_t)$ for $j \in J$, which proves (A.109).

Clearly, for $j \in J$

$$\frac{c^{j*}_t}{k^*_t} = (1-\beta_1)(1+r^*_t)\frac{s^{j*}_{t-1}}{k^*_t} + \frac{w^*_t}{k^*_t},$$

and thus the sequence $c^{j*}_t/k^*_t$ converges to a positive constant as $t \to \infty$. For $j \notin J$, we have simply $c^{j*}_t = w^*_t$. Thus consumption of all agents asymptotically grows at a constant rate. This proves (A.110).

It remains to note that (A.111) follows from the Hotelling rule. □

# B Appendix 2. Public property regime

## B.1 Competitive equilibrium under given extraction rates

Suppose that the economy at time $\tau$ is in the state $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})^L_{j=1}, \hat{R}_{\tau-1}\}$, where $(\hat{s}^j_{\tau-1})^L_{j=1}$ are agents' savings and $\hat{R}_{\tau-1}$ is the stock of natural resources. We suppose that $\mathcal{I}_{\tau-1}$ is a non-degenerate state, i.e.,

$$\hat{s}^j_{\tau-1} \geq 0, \quad j = 1, \ldots, L; \qquad \frac{1}{L}\sum_{j=1}^{L} \hat{s}^j_{\tau-1} > 0; \qquad \hat{R}_{\tau-1} > 0.$$

Suppose we are also given a sequence of extraction rates $\mathbb{E}_\tau = \{\varepsilon_t\}^\infty_{t=\tau}$. We call $\mathbb{E}_\tau$ non-degenerate if $0 < \varepsilon_t < 1$ for all $t \geq \tau$, and

$$0 < \liminf_{t\to\infty} \varepsilon_t \leq \limsup_{t\to\infty} \varepsilon_t < 1.$$

In other words, the sequence of extraction rates is non-degenerate if there exists $\delta > 0$ such that for all $t \geq \tau$ the following property holds:

$$\delta \leq \varepsilon_t \leq 1 - \delta. \tag{B.1}$$

Suppose we are given a sequence of extraction rates $\mathbb{E}_\tau$ and the resource stock $R_{\tau-1} = \hat{R}_{\tau-1}$. Then the volume of extraction $e_t$ and the dynamics of the exhaustible resource stock $R_t$ are recursively determined for $t \geq \tau$:

$$e_t = e_t(\mathbb{E}_\tau) := \frac{\varepsilon_t R_{t-1}}{L}, \qquad R_t = R_t(\mathbb{E}_\tau) := (1-\varepsilon_t)R_{t-1}, \quad t = \tau, \tau+1, \ldots. \tag{B.2}$$

We use this notation to emphasize that the sequence of extraction rates determines the volume of extraction and the dynamics of the resource stock.

**Definition B.1.** *Let $\mathbb{E}_\tau$ be a non-degenerate sequence of extraction rates. A sequence*

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\ldots}$$

*is a competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$ if*

1. *For each $j = 1, \ldots, L$, the sequence $\{c_t^{j**}, s_t^{j**}\}_{t=\tau}^\infty$ is a solution to the following utility maximization problem:*

$$\begin{gathered} \max \sum_{t=\tau}^{\infty} \beta_j^t \ln c_t^j, \\ \text{s. t.} \quad c_t^j + s_t^j \le (1 + r_t)\, s_{t-1}^j + w_t + v_t, \quad t = \tau, \tau+1, \ldots, \\ s_t^j \ge 0, \quad t = \tau, \tau+1, \ldots \end{gathered} \tag{B.3}$$

   *at $r_t = r_t^{**}$, $w_t = w_t^{**}$, $v_t = v_t^{**}$, and $s_{\tau-1}^j = \hat{s}_{\tau-1}^j$;*

2. *Aggregate savings are equal to the capital stock:*

$$\sum_{j=1}^{L} s_{t-1}^{j**} = Lk_t^{**}, \quad t = \tau, \tau+1, \ldots;$$

3. *Capital is paid its marginal product:*

$$1 + r_t^{**} = \alpha_1 A_t (k_t^{**})^{\alpha_1 - 1} (e_t)^{\alpha_3}, \quad t = \tau, \tau+1, \ldots;$$

4. *Labor is paid its marginal product:*

$$w_t^{**} = \alpha_2 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3}, \quad t = \tau, \tau+1, \ldots;$$

5. *The price of natural resources is equal to the marginal product:*

$$q_t^{**} = \alpha_3 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3 - 1}, \quad t = \tau, \tau+1, \ldots;$$

6. *The resource income is given by:*

$$v_t^{**} = q_t^{**} e_t, \quad t = \tau, \tau+1, \ldots.$$

Here we do not suppose that the Hotelling rule holds. The Hotelling rule is an equilibrium condition for the asset market. This is the reason why the Hotelling rule holds in the private property regime, where the stock of natural resources is an asset in which agents can invest. In the public property regime, the resource stock is not an asset, so there is no particular reason for the Hotelling rule to hold. Under some circumstances the rate of change of the resource price is not equal to the interest rate.

It is clear that if

$$\mathcal{E}_0^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots}$$

is a competitive $\mathbb{E}_0$-equilibrium starting from $\{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\}$, then for each $\tau = 1, 2, \ldots$, the sequence

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\ldots}$$

is a competitive $\mathbb{E}_\tau$-equilibrium starting from $\{(s_{\tau-1}^{j**})_{j=1}^L, R_{\tau-1}(\mathbb{E}_0)\}$. In other words, competitive equilibria are time consistent.

The existence of a competitive equilibrium under given non-degenerate extraction rates is established in the following theorem.[22]

**Theorem B.1.** *For any non-degenerate state $\mathcal{I}_{\tau-1}$ there exists a competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$.*

The proof of Theorem B.1 is in many respects similar to the proof of Theorem A.1. However, for the sake of completeness, we provide below a full proof for Theorem B.1.

*Proof.* Without loss of generality, let us consider the case $\tau = 0$ and prove the existence of a competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1} = \{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\}$.

The proof is divided into two steps. First we show the existence of a competitive equilibrium in the finite horizon model. We prove that for any $T > 0$ there exists a finite $T$-period competitive equilibrium under given extraction rates. Second, we construct a candidate for a competitive equilibrium in the infinite horizon model by applying some kind of diagonalization procedure to the sequence of finite $T$-period equilibrium paths, and then prove that this candidate is indeed a competitive equilibrium in the infinite horizon model.

**Step I. Competitive equilibrium under given extraction rates in the finite horizon model**.

Let us define a finite $T$-period competitive equilibrium under given extraction rates along the lines of the above definition. Suppose that $\mathcal{I}_{-1} = \{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\}$ is a non-degenerate initial state, $\mathbb{E}_0 = \{\varepsilon_t\}_{t=0}^\infty$ is a non-degenerate sequence of extraction rates, and recall that $e_t = e_t(\mathbb{E}_0)$, $t = 0, 1, \ldots, T$.

**Definition B.2.** *A sequence*

$$\left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots,T}$$

*is a finite $T$-period competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1}$ if*

1. *For each $j = 1, \ldots, L$, the sequence $\{c_t^{j**}, s_t^{j**}\}_{t=0}^T$ is a solution to the following utility maximization problem:*

$$\begin{aligned} &\max \sum_{t=0}^T \beta_j^t \ln c_t^j, \\ \text{s. t.} \quad & c_t^j + s_t^j \le (1 + r_t)\, s_{t-1}^j + w_t + v_t, \quad t = 0, 1, \ldots, T, \\ & s_t^j \ge 0, \quad t = 0, 1, \ldots, T \end{aligned} \tag{B.4}$$

   *at $r_t = r_t^{**}$, $w_t = w_t^{**}$, $v_t = v_t^{**}$, and $s_{-1}^j = \hat{s}_{-1}^j$;*

2. *Aggregate savings are equal to the capital stock:*

$$\sum_{j=1}^L s_{t-1}^{j**} = Lk_t^{**}, \quad t = 0, 1, \ldots, T;$$

[22] The proof of Theorem B.1 is based on the ideas presented in Becker et al. (2015). The existence of equilibrium in the considered Ramsey-type model can be also proved along the lines of Becker et al. (1991).

3. *Capital is paid its marginal product:*

$$1 + r_t^{**} = \alpha_1 A_t (k_t^{**})^{\alpha_1 - 1} (e_t)^{\alpha_3}, \quad t = 0, 1, \dots, T;$$

4. *Labor is paid its marginal product:*

$$w_t^{**} = \alpha_2 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3}, \quad t = 0, 1, \dots, T;$$

5. *The price of natural resources is equal to the marginal product:*

$$q_t^{**} = \alpha_3 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3 - 1}, \quad t = 0, 1, \dots, T;$$

6. *The resource income is given by:*

$$v_t^{**} = q_t^{**} e_t, \quad t = 0, 1, \dots, T.$$

Clearly, the solution to the problem (B.4), $\{c_t^{j**}, s_t^{j**}\}_{t=0}^T$, satisfies the following conditions:

$$c_t^{j**} + s_t^{j**} = (1 + r_t^{**}) s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \dots, T, \tag{B.5}$$

$$c_{t+1}^{j**} \geq \beta_j (1 + r_{t+1}^{**}) c_t^{j**} \ (= \text{if } s_t^{j**} > 0), \quad t = 0, 1, \dots, T-1, \tag{B.6}$$

$$s_T^{j**} = 0,$$

where $s_{-1}^{j**} = \hat{s}_{-1}^j$.

The existence of a competitive equilibrium under given extraction rates in the finite horizon model is shown via the following steps. First we present some preliminary definitions and results that will be useful in what follows. Second, we reduce our finite horizon model to a game, and show that there exists a Nash equilibrium in this game. Third, we prove that a Nash equilibrium in the game that represents our model determines a competitive equilibrium under given extraction rates in the finite horizon model.

**Step I.1. Preliminaries.**

We use the notation

$$\begin{aligned} f(k, e, A) &:= A k^{\alpha_1} e^{\alpha_3}, \\ 1 + r(k, e, A) &:= \alpha_1 A k^{\alpha_1 - 1} e^{\alpha_3}, \\ w(k, e, A) &:= \alpha_2 A k^{\alpha_1} e^{\alpha_3}, \\ q(k, e, A) &:= \alpha_3 A k^{\alpha_1} e^{\alpha_3 - 1}, \\ v(k, e, A) &:= \alpha_3 A k^{\alpha_1} e^{\alpha_3}, \end{aligned}$$

for the output (production function), interest rate, wage rate, resource price and the resource income as depending on the capital stock $k$, the volume of extraction $e$ and total factor productivity $A$. Clearly,

$$(1 + r(k, e, A))k + w(k, e, A) + v(k, e, A) = f(k, e, A). \tag{B.7}$$

Denote

$$\bar{e} = \frac{\hat{R}_{-1}}{L}.$$

**Claim B.1.** *For all* $t$,

$$\bar{e}\delta^{t+1} \leq e_t \leq \bar{e}, \tag{B.8}$$

*and*

$$\frac{\delta^2}{1-\delta} < \frac{e_{t+1}}{e_t} < \frac{(1-\delta)^2}{\delta}. \tag{B.9}$$

*Proof.* It follows from (B.2) that for all $t > 0$,

$$e_t = \frac{\varepsilon_t R_{t-1}}{L} = \frac{\varepsilon_t(1-\varepsilon_{t-1})R_{t-2}}{L} = \ldots = \frac{\hat{R}_{-1}}{L}\varepsilon_t(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0).$$

It follows from (B.1) that for all $t \geq 0$,

$$\varepsilon_t \geq \delta, \qquad 1-\varepsilon_t \geq \delta. \tag{B.10}$$

Hence

$$e_t = \bar{e}\varepsilon_t(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0) \geq \bar{e}\delta^{t+1} > 0.$$

It is also clear that

$$e_t \leq \bar{e},$$

which proves (B.8).

Moreover, for all $t$,

$$\frac{e_{t+1}}{e_t} = \frac{\bar{e}\varepsilon_{t+1}(1-\varepsilon_t)(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0)}{\bar{e}\varepsilon_t(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0)} = \frac{\varepsilon_{t+1}(1-\varepsilon_t)}{\varepsilon_t}.$$

Using (B.10), we obtain (B.9). □

Denote

$$1+\bar{g} = (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{(1-\delta)^2}{\delta}\right)^{\frac{\alpha_3}{1-\alpha_1}},$$

where $\lambda$ is the growth rate of the total factor productivity:

$$A_t = (1+\lambda)A_{t-1} = (1+\lambda)^t A_0. \tag{B.11}$$

Let also

$$1+\tilde{g} = \min\left\{(1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\delta^2}{1-\delta}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \frac{A_0(\delta\bar{e})^{\alpha_3}}{(\hat{k}_0)^{1-\alpha_1}}\right\},$$

and

$$1+\underline{g} = \beta_L\alpha_1(1+\tilde{g}).$$

It is clear that

$$1+\tilde{g} \leq (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\delta^2}{1-\delta}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \tag{B.12}$$

and

$$1+\bar{g} > 1+\tilde{g} > 1+\underline{g}. \tag{B.13}$$

Suppose that $\bar{\kappa} > 0$ is given by

$$(1+\tilde{g})\bar{\kappa} = (\bar{\kappa})^{\alpha_1}, \tag{B.14}$$

Let the sequence $\{\bar{k}_t\}$ be given by

$$\bar{k}_{t+1} = (1+\bar{g})\bar{k}_t,$$

where

$$\bar{k}_0 = \bar{\kappa}(A_0\bar{e}^{\alpha_3})^{\frac{1}{1-\alpha_1}}.$$

We show that

$$\bar{\kappa}(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}} \leq \bar{k}_t. \tag{B.15}$$

It follows from (B.11), (B.9), and the choice of $\bar{k}_0$ that

$$\begin{aligned}
\bar{\kappa}(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}} &= \bar{\kappa}\left((1+\lambda)^t A_0 \left(\frac{e_t}{e_0}\right)^{\alpha_3} e_0^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} \\
&\leq \bar{\kappa}\left((1+\lambda)^t \left(\frac{(1-\delta)^2}{\delta}\right)^{t\alpha_3} A_0\bar{e}^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} \\
&= \bar{\kappa}\left(A_0\bar{e}^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} (1+\lambda)^{\frac{t}{1-\alpha_1}} \left(\frac{(1-\delta)^2}{\delta}\right)^{\frac{t\alpha_3}{1-\alpha_1}} = (1+\bar{g})^t\bar{k}_0 = \bar{k}_t.
\end{aligned}$$

Furthermore,

$$f(\bar{k}_t, e_t, A_t) < \bar{k}_{t+1}. \tag{B.16}$$

Indeed, by (B.15), (B.14) and (B.13),

$$\begin{aligned}
f(\bar{k}_t, e_t, A_t) - \bar{k}_{t+1} &= (\bar{k}_t)^{\alpha_1} A_t e_t^{\alpha_3} - (1+\bar{g})\bar{k}_t \\
&= \bar{k}_t\left(\frac{A_t e_t^{\alpha_3}}{(\bar{k}_t)^{1-\alpha_1}} - (1+\bar{g})\right) \leq \bar{k}_t\left(\frac{A_t e_t^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t e_t^{\alpha_3}} - (1+\bar{g})\right) \\
&= \bar{k}_t\left(\frac{\bar{\kappa}^{\alpha_1}}{\bar{\kappa}} - (1+\bar{g})\right) = \bar{k}_t\left((1+\tilde{g}) - (1+\bar{g})\right) < 0.
\end{aligned}$$

Denote

$$\bar{c}_t := L\bar{k}_{t+1}. \tag{B.17}$$

Clearly,

$$\bar{c}_{t+1} = (1+\bar{g})\bar{c}_t. \tag{B.18}$$

Let the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ be defined recursively as follows. We take $\tilde{k}_0$ such that $0 < \tilde{k}_0 < \hat{k}_0$. Suppose we are given $\tilde{k}_t > 0$. Consider the following equation in $k$:

$$k + \frac{\bar{c}_{t+1}}{\beta_L(1+r(k, e_{t+1}, A_{t+1}))} = f(\tilde{k}_t, e_t, A_t).$$

The left-hand side of the above equation is increasing in $k$, and equals to 0 when $k=0$. Thus there is a unique positive solution to this equation. We take $\tilde{k}_{t+1} > 0$ as this solution. Clearly, the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ satisfies the following equation:

$$\tilde{k}_{t+1} + \frac{\bar{c}_{t+1}}{\beta_L(1+r(\tilde{k}_{t+1}, e_{t+1}, A_{t+1}))} = f(\tilde{k}_t, e_t, A_t), \quad t = 0, 1, \ldots. \tag{B.19}$$

**Step I.2. A game**.

We reduce our finite horizon model to a game $\Gamma = (X_k, G_k)_{k\in I}$. To specify a game, we need to describe a set of players, $I$, and for each player $k \in I$, define the strategy set $X_k$ and the loss function

$$G_k \ : \ \prod_{i\in I} X_i \to \mathbb{R}.$$

Elements of $\prod_{i\in I} X_i$ are called multistrategies. The equilibrium of the game $\Gamma$ is defined as follows.

**Definition.** *A multistrategy $\left(x_1^*, \ldots, x_{|I|}^*\right)$ is called a Nash equilibrium of the game $\Gamma$ if for each $k \in I$, $x_k^*$ is a solution to*

$$\min_{x_k} G_k\left(x_1^*, \ldots, x_{k-1}^*, x_k, x_{k+1}^*, \ldots, x_{|I|}^*\right), \quad \text{s. t.} \quad x_k \in X_k.$$

The sufficient conditions for the existence of a Nash equilibrium of this game are well-known (see, e.g., Ichiishi, 2014): for each $k \in I$ the set $X_k$ is a convex and compact subset of a finite dimensional space, and the function $G_k(x_1, \ldots, x_k, \ldots, x_{|I|})$ is continuous in all variables and quasi-convex in $x_k$.

Let us specify the game $\Gamma_T$ that represents our model. There are $T+(2T+1)L$ players, and

1. for each $j = 1, \ldots, L$,

   (a) $T$ players determine $s_t^j$, $t = 0, 1, \ldots, T-1$, by solving

   $$\min_s s\left(c_{t+1}^j - \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j\right), \quad \text{s. t.} \quad 0 \le s \le L\bar{k}_{t+1}. \tag{B.20}$$

   (b) $T+1$ players determine $c_t^j$, $t = 0, 1, \ldots, T$, by solving

   $$\min_c \left|c - \left((1 + r(k_t, e_t, A_t))s_{t-1}^j + w(k_t, e_t, A_t) + v(k_t, e_t, A_t) - s_t^j\right)\right|, \quad \text{s. t.} \quad 0 \le c \le \bar{c}_t, \tag{B.21}$$

   where $s_{-1}^j = \hat{s}_{-1}^j$, and $s_T^j = 0$.

2. $T$ players determine $k_t$, $t = 1, 2, \ldots, T$, by solving

$$\min_k \left|k - \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^j\right|, \quad \text{s. t.} \quad \tilde{k}_t \le k \le \bar{k}_t. \tag{B.22}$$

**Lemma.** *There exists a Nash equilibrium in the game $\Gamma_T$ with $T + (2T + 1)L$ players having the strategy sets and loss functions described by (B.20)–(B.22).*

*Proof.* All strategy sets are closed intervals, and for each player the loss function is continuous in all variables and quasi-convex in the player's own strategy variable. Hence the sufficient conditions for the existence of a Nash equilibrium in the game $\Gamma_T$ are satisfied. □

**Step I.3. Nash equilibrium and competitive equilibrium**.

The following lemma maintains that a Nash equilibrium of the game $\Gamma_T$ determines a finite $T$-period competitive $\mathbb{E}_0$-equilibrium.

**Lemma B.1.** *Let*

$$\left\{(c_t^{j**})_{j=1,\ldots,L;t=0,1,\ldots,T},\, (s_t^{j**})_{j=1,\ldots,L;t=0,1,\ldots,T-1},\, (k_t^{**})_{t=1,2,\ldots,T}\right\}$$

*be a Nash equilibrium of the game $\Gamma_T$. Let $k_0^{**} = \hat{k}_0$, and $s_{-1}^{j**} = \hat{s}_{-1}^j$, $s_T^{j**} = 0$ for all $j$. Let also*

$$\begin{aligned} 1 + r_t^{**} &= 1 + r(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T, \\ w_t^{**} &= w(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T, \\ q_t^{**} &= q(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T, \\ v_t^{**} &= v(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T. \end{aligned}$$

*Then*

$$\left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots,T}$$

*is a finite $T$-period competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1}$.*

*Proof.* First, observe that

- if $c_{t+1}^j > \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then the only solution to the problem (B.20) is $s = 0$;
- if $c_{t+1}^j = \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then any $s$ from the interval $[0, L\bar{k}_{t+1}]$ is a solution to the problem (B.20);
- if $c_{t+1}^j < \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then the only solution to the problem (B.20) is $s = L\bar{k}_{t+1}$.

Second, notice that minimization problems (B.21) and (B.22) are of the form

$$\begin{aligned} &\min_x |x - \hat{x}|, \\ \text{s. t.} \quad &a_1 \le x \le a_2. \end{aligned}$$

The unique solution to this problem, $x^*$, is given by

$$x^* = \begin{cases} a_1, & \text{if} \quad \hat{x} < a_1; \\ a_2, & \text{if} \quad \hat{x} > a_2; \\ \hat{x}, & \text{if} \quad a_1 \le \hat{x} \le a_2. \end{cases}$$

**Remark B.1.** *When $\hat{x} \ge a_1$, we have $\hat{x} \ge x^*$.*

**Remark B.2.** *When $\hat{x} \le a_2$, we have $\hat{x} \le x^*$.*

Let $\left\{(c_t^{j**})_{j=1,\ldots,L;t=0,1,\ldots,T},\, (s_t^{j**})_{j=1,\ldots,L;t=0,1,\ldots,T-1},\, (k_t^{**})_{t=1,2,\ldots,T}\right\}$ be a Nash equilibrium of the game $\Gamma_T$. Note that for all $t = 0, 1, \ldots, T$, $k_t^{**} \ge \tilde{k}_t > 0$. It follows that for all $t = 0, 1, \ldots, T$, $w_t^{**} > 0$, $v_t^{**} > 0$, and $0 < 1 + r_t^{**} < \infty$.

We divide the proof of Lemma B.1 into several claims.

**Claim B.2.** *For each $j = 1, \dots, L$,*

$$0 < c_t^{j**} \le (1 + r_t^{**}) s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**}, \quad t = 0, 1, \dots, T, \tag{B.23}$$

*and hence*

$$0 < c_t^{j**} + s_t^{j**} \le (1 + r_t^{**}) s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \dots, T, \tag{B.24}$$

*Proof.* Assume the converse. Then, by the structure of the problem (B.21), there are $j$ and $0 \le \tau \le T$ such that (B.23) holds for $t < \tau$, and

$$0 = c_\tau^{j**} \ge (1 + r_\tau^{**}) s_{\tau-1}^{j**} + w_\tau^{**} + v_\tau^{**} - s_\tau^{j**}. \tag{B.25}$$

Consider two cases. First, let $\tau \le T - 1$. By (B.25),

$$s_\tau^{j**} \ge (1 + r_\tau^{**}) s_{\tau-1}^{j**} + w_\tau^{**} + v_\tau^{**} > 0.$$

Hence, by the structure of the problem (B.20),

$$c_{\tau+1}^{j**} \le \beta_j (1 + r_{\tau+1}^{**}) c_\tau^{j**} = 0,$$

because otherwise we would have $s_\tau^{j**} = 0$. Therefore, using Remark B.1, we conclude that

$$0 = c_{\tau+1}^{j**} \ge (1 + r_{\tau+1}^{**}) s_\tau^{j**} + w_{\tau+1}^{**} + v_{\tau+1}^{**} - s_{\tau+1}^{j**}.$$

Repeating the argument, and using the structure of the problem (B.20), we have

$$\begin{aligned} s_t^{j**} &> 0, \quad t = \tau, \tau+1, \dots, T-1, \\ c_{t+1}^{j**} &= 0, \quad t = \tau, \tau+1, \dots, T-1. \end{aligned}$$

However, $c_T^{j**} = 0$ is impossible, because $s_T^{j**} = 0$, and by the structure of the problem (B.21) we have

$$0 = c_T^{j**} = c_T^{j**} + s_T^{j**} \ge (1 + r_T^{**}) s_{T-1}^{j**} + w_T^{**} + v_T^{**} > 0,$$

a contradiction.

Second, let $\tau = T$. Since $c_{T-1}^{j**} > 0$, and $c_T^{j**} = 0$, we have

$$c_T^{j**} - \beta_j (1 + r_T^{**}) c_{T-1}^{j**} = -\beta_j (1 + r_T^{**}) c_{T-1}^{j**} < 0,$$

and, by the structure of the problem (B.20), $s_{T-1}^{j**} = L\bar{k}_T$. Using the fact that $s_T^{j**} = 0$, by the structure of the problem (B.21) we obtain

$$0 = c_T^{j**} + s_T^{j**} \ge (1 + r_T^{**}) s_{T-1}^{j**} + w_T^{**} + v_T^{**} > 0,$$

a contradiction. □

**Claim B.3.** *For each $j = 1, \dots, L$,*

$$(1 + r_t^{**}) s_{t-1}^{j**} + w_t^{**} + v_t^{**} \le L f(k_t^{**}, e_t, A_t), \quad t = 0, 1, \dots, T, \tag{B.26}$$

*and*

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j**} \le k_t^{**}, \quad t = 0, 1, \dots, T. \tag{B.27}$$

*Proof.* Using (B.24), (B.7), the bounds for $k$ in (B.22), and (B.16), for each $j = 1, \dots, L$, we obtain

$$\begin{aligned} c_0^{j**} + s_0^{j**} &\le (1 + r_0^{**})s_{-1}^{j**} + w_0^{**} + v_0^{**} \\ &\le \sum_{j=1}^{L} \left((1 + r_0^{**})s_{-1}^{j**} + w_0^{**} + v_0^{**}\right) \le L(1 + r_0^{**})k_0^{**} + Lw_0^{**} + Lv_0^{**} \\ &= Lf(k_0^{**}, e_0, A_0) \le Lf(\bar{k}_0, e_0, A_0) < L\bar{k}_1. \end{aligned}$$

Hence

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j**} \le \frac{1}{L}\sum_{j=1}^{L}\left(c_0^{j**} + s_0^{j**}\right) < \bar{k}_1,$$

and, by Remark B.2,

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j**} \le k_1^{**}.$$

Thus, inequalities (B.26) and (B.27) hold for $t = 0$. To obtain these inequalities for all $t \le T$, it is sufficient to repeat the argument. □

**Claim B.4.** *For each $j = 1, \dots, L$,*

$$c_t^{j**} + s_t^{j**} = (1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \dots, T. \tag{B.28}$$

*Proof.* It follows from the constraints in (B.20) that $s_t^{j**} \ge 0$ for all $t = 0, 1, \dots, T$. By (B.26) and (B.17),

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**} < L\bar{k}_{t+1} = \bar{c}_t.$$

Therefore, by the structure of the problem (B.21), for each $j = 1, \dots, L$,

$$c_t^{j**} \ge (1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**}, \quad t = 0, 1, \dots, T.$$

Combining this inequality with (B.23), we obtain (B.28). □

**Claim B.5.** *For each $j = 1, \dots, L$,*

$$c_{t+1}^{j**} \ge \beta_j(1 + r_{t+1}^{**})c_t^{j**} \ (= \textit{if } s_t^{j**} > 0), \quad t = 0, 1, \dots, T.$$

*Proof.* Assume that for some $j$ and $t < T$,

$$c_{t+1}^{j**} < \beta_j(1 + r_{t+1}^{**})c_t^{j**}.$$

Then, by the structure of the problem (B.20), $s_t^{j**} = L\bar{k}_{t+1}$. It follows from (B.26), the bounds for $k$ in (B.22), and (B.16) that

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} \le Lf(k_t^{**}, e_t, A_t) \le Lf(\bar{k}_t, e_t, A_t) < L\bar{k}_{t+1} = s_t^{j**},$$

and thus

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**} \le 0,$$

which contradicts (B.23). Thus we have proved that

$$c_{t+1}^{j**} \ge \beta_j(1 + r_{t+1}^{**})c_t^{j**}.$$

It remains to note that if

$$c_{t+1}^{j**} > \beta_j(1 + r_{t+1}^{**})c_t^{j**},$$

then by the structure of the problem (B.20), $s_t^{j**} = 0$. □

**Claim B.6.** *For all* $t = 0, 1, \ldots, T$,

$$k_t^{**} > \tilde{k}_t, \tag{B.29}$$

*and*

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j**} = k_t^{**}. \tag{B.30}$$

*Proof.* By the choice of $\tilde{k}_0$,

$$\frac{1}{L}\sum_{j=1}^{L} s_{-1}^{j**} = k_0^{**} > \tilde{k}_0.$$

Assume that for some $t = 1, 2, \ldots T$,

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-2}^{j**} = k_{t-1}^{**} > \tilde{k}_{t-1}, \text{ and } \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j**} \le k_t^{**} = \tilde{k}_t.$$

By (B.28),

$$\begin{aligned}\frac{1}{L}\sum_{j=1}^{L}\left(c_{t-1}^{j**} + s_{t-1}^{j**}\right) &= \frac{1}{L}\sum_{j=1}^{L}\left((1 + r_{t-1}^{**})s_{t-2}^{j**} + w_{t-1}^{**} + v_{t-1}^{**}\right) \\ &= (1 + r_{t-1}^{**})k_{t-1}^{**} + w_{t-1}^{**} + v_{t-1}^{**} = f(k_{t-1}^{**}, e_{t-1}, A_{t-1}) > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}).\end{aligned}$$

Hence

$$\frac{1}{L}\sum_{j=1}^{L} c_{t-1}^{j**} > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j**} \ge f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \tilde{k}_t.$$

Therefore, there is $j$ such that

$$c_{t-1}^{j**} > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \tilde{k}_t > 0. \tag{B.31}$$

Using (B.19), the bounds for $c$ in (B.21), and taking into account that $\tilde{k}_t = k_t^{**}$, we get

$$\begin{aligned}c_{t-1}^{j**} &> f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \tilde{k}_t \\ &= \frac{\bar{c}_t}{\beta_L(1 + r(\tilde{k}_t, e_t, A_t))} \ge \frac{\bar{c}_t}{\beta_j(1 + r(\tilde{k}_t, e_t, A_t))} \ge \frac{c_t^{j**}}{\beta_j(1 + r_t^{**})},\end{aligned}$$

and hence

$$c_t^{j**} \le \beta_j(1 + r_t^{**})c_{t-1}^{j**}.$$

It follows from the structure of the problem (B.20) that for this $j$ we have $s_{t-1}^{j**} = L\bar{k}_t$. By (B.28), (B.26), the bounds for $k$ in (B.22), and (B.16), we have

$$\begin{aligned}c_{t-1}^{j**} &= (1 + r_{t-1}^{**})s_{t-2}^{j**} + w_{t-1}^{**} + v_{t-1}^{**} - s_{t-1}^{j**} \\ &\le Lf(k_{t-1}^{**}, e_{t-1}, A_{t-1}) - s_{t-1}^{j**} \le Lf(\bar{k}_{t-1}, e_{t-1}, A_{t-1}) - L\bar{k}_t < 0,\end{aligned}$$

a contradiction of (B.31). This proves (B.29).

Now (B.30) follows from (B.27), (B.29), and the structure of the problem (B.22). □

Claims B.2–B.6 complete the proof of Lemma B.1. □

**Step II. Competitive equilibrium under given extraction rates in the infinite horizon model**.

**Step II.1. A candidate for an equilibrium path**.

Let for $T = 1, 2, \ldots$,

$$\mathcal{E}^{**}_{0,T} = \left\{(c^{j**}_t(T))^L_{j=1}, (s^{j**}_t(T))^L_{j=1}, k^{**}_t(T), r^{**}_t(T), w^{**}_t(T), q^{**}_t(T), v^{**}_t(T)\right\}_{t=0,1,\ldots,T}$$

be a finite $T$-period equilibrium path. Let us apply the following procedure to the sequence $\{\mathcal{E}^{**}_{0,T}\}_{T=1,2,\ldots}$.

At the first step of the process we take a cluster point of the sequence

$$\left\{(c^{j**}_0(T))^L_{j=1}, (s^{j**}_0(T))^L_{j=1}, k^{**}_0(T), r^{**}_0(T), w^{**}_0(T), q^{**}_0(T), v^{**}_0(T)\right\}_{T=1,2,\ldots},$$

denote it as

$$\left\{(c^{j**}_0)^L_{j=1}, (s^{j**}_0)^L_{j=1}, k^{**}_0, r^{**}_0, w^{**}_0, q^{**}_0, v^{**}_0\right\},$$

and extract a subsequence $\{T_{0n}\}^\infty_{n=1}$ from $\{T\}_{T=1,2,\ldots}$ such that

$$\left\{(c^{j**}_0(T_{0n}))^L_{j=1}, (s^{j**}_0(T_{0n}))^L_{j=1}, k^{**}_0(T_{0n}), r^{**}_0(T_{0n}), w^{**}_0(T_{0n}), q^{**}_0(T_{0n}), v^{**}_0(T_{0n})\right\}^\infty_{n=1}$$

converges to $\left\{(c^{j**}_0)^L_{j=1}, (s^{j**}_0)^L_{j=1}, k^{**}_0, r^{**}_0, w^{**}_0, q^{**}_0, v^{**}_0\right\}$.

At the second step we take a cluster point of the sequence

$$\left\{(c^{j**}_1(T_{0n}))^L_{j=1}, (s^{j**}_1(T_{0n}))^L_{j=1}, k^{**}_1(T_{0n}), r^{**}_1(T_{0n}), w^{**}_1(T_{0n}), q^{**}_1(T_{0n}), v^{**}_1(T_{0n})\right\}^\infty_{n=1},$$

denote it as

$$\left\{(c^{j**}_1)^L_{j=1}, (s^{j**}_1)^L_{j=1}, k^{**}_1, r^{**}_1, w^{**}_1, q^{**}_1, v^{**}_1\right\},$$

and extract a subsequence $\{T_{1n}\}^\infty_{n=1}$ from the sequence $\{T_{0n}\}^\infty_{n=1}$ such that $T_{11} > 1$, and

$$\left\{(c^{j**}_1(T_{1n}))^L_{j=1}, (s^{j**}_1(T_{1n}))^L_{j=1}, k^{**}_1(T_{0n}), r^{**}_1(T_{1n}), w^{**}_1(T_{1n}), q^{**}_1(T_{1n}), v^{**}_1(T_{1n})\right\}^\infty_{n=1}$$

converges to $\left\{(c^{j**}_1)^L_{j=1}, (s^{j**}_1)^L_{j=1}, k^{**}_1, r^{**}_1, w^{**}_1, q^{**}_1, v^{**}_1\right\}$. This procedure continues ad infinitum.

As a result, we obtain an infinite sequence

$$\mathcal{E}^{**}_{0,\infty} = \left\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=0,1,\ldots}. \tag{B.32}$$

This sequence is a natural candidate for a competitive equilibrium under given extraction rates in our model.

**Step II.2. Bounds of the $T$-period equilibrium capital sequence**.

We already know that every element of the capital sequence for any $T$-period finite equilibrium, $k^{**}_t$, is bounded from below by $\tilde{k}_t$. However, we need to establish a more precise estimate for the lower bound of the capital sequence of a $T$-period finite equilibrium. Again, we begin with some preliminary definitions.

Let the value $1 + r'$ be such that

$$\beta_L(1 + r') > 2(1 + \bar{g}), \tag{B.33}$$

and $k'$ be given by

$$\alpha_1(\delta\bar{e})^{\alpha_3}A_0(k')^{\alpha_1-1} = 1 + r'. \tag{B.34}$$

Let further the sequence $\{k'_t\}$ be given by

$$k'_{t+1} = (1+\underline{g})k'_t, \tag{B.35}$$

where

$$0 < k'_0 < \min\{\hat{k}_0, k'\}.$$

**Claim B.7.** *For all $t$,*

$$(1+\underline{g}) < \frac{f(k'_{t+1}, e_{t+1}, A_{t+1})}{f(k'_t, e_t, A_t)} < (1+\bar{g}). \tag{B.36}$$

*Proof.* By (B.35), (B.11), (B.9), (B.12) and (B.13),

$$\begin{aligned}\frac{f(k'_{t+1}, e_{t+1}, A_{t+1})}{f(k'_t, e_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}}\frac{A_{t+1}}{A_t}\frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} = \left(1+\underline{g}\right)^{\alpha_1}(1+\lambda)\frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} \\ &> \left(1+\underline{g}\right)^{\alpha_1}(1+\lambda)\left(\frac{\delta^2}{1-\delta}\right)^{\alpha_3} \geq \left(1+\underline{g}\right)^{\alpha_1}(1+\tilde{g})^{1-\alpha_1} > (1+\underline{g}).\end{aligned}$$

Analogously, using (B.35), (B.11), (B.13) and (B.9), we have

$$\begin{aligned}\frac{f(k'_{t+1}, e_{t+1}, A_{t+1})}{f(k'_t, e_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}}\frac{A_{t+1}}{A_t}\frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} = \left(1+\underline{g}\right)^{\alpha_1}(1+\lambda)\frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} \\ &< (1+\bar{g})^{\alpha_1}(1+\lambda)\frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} < (1+\bar{g})^{\alpha_1}(1+\lambda)\left(\frac{(1-\delta)^2}{\delta}\right)^{\alpha_3} = (1+\bar{g}).\end{aligned}$$

□

**Claim B.8.** *For all $t = 0, 1, \ldots, T$,*

$$1 + r(k'_t, e_t, A_t) > 1 + r'. \tag{B.37}$$

*Proof.* It follows from (B.35) and (B.36) that

$$\begin{aligned}1 + r(k'_t, e_t, A_t) &= \alpha_1 A_t e_t^{\alpha_3}(k'_t)^{\alpha_1-1} \\ &= \alpha_1\frac{f(k'_t, e_t, A_t)}{k'_t} > \alpha_1\frac{f(k'_{t-1}, e_{t-1}, A_{t-1})}{k'_{t-1}} = 1 + r(k'_{t-1}, e_{t-1}, A_{t-1}).\end{aligned}$$

Repeating the argument, and using (B.34), we get

$$\begin{aligned}1 + r(k'_t, e_t, A_t) > 1 + r(k'_0, e_0, A_0) &> 1 + r(k', e_0, A_0) \\ &= \alpha_1 A_0 e_0^{\alpha_3}(k')^{\alpha_1-1} > \alpha_1 A_0(\delta\bar{e})^{\alpha_3}(k')^{\alpha_1-1} = 1 + r'.\end{aligned}$$

□

Let

$$\begin{aligned}w'_{t+1} &= (1+\underline{g})w'_t, \quad t = 0, 1, \ldots, \\ v'_{t+1} &= (1+\underline{g})v'_t, \quad t = 0, 1, \ldots,\end{aligned}$$

where

$$\begin{aligned}w'_0 &= \alpha_2 A_0(k'_0)^{\alpha_1}(\delta\bar{e})^{\alpha_3} > 0, \\ v'_0 &= \alpha_3 A_0(k'_0)^{\alpha_1}(\delta\bar{e})^{\alpha_3} > 0.\end{aligned}$$

**Claim B.9.** *In any finite $T$-period competitive $\mathbb{E}_0$-equilibrium*

$$\left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots,T},$$

*for $t \leq T-1$,*

$$k_t^{**} > k_t' > 0, \tag{B.38}$$

*and*

$$\begin{aligned} w_t^{**} > w_t' > 0, \\ v_t^{**} > v_t' > 0. \end{aligned}$$

*Proof.* First let us prove (B.38). Assume the converse. Then there is $\tau > 0$ such that $k_\tau^{**} > k_\tau'$, and $k_{\tau+1}^{**} \leq k_{\tau+1}'$. It follows from (B.6) and (B.37), that for all $j$,

$$\begin{aligned} c_{\tau+1}^{j**} \geq \beta_j(1+r_{\tau+1}^{**})c_\tau^{j**} \geq \beta_L(1+r_{\tau+1}^{**})c_\tau^{j**} \\ = \beta_L(1+r(k_{\tau+1}^{**}, e_{\tau+1}, A_{\tau+1}))c_\tau^{j**} \geq \beta_L(1+r(k_{\tau+1}', e_{\tau+1}, A_{\tau+1}))c_\tau^{j**} \\ > \beta_L(1+r')c_\tau^{j**}. \end{aligned}$$

By (B.33),

$$c_{\tau+1}^{j**} > 2(1+\bar{g})c_\tau^{j**}. \tag{B.39}$$

Adding together the budget constraints of all agents at time $t$ in (B.5), and using condition 2 in Definition B.2, we get

$$\frac{1}{L}\sum_{j=1}^{L} c_t^{j**} + k_{t+1}^{**} = (1+r_t^{**})\, k_t^{**} + w_t^{**} + v_t^{**} = f(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T. \tag{B.40}$$

Applying (B.40) for $t = \tau+1$ and $t = \tau$, and using (B.39), we have

$$\begin{aligned} f(k_{\tau+1}^{**}, e_{\tau+1}, A_{\tau+1}) - k_{\tau+2}^{**} &= \frac{1}{L}\sum_{j=1}^{L} c_{\tau+1}^{j**} \\ &> 2(1+\bar{g})\frac{1}{L}\sum_{j=1}^{L} c_\tau^{j**} = 2(1+\bar{g})\left(f(k_\tau^{**}, e_\tau, A_\tau) - k_{\tau+1}^{**}\right). \end{aligned}$$

Hence, by the choice of $\tau$,

$$\begin{aligned} k_{\tau+2}^{**} < 2(1+\bar{g})k_{\tau+1}^{**} + f(k_{\tau+1}^{**}, e_{\tau+1}, A_{\tau+1}) - 2(1+\bar{g})f(k_\tau^{**}, e_\tau, A_\tau) \\ \leq (1+\bar{g})\left(2k_{\tau+1}' - f(k_\tau', e_\tau, A_\tau)\right) + f(k_{\tau+1}', e_{\tau+1}, A_{\tau+1}) - (1+\bar{g})f(k_\tau', e_\tau, A_\tau). \end{aligned} \tag{B.41}$$

It follows from (B.36) that

$$f(k_{\tau+1}', e_{\tau+1}, A_{\tau+1}) < (1+\bar{g})f(k_\tau', e_\tau, A_\tau). \tag{B.42}$$

Moreover, using (B.37) and (B.33), we get

$$\frac{f(k_\tau', e_\tau, A_\tau)}{k_\tau'} = \frac{1+r(k_\tau', e_\tau, A_\tau)}{\alpha_1} > \frac{1+r'}{\alpha_1} > \frac{2(1+\bar{g})}{\alpha_1\beta_L} > 2(1+\bar{g}),$$

and hence, by (B.35) and (B.13), we get

$$2k'_{\tau+1} = 2(1+\underline{g})k'_\tau < 2(1+\bar{g})k'_\tau < f(k'_\tau, e_\tau, A_\tau). \tag{B.43}$$

Combining (B.42) and (B.43), we have

$$(1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau, e_\tau, A_\tau)\right) + f(k'_{\tau+1}, e_{\tau+1}, A_{\tau+1}) - (1+\bar{g})f(k'_\tau, e_\tau, A_\tau) < 0.$$

Now it follows from (B.41) that $k^{**}_{\tau+2} < 0$, which is impossible. Hence $\tau + 2 > T$, and therefore the inequality (B.38) holds for $t \le T-1$.

Using (B.38), (B.11), (B.9), (B.12) and (B.13), we obtain that for all $t = 0, 1, \ldots, T-1$,

$$\begin{aligned} w^{**}_t &= \alpha_2 A_t (k^{**}_t)^{\alpha_1}(e_t)^{\alpha_3} > \alpha_2 A_t (k'_t)^{\alpha_1}\left(\frac{e_t}{e_0}\right)^{\alpha_3} e_0^{\alpha_3} \\ &> \alpha_2(1+\lambda)^t A_0 (1+\underline{g})^{t\alpha_1}(k'_0)^{\alpha_1}\left(\frac{\delta^2}{1-\delta}\right)^{t\alpha_3}(\delta\bar{e})^{\alpha_3} \\ &= (1+\underline{g})^{t\alpha_1}(1+\lambda)^t\left(\frac{\delta^2}{1-\delta}\right)^{t\alpha_3}\alpha_2 A_0 (k'_0)^{\alpha_1}(\delta\bar{e})^{\alpha_3} \\ &\ge (1+\underline{g})^{t\alpha_1}(1+\tilde{g})^{t(1-\alpha_1)}w'_0 > (1+\underline{g})^t w'_0 = w'_t. \end{aligned}$$

Applying the same argument, it can be easily seen that for $t \le T-1$,

$$v^{**}_t > v'_t > 0.$$

□

**Step II.3. Existence of an equilibrium**.

Now we are ready to prove the following lemma which maintains that the sequence $\mathcal{E}^{**}_{0,\infty}$ defined by (B.32) is a competitive equilibrium under given extraction rates in our model.

**Lemma B.2.** *The sequence $\mathcal{E}^{**}_{0,\infty}$ defined by (B.32) is a competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1}$.*

*Proof.* It is clear that by construction $\mathcal{E}^{**}_{0,\infty}$ satisfies conditions 2–6 of Definition B.1. Thus to prove that $\mathcal{E}^{**}_{0,\infty}$ is a competitive $\mathbb{E}_0$-equilibrium it is sufficient to show that $\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}\}^\infty_{t=0}$ is a solution to the problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta_j^t \ln c^j_t, \\ \text{s. t.}\quad & c^j_t + s^j_t \le (1+r^{**}_t)\, s^j_{t-1} + w^{**}_t + v^{**}_t, \quad t = 0, 1, \ldots, \\ & s^j_t \ge 0, \quad t = 0, 1, \ldots. \end{aligned} \tag{B.44}$$

Let $c'_t$ be such that

$$c'_t = \frac{1}{2}(w'_t + v'_t), \quad t = 0, 1, \ldots. \tag{B.45}$$

It is clear that

$$c'_{t+1} = (1+\underline{g})c'_t,$$

and hence

$$\sum_{t=0}^{\infty} \beta^t \ln c'_t = \frac{\ln c'_0}{1-\beta} + \ln(1+\underline{g}) \sum_{t=0}^{\infty} t\beta^t = \frac{\ln c'_0}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g}).$$

Consider the instantaneous utility function

$$u_t(c) = \ln c - \ln c'_t.$$

Clearly, the solution to the problem (B.44) will not change if we replace the instantaneous utility function $\ln c$ with the function $u_t(c)$. It is also clear that $u_t(c'_t) = 0$.

Note that for all $t$,

$$\bar{c}_t > Lf(\bar{k}_t, e_t, A_t) > Lf(k'_t, e_t, A_t) > c'_t,$$

and hence $u_t(\bar{c}_t) > 0$. Moreover, it follows from (B.18) that

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u_t(\bar{c}_t) &= \frac{\ln \bar{c}_0}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\bar{g}) - \frac{\ln c'_0}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g}) \\ &= \frac{1}{1-\beta} \ln\left(\frac{\bar{c}_0}{c'_0}\right) + \frac{\beta}{(1-\beta)^2} \ln\left(\frac{1+\bar{g}}{1+\underline{g}}\right). \end{aligned}$$

Now assume that $\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L\}_{t=0}^{\infty}$ is not a solution to the problem (B.44). Then for some $j$ (we fix this $j$ and omit it in the remaining part of the proof for the simplicity of notation) there is a feasible sequence $\{\widehat{c}_t, \widehat{s}_t\}_{t=0}^{\infty}$ such that

$$\widehat{U} > U^*, \text{ where } \widehat{U} = \sum_{t=0}^{\infty} \beta^t u_t(\widehat{c}_t), \text{ and } U^* = \sum_{t=0}^{\infty} \beta^t u_t(c_t^{**}).$$

Let $0 < \Delta < \widehat{U} - U^*$, and let $\Theta$ be such that

$$\sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \min\left\{\frac{\Delta}{2}, \ln 2\right\}.$$

Further, let

$$U^{*\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}), \quad \widehat{U}^{\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t),$$

and

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)), \quad U^{*\Theta}(T) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)),$$

for $T = \Theta+1, \Theta+2, \ldots$.

**Claim B.10.** *There is a sequence $\{T_{\Theta n}\}_{n=1}^{\infty}$ such that*

$$U^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} U^{*\Theta}.$$

*Proof.* It is sufficient to note that since $\mathcal{E}^{**}_{0,\infty}$ is obtained as a result of the application of the process described at Step II.1 to the sequence $\{\mathcal{E}^{**}_{0,T}\}_{T=1,2,\ldots}$, there is a sequence $\{T_{\Theta n}\}_{n=1}^{\infty}$ such that for $t = 0, 1, \ldots, \Theta$:

$$\lim_{n\to\infty} k_t^{**}(T_{\Theta n}) = k_t^{**}, \quad \lim_{n\to\infty} w_t^{**}(T_{\Theta n}) = w_t^{**}, \quad \lim_{n\to\infty} v_t^{**}(T_{\Theta n}) = v_t^{**},$$
$$\lim_{n\to\infty} r_t^{**}(T_{\Theta n}) = r_t^{**}, \quad \lim_{n\to\infty} c_t^{**}(T_{\Theta n}) = c_t^{**}, \quad \lim_{n\to\infty} s_t^{**}(T_{\Theta n}) = s_t^{**}.$$

□

Let $W^{*\Theta}$ be the maximum value of utility in the problem

$$\max \sum_{t=0}^{\Theta} \beta^t u_t(c_t),$$
$$\text{s. t.} \quad c_t + s_t \le (1 + r_t^{**})\, s_{t-1} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \ldots, \Theta,$$
$$s_t \ge 0, \quad t = 0, 1, \ldots, \Theta,$$

and $W^{*\Theta}(T)$ be the maximum value of utility in the problem

$$\max \sum_{t=0}^{\Theta} \beta^t u_t(c_t),$$
$$\text{s. t.} \quad c_t + s_t \le (1 + r_t^{**}(T))\, s_{t-1} + w_t^{**}(T) + v_t^{**}(T), \tag{B.46}$$
$$s_t \ge 0, \quad t = 0, 1, \ldots, \Theta,$$

for $T = \Theta + 1, \Theta + 2, \ldots$.

**Claim B.11.**

$$W^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} W^{*\Theta}.$$

*Proof.* Consider the correspondence that takes to each

$$\{(1 + r_0, w_0, v_0), \ldots, (1 + r_\Theta, w_\Theta, v_\Theta)\} \in \prod_{t=0}^{\Theta} \Big( [1 + r(\bar{k}_t, e_t, A_t), 1 + r(\tilde{k}_t, e_t, A_t)]$$
$$\times [w(\tilde{k}_t, e_t, A_t), w(\bar{k}_t, e_t, A_t)] \times [v(\tilde{k}_t, e_t, A_t), v(\bar{k}_t, e_t, A_t)] \Big)$$

the set

$$\{(c_0, s_0), \ldots, (c_\Theta, s_\Theta\} \in \mathbb{R}^{2(\Theta+1)}$$

which is such that, with $s_{-1} = \hat{s}_{-1}$ being given,

$$c_t + s_t \le (1 + r_t^{**}(T))\, s_{t-1} + w_t^{**}(T) + v_t^{**}(T), \text{ and } s_t \ge 0,$$

hold for all $t = 0, 1, \ldots, \Theta$.

By Statement 1, this correspondence is lower- and upper-semicontinuous, and it is sufficient to apply the Maximum Theorem. □

**Claim B.12.**

$$U^*(T) \ge W^{*\Theta}(T).$$

*Proof.* Let for some $T > \Theta + 1$, the sequence $\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_\Theta, \breve{s}_\Theta)\}$ be a solution to (B.46). Let further for $t = \Theta + 1, \ldots, T$, $\{(\breve{c}_t, \breve{s}_t)\}$ be defined recursively by

$$\breve{c}_t = c'_t, \quad \breve{s}_t = (1 + r^{**}_t(T))\, \breve{s}_{t-1} + w^{**}_t(T) + v^{**}_t(T) - \breve{c}_t. \tag{B.47}$$

We show that given $s_{-1} = \hat{s}_{-1}$, the sequence

$$\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_\Theta, \breve{s}_\Theta), (\breve{c}_{\Theta+1}, \breve{s}_{\Theta+1}), \ldots, (\breve{c}_T, \breve{s}_T)\} \tag{B.48}$$

is feasible for the problem

$$\begin{aligned} &\max \sum_{t=0}^{T} \beta^t u_t(c_t), \\ \text{s. t.} \quad & c_t + s_t \le (1 + r^{**}_t(T))\, s_{t-1} + w^{**}_t(T) + v^{**}_t(T), \\ & s_t \ge 0, \quad t = 0, 1, \ldots, T. \end{aligned} \tag{B.49}$$

It is sufficient to check that $\breve{s}_t \ge 0$ for $t = \Theta + 1, \ldots, T$. By Claim B.9, we have for $\Theta + 1 \le t \le T - 1$,

$$\breve{c}_t = c'_t = \frac{1}{2}(w'_t + v'_t) < w^{**}_t(T) + v^{**}_t(T).$$

We prove that $\breve{s}_t > 0$ for $t = \Theta + 1, \ldots, T - 1$ recursively. Clearly, $\breve{s}_\Theta = 0$. Suppose that $\breve{s}_{t-1} \ge 0$ for $\Theta + 1 \le t < T - 2$. Then

$$\breve{s}_t = (1 + r^{**}_t(T))\, \breve{s}_{t-1} + w^{**}_t(T) + v^{**}_t(T) - \breve{c}_t \ge w^{**}_t(T) + v^{**}_t(T) - c'_t > 0.$$

In particular, $\breve{s}_{T-2} > 0$. For $t = T - 1$ we have

$$\begin{aligned} \breve{s}_{T-1} = \big(1 + r^{**}_{T-1}(T)\big)\, \breve{s}_{T-2} + w^{**}_{T-1}(T) + v^{**}_{T-1}(T) - \breve{c}_{T-1} \\ \ge w^{**}_{T-1}(T) + v^{**}_{T-1}(T) - c'_{T-1} > 2c'_{T-1} - c'_{T-1} = c'_{T-1}. \end{aligned}$$

For $t = T$ we know from Claim B.9 that either

$$w^{**}_T(T) + v^{**}_T(T) > c'_T,$$

or

$$1 + r^{**}_T \ge 1 + r'.$$

In the first case, we can apply the same reasoning as before:

$$\breve{s}_T = (1 + r^{**}_T(T))\, \breve{s}_{T-1} + w^{**}_T(T) + v^{**}_T(T) - \breve{c}_T \ge w^{**}_T(T) + v^{**}_T(T) - c'_T > 0.$$

In the second case, using (B.33) and (B.13), we have

$$\begin{aligned} \breve{s}_T = (1 + r^{**}_T(T))\, \breve{s}_{T-1} + w^{**}_T(T) + v^{**}_T(T) - \breve{c}_T > (1 + r')c'_{T-1} - c'_T \\ > \frac{2}{\beta_L}(1 + \bar{g})c'_{T-1} - c'_T > (1 + \bar{g})c'_{T-1} - c'_T > (1 + \underline{g})c'_{T-1} - c'_T = 0. \end{aligned}$$

Thus we have proved that the sequence (B.48) is feasible for the problem (B.49). Since the sequence

$$\{(c^{**}_0(T), s^{**}_0(T)), \ldots, (c^{**}_T(T), s^{**}_T(T))\}$$

is the solution to this problem, we have

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c^{**}_t(T)) \ge \sum_{t=0}^{\Theta} \beta^t u_t(\breve{c}_t) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c'_t) = \sum_{t=0}^{\Theta} \beta^t u_t(\breve{c}_t) = W^{*\Theta}(T).$$

$\square$

Let us prove another useful claim.

**Claim B.13.** *For all* $t = 0, 1, \dots, T$,

$$k_t^{**}(T) \le \bar{\kappa}(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}. \tag{B.50}$$

*Proof.* It is sufficient to show that

$$\kappa_t^{**} \le \bar{\kappa}, \quad t = 0, 1, \dots, T, \tag{B.51}$$

where

$$\kappa_t^{**} := \frac{k_t^{**}(T)}{(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}}.$$

By (B.14), the choice of $1 + \tilde{g}$, and (B.8),

$$\bar{\kappa} = \frac{1}{(1+\tilde{g})^{\frac{1}{1-\alpha_1}}} \ge \frac{\hat{k}_0}{(A_0(\delta \bar{e})^{\alpha_3})^{\frac{1}{1-\alpha_1}}} \ge \frac{\hat{k}_0}{(A_0 e_0^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_0^{**},$$

which proves (B.51) for $t = 0$. We prove it for $t = 1, \dots, T$ recursively. Suppose that $\kappa_t^{**} \le \bar{\kappa}$. If follows from (B.40) that for all $t$,

$$k_{t+1}^{**}(T) \le f(k_t^{**}(T), e_t, A_t) = (k_t^{**}(T))^{\alpha_1} \left( (A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}} \right)^{1-\alpha_1},$$

and hence, due to (B.11), (B.9) and (B.12),

$$\begin{aligned}(\kappa_t^{**})^{\alpha_1} &\ge \frac{k_{t+1}^{**}(T)}{(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_{t+1}^{**} \frac{(A_{t+1} e_{t+1}^{\alpha_3})^{\frac{1}{1-\alpha_1}}}{(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_{t+1}^{**} \left( (1+\lambda) \left( \frac{e_{t+1}}{e_t} \right)^{\alpha_3} \right)^{\frac{1}{1-\alpha_1}} \\ &> \kappa_{t+1}^{**} (1+\lambda)^{\frac{1}{1-\alpha_1}} \left( \frac{\delta^2}{1-\delta} \right)^{\frac{\alpha_3}{1-\alpha_1}} \ge (1+\tilde{g}) \kappa_{t+1}^{**}.\end{aligned}$$

Therefore, by (B.14),

$$\kappa_{t+1}^{**} \le \frac{(\kappa_t^{**})^{\alpha_1}}{1+\tilde{g}} \le \frac{(\bar{\kappa})^{\alpha_1}}{1+\tilde{g}} = \bar{\kappa}.$$

Thus (B.51) holds for all $t = 0, 1, \dots, T$. □

Denote

$$1 + \bar{r} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}}.$$

By (B.14) and the choice of $1 + \underline{g}$,

$$\beta_L(1+\bar{r}) = \beta_L \alpha_1 \frac{(\bar{\kappa})^{\alpha_1}}{\bar{\kappa}} = \beta_L \alpha_1 (1+\tilde{g}) = (1+\underline{g}), \tag{B.52}$$

and hence, by (B.50), for all $t = 0, 1, \dots, T$, we have

$$1 + r_t^{**}(T) = \alpha_1 \frac{A_t e_t^{\alpha_3}}{(k_t^{**}(T))^{1-\alpha_1}} \ge \alpha_1 \frac{A_t e_t^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t e_t^{\alpha_3}} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}} = 1 + \bar{r}. \tag{B.53}$$

**Claim B.14.**

$$U^* \ge U^{*\Theta}.$$

*Proof.* Let us prove that for any $T > \Theta + 1$,

$$c_\Theta^{**}(T) \geq c'_\Theta. \tag{B.54}$$

Assume that $c_\Theta^{**}(T) < c'_\Theta$. We show that this inequality implies $c_t^{**}(T) < c'_t$ for all $t \leq \Theta$. Indeed, if $c_t^{**}(T) < c'_t$ for some $t < \Theta$, then it follows from (B.6), (B.53) and (B.52) that

$$c_t^{**}(T) \geq \beta_j(1 + r_t^{**}(T))c_{t-1}^{**}(T) \geq \beta_L(1 + \bar{r})c_{t-1}^{**}(T) = (1 + \underline{g})c_{t-1}^{**}(T), \tag{B.55}$$

and thus

$$c_{t-1}^{**}(T) \leq \frac{c_t^{**}(T)}{1 + \underline{g}} < \frac{c'_t}{1 + \underline{g}} = c'_{t-1}.$$

Hence

$$\sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)) = \sum_{t=0}^{\Theta} \beta^t \left(\ln c_t^{**}(T) - \ln c'_t\right) < 0.$$

At the same time, by the choice of $\Theta$, we have

$$\sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^{**}(T)) \leq \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \leq \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \ln 2.$$

Therefore

$$\sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^{**}(T)) < \ln 2. \tag{B.56}$$

Consider the sequence

$$\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_T, \breve{s}_T)\}, \tag{B.57}$$

defined as follows: for $t \leq \Theta$

$$\breve{c}_t = w_t^{**}(T) + v_t^{**}(T), \quad \breve{s}_t = 0,$$

and for $t = \Theta + 1, \ldots, T$, $\{(\breve{c}_t, \breve{s}_t)\}$ is given by (B.47). It follows from Claim B.9 that for $t \leq \Theta$,

$$w_t^{**}(T) + v_t^{**}(T) > w'_t + v'_t > c'_t. \tag{B.58}$$

Repeating the argument from the proof of Claim B.12, we obtain that the sequence (B.57) is feasible for the problem (B.49). At the same time, the sequence

$$\{(c_0^{**}(T), s_0^{**}(T)), \ldots, (c_T^{**}(T), s_T^{**}(T))\}$$

is the solution to this problem. Hence, using (B.58), we get

$$\begin{aligned}
\sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)) &\geq \sum_{t=0}^{T} \beta^t u_t(\breve{c}_t) = \sum_{t=0}^{\Theta} \beta^t u_t(w_t^{**}(T) + v_t^{**}(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c'_t) \\
&= u_0(w_0^{**}(T) + v_0^{**}(T)) + \sum_{t=1}^{\Theta} \beta^t u_t(\breve{c}_t) > u_0(w_0^{**}(T) + v_0^{**}(T)) \\
&= \ln(w_0^{**}(T) + v_0^{**}(T)) - \ln c'_0 > \ln(w'_0 + v'_0) - \ln c'_0 = \ln\left(\frac{w'_0 + v'_0}{c'_0}\right) = \ln 2,
\end{aligned}$$

a contradiction of (B.56).

Thus (B.54) holds, and using the fact that $c^{**}_\Theta$ is a limit of the sequence $\{c^{**}_\Theta(T_{\Theta n})\}_{n=1}^\infty$, we have

$$c^{**}_\Theta \geq c'_\Theta.$$

It immediately follows from (B.55) that for all $\Theta + 1 \leq t \leq T$,

$$c^{**}_t(T) \geq c'_t.$$

Since every $c^{**}_t$ is a cluster point of the sequence $\{c^{**}_t(T)\}_{T=1,2,\ldots}$, we get

$$c^{**}_t \geq c'_t, \quad t = \Theta + 1, \Theta + 2, \ldots.$$

It follows that

$$U^* - U^{*\Theta} = \sum_{t=\Theta+1}^{\infty} \beta^t u_t(c^{**}_t) = \sum_{t=\Theta+1}^{\infty} \beta^t \left(\ln(c^{**}_t) - \ln c'_t\right) \geq 0,$$

which completes the proof. □

**Claim B.15.**

$$U^{*\Theta}(T) > U^*(T) - \frac{\Delta}{2}, \quad T = \Theta + 1, \Theta + 2, \ldots; \tag{B.59}$$

$$W^{*\Theta} > \widehat{U} - \frac{\Delta}{2}. \tag{B.60}$$

*Proof.* Clearly, $\bar{c}_t > c^{**}_t(T)$ and $\bar{c}_t > \widehat{c}_t$ for all $t$. It follows from the choice of $\Theta$ that

$$\begin{aligned} U^*(T) - U^{*\Theta}(T) &= \sum_{t=0}^{T} \beta^t u_t(c^{**}_t(T)) - \sum_{t=0}^{\Theta} \beta^t u_t(c^{**}_t(T)) = \sum_{t=\Theta+1}^{T} \beta^t u_t(c^{**}_t(T)) \\ &\leq \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \leq \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \frac{\Delta}{2}, \end{aligned}$$

which proves (B.59).

Due to the definition of $W^{*\Theta}$, we have $W^{*\Theta} \geq \widehat{U}^\Theta$. Now it is easily seen that

$$W^{*\Theta} \geq \widehat{U}^\Theta = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t) = \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\widehat{c}_t) \geq \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) > \widehat{U} - \frac{\Delta}{2},$$

which proves (B.60). □

Now, combining Claims B.10–B.12 and B.14–B.15, we obtain

$$\begin{aligned} U^* \geq U^{*\Theta} &= \lim_{n\to\infty} U^{*\Theta}(T_{\Theta n}) \geq \lim_{n\to\infty} U^*(T_{\Theta n}) - \frac{\Delta}{2} \\ &\geq \lim_{n\to\infty} W^{*\Theta}(T_{\Theta n}) - \frac{\Delta}{2} = W^{*\Theta} - \frac{\Delta}{2} > \widehat{U} - \Delta, \end{aligned}$$

which contradicts the choice of $\Delta$. This contradiction completes the proof of the lemma. □

Thus the proof of Theorem B.1 is finally complete, and there exists a competitive equilibrium under given extraction rates. □

The issue with uniqueness of a competitive equilibrium under given extraction rates is more subtle. We can only conjecture that the competitive equilibrium is unique, but we have no proof of this fact. At the same time, the following proposition maintains that the competitive equilibrium starting from the state where the whole stock of physical capital is owned by the most patient agents is unique.

**Proposition B.1.** *Suppose that $\mathcal{I}_{\tau-1}$ is such that $\hat{s}^j_{\tau-1} = 0$ $(j \notin J)$. Then there exists a unique competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$,*

$$\mathcal{E}^{**}_\tau = \left\{ (c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t \right\}_{t=\tau,\tau+1,\ldots},$$

*which is given for $t = \tau, \tau+1, \ldots$ by*

$$\begin{gathered} c^{j**}_t = (1-\beta_1)(1+r^{**}_t)s^{j**}_{t-1} + w^{**}_t + v^{**}_t, \qquad s^{j**}_t = \beta_1(1+r^{**}_t)s^{j**}_{t-1} \quad (j \in J), \\ c^{j**}_t = w^{**}_t + v^{**}_t, \qquad s^{j**}_t = 0 \quad (j \notin J), \\ k^{**}_{t+1} = \beta_1(1+r^{**}_t)k^{**}_t, \qquad 1 + r^{**}_t = \alpha_1 A_t (k^{**}_t)^{\alpha_1 - 1}(e_t)^{\alpha_3}, \\ w^{**}_t = \alpha_2 A_t (k^{**}_t)^{\alpha_1}(e_t)^{\alpha_3}, \qquad q^{**}_t = \alpha_3 A_t (k^{**}_t)^{\alpha_1}(e_t)^{\alpha_3 - 1}, \qquad v^{**}_t = q^{**}_t e_t, \end{gathered}$$

*where $s^{j**}_{\tau-1} = \hat{s}^j_{\tau-1}$, and $e_t = e_t(\mathbb{E}_\tau)$.*

This case is important because in every competitive $\mathbb{E}_\tau$-equilibrium less patient agents inevitably lose their capital with time. The following proposition verifies that the whole capital stock eventually belongs to the most patient agents.

**Proposition B.2.** *Suppose that*

$$\mathcal{E}^{**}_\tau = \left\{ (c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t \right\}_{t=\tau,\tau+1,\ldots}$$

*is a competitive $\mathbb{E}_\tau$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$. Then there exists a point in time $T$ such that for all $t \geq T$,*

$$s^{j**}_t = 0 \quad (j \notin J).$$

***Proof of Propositions B.1 and B.2*** is very similar to that of Propositions A.1 and A.2. Without loss of generality, let us consider a competitive $\mathbb{E}_0$-equilibrium

$$\mathcal{E}^{**}_0 = \left\{ (c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t \right\}_{t=0,1,\ldots}$$

starting from $\mathcal{I}_{-1} = \{(\hat{s}^j_{-1})^L_{j=1}, \hat{R}_{-1}\}$, and give a sketch of the proof.

**Lemma B.3.** *Let $\beta > 0$ be such that for some $T$*

$$k^{**}_{t+1} > \beta(1+r^{**}_t)k^{**}_t = \beta\alpha_1 A_t (k^{**}_t)^{\alpha_1}(e_t)^{\alpha_3}, \quad t > T.$$

*If $\beta_j < \beta$, then $s^{j**}_t = 0$ for all sufficiently large $t$.*

*Proof.* This lemma can be proved in the same way as Lemma A.3. □

**Lemma B.4.**

$$k^{**}_{t+1} \leq \beta_1(1+r^{**}_t)k^{**}_t, \quad t = 0, 1, \ldots.$$

*Proof.* It is sufficient to repeat the argument used in the proof of Lemma A.4. □

**Lemma B.5.**

$$w_{t+1}^{**} \leq \beta_1(1 + r_{t+1}^{**})w_t^{**}, \quad t = 0, 1, \ldots,$$
$$v_{t+1}^{**} \leq \beta_1(1 + r_{t+1}^{**})v_t^{**}, \quad t = 0, 1, \ldots.$$

*Proof.* This statement follows from Lemma B.4. □

**Lemma B.6.**

$$s_{t+1}^{j**} \geq \beta_1(1 + r_{t+1}^{**})s_t^{j**} \quad (j \in J), \quad t = -1, 0, \ldots.$$

*Proof.* This lemma can be proved in the same way as Lemma A.6. □

**Lemma B.7.** *For all $\delta > 0$ there exists a point in time $T$ such that for all $t > T$,*

$$k_{t+1}^{**} > \beta_1(1 - \delta)(1 + r_t^{**})k_t^{**}.$$

*Proof.* This lemma can be proved in the same way as Lemma A.7. □

Proposition B.2 follows from Lemmas B.3 and B.7. Proposition B.1 follows directly from Lemma B.8 which explicitly constructs a competitive $\mathbb{E}_0$-equilibrium starting from the state $\mathcal{I}_{-1}$ such that $\hat{s}_{-1}^j = 0 \ \ (j \notin J)$.

**Lemma B.8.** *If*

$$k_0^{**} = \frac{1}{L}\sum_{j \in J} \hat{s}_{-1}^j, \ \ i.e., \ \ \hat{s}_{-1}^j = 0 \ \ (j \notin J),$$

*then for all $t = 0, 1, \ldots$,*

$$k_{t+1}^{**} = \beta_1(1 + r_t^{**})k_t^{**},$$
$$c_t^{j**} = (1 - \beta_1)(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \qquad s_t^{j**} = \beta_1(1 + r_t^{**})s_{t-1}^{j**} \quad (j \in J),$$
$$c_t^{j**} = w_t^{**} + v_t^{**}, \qquad s_t^{j**} = 0 \quad (j \notin J).$$

*Proof.* By Lemma B.6,

$$\beta_1(1 + r_0^{**})k_0^{**} = \beta_1(1 + r_0^{**})\frac{1}{L}\sum_{j \in J} \hat{s}_{-1}^j \leq \frac{1}{L}\sum_{j \in J} s_0^{j**} \leq k_1^{**}.$$

At the same time, by Lemma B.4,

$$k_1^{**} \leq \beta_1(1 + r_0^{**})k_0^{**}.$$

Therefore, $k_1^{**} = \beta_1(1 + r_0^{**})k_0^{**}$, and hence $s_0^{j**} = \beta_1(1 + r_0^{**})\hat{s}_{-1}^j \quad (j \in J)$, while $s_0^{j**} = 0 \ \ (j \notin J)$. We have proved the lemma for $t = 0$. To prove it for $t = 1, 2, \ldots$, it is sufficient to repeat the argument. □

This completes the proof of Propositions B.1 and B.2. □

## B.2 Balanced-growth equilibrium under given extraction rate

Suppose that the sequence of extraction rates is constant,

$$\mathbb{E}^{\varepsilon}_{\tau} = \mathbb{E}^{\varepsilon} := \{\varepsilon, \varepsilon, \varepsilon, \ldots\}.$$

Then, clearly,

$$R_t = (1-\varepsilon)^{t+1-\tau}\hat{R}_{\tau-1}, \qquad e_t = (1-\varepsilon)^{t-\tau}\frac{\varepsilon \hat{R}_{\tau-1}}{L}, \quad t = \tau, \tau+1, \ldots.$$

**Definition.** *A competitive $\mathbb{E}^{\varepsilon}_{\tau}$-equilibrium*

$$\mathcal{E}^{**}_{\tau} = \left\{(c_t^{j**})_{j=1}^{L}, (s_t^{j**})_{j=1}^{L}, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\ldots}$$

*starting from $\mathcal{I}_{\tau-1}$ is called a balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium if there exist an equilibrium rate of balanced growth $\gamma^{**}$ and the rate of change of the resource price, $\pi^{**}$, such that for $t = \tau, \tau+1, \ldots$,*

$$c_{t+1}^{j**} = (1+\gamma^{**})c_t^{j**}, \qquad s_t^{j**} = (1+\gamma^{**})s_{t-1}^{j**}, \quad j = 1, \ldots, L, \tag{B.61}$$

$$k_{t+1}^{**} = (1+\gamma^{**})k_t^{**}, \qquad w_{t+1}^{**} = (1+\gamma^{**})w_t^{**}, \qquad v_{t+1}^{**} = (1+\gamma^{**})v_t^{**}, \tag{B.62}$$

$$q_{t+1}^{**} = (1+\pi^{**})\, q_t^{**}, \qquad 1 + r_t^{**} = 1 + r^{**}. \tag{B.63}$$

The following proposition provides necessary and sufficient conditions for the existence of a balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium. In particular, this proposition maintains that in a balanced-growth equilibrium only the most patient agents make positive savings and own the whole capital stock.

**Proposition B.3.** *Suppose that a constant sequence of extraction rates $\mathbb{E}^{\varepsilon}$ is given. A balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium*

$$\mathcal{E}^{**}_{\tau} = \left\{(c_t^{j**})_{j=1}^{L}, (s_t^{j**})_{j=1}^{L}, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\ldots}$$

*starting from a non-degenerate state $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})_{j=1}^{L}, \hat{R}_{\tau-1}\}$ exists if and only if*

$$\hat{s}^j_{\tau-1} = 0, \quad j \notin J,$$

$$\alpha_1 A_\tau \left(\frac{1}{L}\sum_{j=1}^{L}\hat{s}^j_{\tau-1}\right)^{\alpha_1-1}\left(\frac{\varepsilon\hat{R}_{\tau-1}}{L}\right)^{\alpha_3} = (1+\lambda)^{\frac{1}{1-\alpha_1}}(1-\varepsilon)^{\frac{\alpha_3}{1-\alpha_1}}\frac{1}{\beta_1},$$

*and (B.61)–(B.63) hold.*

*Proof.* It can be proved exactly in the same way as Proposition A.3. □

The following proposition maintains that the interest rate, the equilibrium rate of balanced growth, and the rate of change of the resource price are determined by the parameters of the model and by the constant over time extraction rate $\varepsilon$.

**Proposition B.4.** *Suppose that a constant sequence of extraction rates $\mathbb{E}^{\varepsilon}$ is given. In a balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium, the interest rate, the equilibrium rate of balanced growth, and the rate of change of the resource price are determined as follows:*

$$1+\gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}}(1-\varepsilon)^{\frac{\alpha_3}{1-\alpha_1}},$$

$$1+\pi^{**} = \frac{1+\gamma^{**}}{1-\varepsilon},$$

$$1+r^{**} = \frac{1+\gamma^{**}}{\beta_1}.$$

*Proof.* The proof is similar to that of Proposition A.4. □

The following proposition maintains that every competitive $\mathbb{E}^{\varepsilon}_{\tau}$-equilibrium under given constant sequence of extraction rates converges in some sense to a balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium.

**Proposition B.5.** *Every competitive $\mathbb{E}^{\varepsilon}_{\tau}$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$ satisfies the following asymptotic properties:*

$$\lim_{t\to\infty} 1 + r^{**}_t = 1 + r^{**} = \frac{1+\gamma^{**}}{\beta_1},$$

$$\lim_{t\to\infty} \frac{k^{**}_{t+1}}{k^{**}_t} = \lim_{t\to\infty} \frac{w^{**}_{t+1}}{w^{**}_t} = \lim_{t\to\infty} \frac{v^{**}_{t+1}}{v^{**}_t} = 1+\gamma^{**},$$

$$\lim_{t\to\infty} \frac{s^{j**}_{t+1}}{s^{j**}_t} = 1+\gamma^{**} \quad (j \in J),$$

$$\lim_{t\to\infty} \frac{c^{j**}_{t+1}}{c^{j**}_t} = 1+\gamma^{**}, \quad j = 1,\dots,L,$$

$$\lim_{t\to\infty} \frac{q^{**}_{t+1}}{q^{**}_t} = 1+\pi^{**},$$

*where* $1+\gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1-\varepsilon)^{\frac{\alpha_3}{1-\alpha_1}}$, *and* $1+\pi^{**} = \frac{1+\gamma^{**}}{1-\varepsilon}$.

*Proof.* The proof is similar to that of Proposition A.5. □

## B.3 Time $\tau$ extraction rate

Before making extraction rates endogenous, let us explore the dependence of a competitive $\mathbb{E}_{\tau}$-equilibrium on the time $\tau$ extraction rate.

Suppose we are given a non-degenerate sequence of extraction rates $\mathbb{E}^0_{\tau} = \{\varepsilon^0_t\}^{\infty}_{t=\tau}$. Assume that $\varepsilon^0_{\tau}$ is replaced by some other extraction rate $\varepsilon_{\tau}$, while all future extraction rates remain intact. Clearly, the volumes of extraction and the dynamics of the resource stock before and after this replacement are linked in the following way:

$$\tilde{R}_t = \frac{1-\varepsilon_{\tau}}{1-\varepsilon^0_{\tau}} R_t, \qquad t = \tau, \tau+1, \dots,$$

$$\tilde{e}_{\tau} = \frac{\varepsilon_{\tau}}{\varepsilon^0_{\tau}} e_{\tau}, \quad \tilde{e}_t = \frac{1-\varepsilon_{\tau}}{1-\varepsilon^0_{\tau}} e_t, \qquad t = \tau+1, \tau+2, \dots.$$

A competitive $\mathbb{E}^0_{\tau}$-equilibrium should also change. The change of a competitive $\mathbb{E}^0_{\tau}$-equilibrium and the dependence of a new equilibrium on $\varepsilon_{\tau}$ is characterized in the following lemma.

**Lemma B.9.** *Suppose that for a non-degenerate sequence of extraction rates $\mathbb{E}^0_{\tau} = \{\varepsilon^0_t\}^{\infty}_{t=\tau}$ the sequence*

$$\mathcal{E}^{**}_{\tau} = \left\{ (c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t \right\}_{t=\tau,\tau+1,\dots}$$

*is a competitive $\mathbb{E}^0_{\tau}$-equilibrium starting from $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})^L_{j=1}, \hat{R}_{\tau-1}\}$.*

*Let*

$$\mathbb{E}_{\tau} = \{\varepsilon_{\tau}, \varepsilon^0_{\tau+1}, \varepsilon^0_{\tau+2}, \dots\},$$

*and*

$$\nu_\tau = \frac{1-\varepsilon_\tau}{1-\varepsilon^0_\tau}.$$

*Consider the sequence*

$$\tilde{\mathcal{E}}_\tau(\varepsilon_\tau) = \left\{(\tilde{c}^j_t(\varepsilon_\tau))^L_{j=1}, (\tilde{s}^j_t(\varepsilon_\tau))^L_{j=1}, \tilde{k}_t(\varepsilon_\tau), \tilde{r}_t(\varepsilon_\tau), \tilde{w}_t(\varepsilon_\tau), \tilde{q}_t(\varepsilon_\tau), \tilde{v}_t(\varepsilon_\tau)\right\}_{t=\tau,\tau+1,\ldots},$$

*given by*

$$\tilde{k}_{\tau+1}(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} k^{**}_{\tau+1}, \tag{B.64}$$

$$\tilde{k}_{t+1}(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} k^{**}_{t+1}, \quad t=\tau+1,\tau+2,\ldots, \tag{B.65}$$

$$\tilde{w}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} w^{**}_\tau, \tag{B.66}$$

$$\tilde{w}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} w^{**}_t, \quad t=\tau+1,\tau+2,\ldots, \tag{B.67}$$

$$\tilde{v}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} v^{**}_\tau, \tag{B.68}$$

$$\tilde{v}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} v^{**}_t, \quad t=\tau+1,\tau+2,\ldots, \tag{B.69}$$

$$\tilde{c}^j_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} c^{j**}_\tau, \quad j=1,\ldots,L, \tag{B.70}$$

$$\tilde{c}^j_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} c^{j**}_t, \quad t=\tau+1,\tau+2,\ldots, \quad j=1,\ldots,L, \tag{B.71}$$

$$\tilde{s}^j_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} s^{j**}_\tau, \quad j=1,\ldots,L, \tag{B.72}$$

$$\tilde{s}^j_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} s^{j**}_t, \quad t=\tau+1,\tau+2,\ldots, \quad j=1,\ldots,L, \tag{B.73}$$

$$1+\tilde{r}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} (1+r^{**}_\tau), \tag{B.74}$$

$$1+\tilde{r}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1-1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3\alpha_1^{t-\tau-1}} (1+r^{**}_t), \quad t=\tau+1,\tau+2,\ldots, \tag{B.75}$$

$$\tilde{q}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3-1} q^{**}_\tau, \tag{B.76}$$

$$\tilde{q}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})-1} q^{**}_t, \quad t=\tau+1,\tau+2,\ldots. \tag{B.77}$$

*The sequence $\tilde{\mathcal{E}}_\tau(\varepsilon_\tau)$ is a competitive $\mathbb{E}_\tau$-equilibrium starting from the same state $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})^L_{j=1}, \hat{R}_{\tau-1}\}$.*

This lemma plays a very important role in our further considerations. If both the competitive $\mathbb{E}^0_\tau$-equilibrium and the competitive $\mathbb{E}_\tau$-equilibrium are unique, then (B.64)–(B.77) provides formulas of transition from the equilibrium before the change of the time $\tau$ extraction rate to the equilibrium after the change. If we cannot guarantee the uniqueness of a competitive $\mathbb{E}_\tau$-equilibrium, then the interpretation of this lemma is slightly different. It maintains that after the change of the time $\tau$ extraction rate, there exists a competitive equilibrium which is given by (B.64)–(B.77).

*Proof.* For the simplicity of exposition let us slightly abuse the notation and write simply $\tilde{k}_t$, $\tilde{w}_t$, etc., instead of $\tilde{k}_t(\varepsilon_\tau)$, $\tilde{w}_t(\varepsilon_\tau)$, etc.

Obviously, $\tilde{k}_\tau = k^{**}_\tau$, as the initial state is the same.

Directly from (B.64)–(B.65) and (B.72)–(B.73), we get

$$\tilde{k}_{t+1} = \sum_{j=1}^{L} \tilde{s}^j_t, \quad t = \tau, \tau+1, \ldots.$$

It is also clear that capital, labor and natural resources are paid their marginal products:

$$1 + \tilde{r}_\tau = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} (1 + r^{**}_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} \alpha_1 A_\tau (e^{**}_\tau)^{\alpha_3} (k^{**}_\tau)^{\alpha_1 - 1} = \alpha_1 A_\tau (\tilde{e}_\tau)^{\alpha_3} (\tilde{k}_\tau)^{\alpha_1 - 1},$$

$$\begin{aligned}
1 + \tilde{r}_{\tau+1} &= \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1 - 1)} \nu_\tau^{\alpha_3} (1 + r^{**}_{\tau+1}) \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1 - 1)} \nu_\tau^{\alpha_3} \alpha_1 A_{\tau+1} (e^{**}_{\tau+1})^{\alpha_3} (k^{**}_{\tau+1})^{\alpha_1 - 1} \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1 - 1)} \nu_\tau^{\alpha_3} \alpha_1 A_{\tau+1} (\tilde{e}_{\tau+1})^{\alpha_3} (\tilde{k}_{\tau+1})^{\alpha_1 - 1} \nu_\tau^{-\alpha_3} \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(1 - \alpha_1)} \\
&= \alpha_1 A_{\tau+1} (\tilde{e}_{\tau+1})^{\alpha_3} (\tilde{k}_{\tau+1})^{\alpha_1 - 1},
\end{aligned}$$

$$\begin{aligned}
1 + \tilde{r}_t &= \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1 - 1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3 \alpha_1^{t-\tau-1}} (1 + r^{**}_t) \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1 - 1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3 \alpha_1^{t-\tau-1}} \alpha_1 A_t (e^{**}_t)^{\alpha_3} (k^{**}_t)^{\alpha_1 - 1} \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3(\alpha_1 - 1)\alpha_1^{t-\tau-1}} \cdot \nu_\tau^{\alpha_3 \alpha_1^{t-\tau-1}} \cdot \alpha_1 A_t (\tilde{e}_t)^{\alpha_3} (\tilde{k}_t)^{\alpha_1 - 1} \cdot \nu_\tau^{-\alpha_3} \cdot \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{(\alpha_3 \alpha_1^{t-\tau-1})(1 - \alpha_1)} \cdot \\
&\quad \cdot \nu_\tau^{\alpha_3(1 + \alpha_1 + \ldots + \alpha_1^{t-\tau-2})(1 - \alpha_1)} = \alpha_1 A_t (\tilde{e}_t)^{\alpha_3} (\tilde{k}_t)^{\alpha_1 - 1}, \quad t = \tau+2, \tau+3, \ldots;
\end{aligned}$$

$$\tilde{w}_\tau = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} w^{**}_\tau = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} \alpha_2 A_\tau (k^{**}_\tau)^{\alpha_1} (e^{**}_\tau)^{\alpha_3} = \alpha_2 A_\tau (\tilde{k}_\tau)^{\alpha_1} (\tilde{e}_\tau)^{\alpha_3},$$

$$\tilde{w}_{\tau+1} = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3} w_{\tau+1}^{**}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3} \alpha_2 A_{\tau+1} (k_{\tau+1}^{**})^{\alpha_1} (e_{\tau+1}^{**})^{\alpha_3}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3} \alpha_2 A_{\tau+1} (\tilde{k}_{\tau+1})^{\alpha_1} (\tilde{e}_{\tau+1})^{\alpha_3} \nu_\tau^{-\alpha_3} \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(-\alpha_1)}$$

$$= \alpha_2 A_{\tau+1} (\tilde{k}_{\tau+1})^{\alpha_1} (\tilde{e}_{\tau+1})^{\alpha_3},$$

$$\tilde{w}_t = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} w_t^{**}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} \alpha_2 A_t (k_t^{**})^{\alpha_1} (e_t^{**})^{\alpha_3}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \cdot \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})} \cdot \alpha_2 A_t (\tilde{k}_t)^{\alpha_1} (\tilde{e}_t)^{\alpha_3} \cdot \nu_\tau^{-\alpha_3} \cdot \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{(\alpha_3\alpha_1^{t-\tau-1})(-\alpha_1)} \cdot$$

$$\cdot \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-2})(-\alpha_1)} = \alpha_2 A_t (\tilde{k}_t)^{\alpha_1} (\tilde{e}_t)^{\alpha_3}, \quad t = \tau+2, \tau+3, \ldots,$$

$$\tilde{q}_\tau = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3-1} q_\tau^{**} = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3-1} \alpha_3 A_\tau (k_\tau^{**})^{\alpha_1} (e_\tau^{**})^{\alpha_3-1} = \alpha_3 A_\tau (\tilde{k}_\tau)^{\alpha_1} (\tilde{e}_\tau)^{\alpha_3-1},$$

$$\tilde{q}_{\tau+1} = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3-1} q_{\tau+1}^{**}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3-1} \alpha_3 A_{\tau+1} (k_{\tau+1}^{**})^{\alpha_1} (e_{\tau+1}^{**})^{\alpha_3-1}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3-1} \alpha_3 A_{\tau+1} (\tilde{k}_{\tau+1})^{\alpha_1} (\tilde{e}_{\tau+1})^{\alpha_3-1} \nu_\tau^{-\alpha_3+1} \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(-\alpha_1)}$$

$$= \alpha_3 A_{\tau+1} (\tilde{k}_{\tau+1})^{\alpha_1} (\tilde{e}_{\tau+1})^{\alpha_3-1},$$

$$\tilde{q}_t = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})-1} q_t^{**}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})-1} \alpha_3 A_t (k_t^{**})^{\alpha_1} (e_t^{**})^{\alpha_3-1}$$

$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})-1} \cdot \alpha_3 A_t (\tilde{k}_t)^{\alpha_1} (\tilde{e}_t)^{\alpha_3-1} \cdot \nu_\tau^{-\alpha_3+1} \cdot \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{(\alpha_3\alpha_1^{t-\tau-1})(-\alpha_1)} \cdot$$

$$\cdot \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-2})(-\alpha_1)} = \alpha_3 A_t (\tilde{k}_t)^{\alpha_1} (\tilde{e}_t)^{\alpha_3-1}, \quad t = \tau+2, \tau+3, \ldots.$$

Now it can be easily seen that

$$\tilde{v}_t = \tilde{q}_t \tilde{e}_t, \quad t = \tau, \tau+1, \dots.$$

It remains to show that the sequence $\{(\tilde{c}_t^j)_{j=1}^L, (\tilde{s}_t^j)_{j=1}^L\}_{t=\tau}^\infty$ is a solution to the following problem:

$$\max \sum_{t=\tau}^{\infty} \beta_j^t \ln c_t^j$$
$$\text{s. t.} \quad c_t^j + s_t^j = (1+r_t)s_{t-1}^j + w_t + v_t, \quad t = \tau, \tau+1, \dots,$$
$$s_t^j \geq 0, \quad t = \tau, \tau+1, \dots,$$

at $r_t = \tilde{r}_t$, $w_t = \tilde{w}_t$, and $v_t = \tilde{v}_t$.

Equivalently, it remains to show that the following conditions hold ($j = 1, \dots, L$):

$$\tilde{c}_t^j + \tilde{s}_t^j = (1+\tilde{r}_t)\tilde{s}_{t-1}^j + \tilde{w}_t + \tilde{v}_t, \quad t = \tau, \tau+1, \dots, \tag{B.78}$$

$$\tilde{c}_{t+1}^j \geq \beta_j (1+\tilde{r}_{t+1})\tilde{c}_t^j \ (= \text{if } \tilde{s}_t^j > 0), \quad t = \tau, \tau+1, \dots, \tag{B.79}$$

$$\frac{\beta_j^t \tilde{s}_t^j}{\tilde{c}_t^j} \xrightarrow[t\to\infty]{} 0. \tag{B.80}$$

Note that the sequence $\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L\}_{t=\tau}^\infty$ is a solution to maximization problem (B.3) and hence satisfies the following conditions:

$$c_t^{j**} + s_t^{j**} = (1+r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = \tau, \tau+1, \dots,$$
$$c_{t+1}^{j**} \geq \beta_j (1+r_{t+1}^{**})c_t^{j**} \ (= \text{if } s_t^{j**} > 0), \quad t = \tau, \tau+1, \dots,$$
$$\frac{\beta_j^t s_t^{j**}}{c_t^{j**}} \xrightarrow[t\to\infty]{} 0.$$

Consider (B.78) for $t = \tau$. We have

$$\tilde{c}_\tau^j + \tilde{s}_\tau^j = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} (c_\tau^{j**} + s_\tau^{j**}) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} \left((1+r_\tau^{**})\hat{s}_{\tau-1}^j + w_\tau^{**} + v_\tau^{**}\right)$$
$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} (1+r_\tau^{**})\hat{s}_{\tau-1}^j + \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} w_\tau^{**} + \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} v_\tau^{**} = (1+\tilde{r}_\tau)\hat{s}_{\tau-1}^j + \tilde{w}_\tau + \tilde{v}_\tau.$$

Consider (B.78) for $t = \tau+1$:

$$\tilde{c}_{\tau+1}^j + \tilde{s}_{\tau+1}^j = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3}(c_{\tau+1}^{j**} + s_{\tau+1}^{j**}) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3}\left((1+r_{\tau+1}^{**})s_\tau^{j**} + w_{\tau+1}^{**} + v_{\tau+1}^{**}\right)$$
$$= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)} \nu_\tau^{\alpha_3}(1+r_{\tau+1}^{**}) \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} s_\tau^{j**} + \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3}(w_{\tau+1}^{**} + v_{\tau+1}^{**})$$
$$= (1+\tilde{r}_{\tau+1})\tilde{s}_\tau^j + \tilde{w}_{\tau+1} + \tilde{v}_{\tau+1}.$$

The validity of conditions (B.78) for $t \geq \tau+2$ and (B.79) for $t \geq \tau$ can be proved similarly. It remains to notice that

$$\lim_{t\to\infty} \frac{\beta_j^t \tilde{s}_t^j}{\tilde{c}_t^j} = \lim_{t\to\infty} \beta_j^t \frac{\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}}}{\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}}} \frac{\nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})}}{\nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})}} \frac{s_t^{j**}}{c_t^{j**}} = \lim_{t\to\infty} \frac{\beta_j^t s_t^{j**}}{c_t^{j**}} = 0.$$

□

## B.4 Time $\tau$ voting equilibrium

We have characterized a competitive equilibrium and a balanced-growth equilibrium under given sequence of extraction rates. Now we make extraction rates endogenous and introduce voting into our model.

Suppose that we start at time $\tau$. The economy is in the non-degenerate state $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})^L_{j=1}, \hat{R}_{\tau-1}\}$. Suppose that agents have non-degenerate expectations about future extraction rates, $\{\varepsilon^e_t\}^\infty_{t=\tau+1}$, and they vote on the time $\tau$ extraction rate.

For any $\varepsilon_\tau \in (0,1)$ consider the sequence of extraction rates

$$\mathbb{E}_\tau(\varepsilon_\tau) = \{\varepsilon_\tau, \varepsilon^e_{\tau+1}, \varepsilon^e_{\tau+2}, \ldots\}.$$

Let us assume that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$:

$$\mathcal{E}^{**}_\tau(\varepsilon_\tau) = \left\{(c^{j**}_t(\varepsilon_\tau))^L_{j=1}, (s^{j**}_t(\varepsilon_\tau))^L_{j=1}, k^{**}_t(\varepsilon_\tau), r^{**}_t(\varepsilon_\tau), w^{**}_t(\varepsilon_\tau), q^{**}_t(\varepsilon_\tau), v^{**}_t(\varepsilon_\tau)\right\}_{t=\tau,\ldots}.$$

It is clear that $\mathcal{E}^{**}_\tau(\varepsilon_\tau)$ depends on the expectations and on the parameters of the model as well. However, here we underline its dependence only on $\varepsilon_\tau$, as it is the value on which agents vote.

Under the uniqueness assumption, agents' preferences over the time $\tau$ extraction rate are represented by the following indirect utility functions:

$$\mathcal{U}^j_\tau(\varepsilon_\tau) = \ln c^{j**}_\tau(\varepsilon_\tau) + \beta_j \ln c^{j**}_{\tau+1}(\varepsilon_\tau) + \ldots, \quad j = 1, \ldots, L.$$

**Definition.** *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$, $\mathcal{E}^{**}_\tau(\varepsilon_\tau)$. We call a couple $\{\varepsilon^{**}_\tau, \mathcal{E}^{**}_\tau\}$ a time $\tau$ voting equilibrium if $\varepsilon^{**}_\tau$ is a Condorcet winner in voting on the time $\tau$ extraction rate, and $\mathcal{E}^{**}_\tau = \mathcal{E}^{**}_\tau(\varepsilon^{**}_\tau)$.*

Since the functions $(\mathcal{U}^j(\varepsilon_\tau))^L_{j=1}$ are strictly concave, the agents' preferences are single-peaked. Hence the median voter theorem applies, and at each point in time there exists a Condorcet winner. Note that the time $\tau$ voting equilibrium consists of the time $\tau$ voting equilibrium extraction rate $\varepsilon^{**}_\tau$ and the corresponding competitive equilibrium.

In order to determine a Condorcet winner, let us consider *the preferred time $\tau$ extraction rate for agent $j$*. This is the value $\varepsilon^j_\tau$ such that

$$\mathcal{U}^j_\tau(\varepsilon^j_\tau) > \mathcal{U}^j_\tau(\varepsilon_\tau) \quad \forall\ \varepsilon_\tau \neq \varepsilon^j_\tau.$$

From Lemma B.9 we know how the consumption stream of every agent depends on the time $\tau$ extraction rate, which allows us to obtain agents' preferred values of time $\tau$ extraction rate.

**Proposition B.6.** *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$. The preferred time $\tau$ extraction rate for each agent $j$ is given by*

$$\varepsilon^j_\tau = 1 - \beta_j. \tag{B.81}$$

*Proof.* Let us take some $\varepsilon^0_\tau \in (0,1)$, and consider the sequence

$$\mathbb{E}_\tau(\varepsilon^0_\tau) = \{\varepsilon^0_\tau, \varepsilon^e_{\tau+1}, \varepsilon^e_{\tau+2}, \ldots\}.$$

By assumption, there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau^0)$-equilibrium

$$\mathcal{E}_\tau^{**}(\varepsilon_\tau^0) = \big\{(c_t^{j**}(\varepsilon_\tau^0))_{j=1}^L, (s_t^{j**}(\varepsilon_\tau^0))_{j=1}^L, k_t^{**}(\varepsilon_\tau^0), r_t^{**}(\varepsilon_\tau^0), w_t^{**}(\varepsilon_\tau^0), q_t^{**}(\varepsilon_\tau^0), v_t^{**}(\varepsilon_\tau^0)\big\}_{t=\tau,\ldots}$$

starting from $\mathcal{I}_{\tau-1}$. We use this equilibrium as a benchmark.

Further, for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium

$$\mathcal{E}_\tau^{**}(\varepsilon_\tau) = \big\{(c_t^{j**}(\varepsilon_\tau))_{j=1}^L, (s_t^{j**}(\varepsilon_\tau))_{j=1}^L, k_t^{**}(\varepsilon_\tau), r_t^{**}(\varepsilon_\tau), w_t^{**}(\varepsilon_\tau), q_t^{**}(\varepsilon_\tau), v_t^{**}(\varepsilon_\tau)\big\}_{t=\tau,\ldots}$$

starting from $\mathcal{I}_{\tau-1}$.

From Lemma B.9 we know that if the time $\tau$ extraction rate changes from $\varepsilon_\tau^0$ to $\varepsilon_\tau$, the benchmark equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau^0)$ transforms to the "new" equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ according to the formulas (B.64)–(B.77). In particular, the consumption stream of agent $j$ is given by (B.70)–(B.71). Therefore, the indirect utility function of agent $j$ in this equilibrium is given by:

$$\begin{aligned}
\mathcal{U}_\tau^j(\varepsilon_\tau) &= \ln c_\tau^{j**}(\varepsilon_\tau) + \beta_j \ln c_{\tau+1}^{j**}(\varepsilon_\tau) + \ldots \\
&= \alpha_3 \ln\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right) + \ln c_\tau^{j**}(\varepsilon_\tau^0) + \beta_j\alpha_3\alpha_1 \ln\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right) + \beta_j\alpha_3 \ln\left(\frac{1-\varepsilon_\tau}{1-\varepsilon_\tau^0}\right) + \beta_j \ln c_{\tau+1}^{j**}(\varepsilon_\tau^0) + \ldots \\
&= \Gamma^j + \alpha_3 \ln \varepsilon_\tau (1 + \beta_j\alpha_1 + \beta_j^2\alpha_1^2 + \ldots) \\
&\quad + \beta_j\alpha_3 \ln(1-\varepsilon_\tau)\big(1 + \beta_j(1+\alpha_1) + \beta_j^2(1+\alpha_1+\alpha_1^2) + \ldots\big) \\
&= \Gamma^j + \frac{\alpha_3}{1-\beta_j\alpha_1}\ln\varepsilon_\tau + \frac{\beta_j\alpha_3}{1-\beta_j}(1 + \beta_j\alpha_1 + \beta_j^2\alpha_1^2 + \ldots)\ln(1-\varepsilon_\tau) \\
&= \Gamma^j + \frac{\alpha_3}{1-\beta_j\alpha_1}\ln\varepsilon_\tau + \frac{\alpha_3}{1-\beta_j\alpha_1}\frac{\beta_j}{1-\beta_j}\ln(1-\varepsilon_\tau),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^j = \ln\left[\left(\frac{1}{\varepsilon_\tau^0}\right)^{\alpha_3} c_\tau^{j**}(\varepsilon_\tau^0)\right] &+ \beta_j \ln\left[\left(\frac{1}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1}\left(\frac{1}{1-\varepsilon_\tau^0}\right)^{\alpha_3} c_{\tau+1}^{j**}(\varepsilon_\tau^0)\right] + \ldots \\
&+ \beta_j^t \ln\left[\left(\frac{1}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^t}\left(\frac{1}{1-\varepsilon_\tau^0}\right)^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-1})} c_{\tau+t}^{j**}(\varepsilon_\tau^0)\right] + \ldots.
\end{aligned}$$

Note that $\Gamma^j$ does not depend on $\varepsilon_\tau$, on which agents vote. $\Gamma^j$ depends on the parameters of the model and on the characteristics (the extraction rate $\varepsilon_\tau^0$ and the consumption stream) of the benchmark equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau^0)$.

When voting on $\varepsilon_\tau$, agent $j$ maximizes her indirect utility $\mathcal{U}_\tau^j(\varepsilon_\tau)$, i.e., solves

$$\frac{\partial \mathcal{U}_\tau^j(\varepsilon_\tau)}{\partial \varepsilon_\tau} = 0.$$

This equation can be rewritten as

$$\frac{1}{\varepsilon_\tau} - \frac{\beta_j}{1-\beta_j}\frac{1}{1-\varepsilon_\tau} = 0.$$

The solution to this equation is $\varepsilon_\tau^j = 1 - \beta_j$. □

Proposition B.6 maintains that the preferred time $\tau$ extraction rate for every agent is constant over time and depends only on this agent's discount factor. In particular, the preferred time $\tau$ extraction rate for agent $j$ is time- and expectations-independent.

Now it is straightforward to see that the Condorcet winner in voting on the time $\tau$ extraction rate is

$$\varepsilon_\tau^{**} = 1 - \beta_{med},$$

where $\beta_{med}$ is the median discount factor. Thus the following theorem takes place.

**Theorem B.2.** *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$. Then there exists a unique time $\tau$ voting equilibrium $\{\varepsilon_\tau^{**}, \mathcal{E}_\tau^{**}\}$. The equilibrium extraction rate is constant over time and given by*

$$\varepsilon_\tau^{**} = \varepsilon^{**} = 1 - \beta_{med}. \tag{B.82}$$

## B.5 Intertemporal voting equilibrium

Suppose we are given an initial state $\mathcal{I}_{-1} = \{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\}$ and a non-degenerate sequence of extraction rates $\mathbb{E}^{**} = \mathbb{E}_0^{**} = \{\varepsilon_t^{**}\}_{t=0}^\infty$. Therefore, the volumes of extraction and the dynamics of the resource stock are also known:

$$e_t^{**} = e_t(\mathbb{E}^{**}), \qquad R_t^{**} = R_t(\mathbb{E}^{**}), \quad t = 0, 1, \ldots.$$

Let

$$\mathcal{E}_0^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots}$$

be a competitive $\mathbb{E}^{**}$-equilibrium starting from $\mathcal{I}_{-1}$. Let for $\tau = 1, 2, \ldots$

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\ldots}$$

be the corresponding tail of $\mathcal{E}_0^{**}$, which is a competitive $\mathbb{E}_\tau^{**}$-equilibrium starting from $\mathcal{I}_{\tau-1}^{**} = \{(s_{\tau-1}^{j**})_{j=1}^L, R_{\tau-1}^{**}\}$.

**Definition.** *We call a couple $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ an intertemporal voting equilibrium starting from $\mathcal{I}_{-1}$ if for each time $\tau = 0, 1, \ldots$, a couple $\{\varepsilon_\tau^{**}, \mathcal{E}_\tau^{**}\}$ is a time $\tau$ voting equilibrium starting from $\mathcal{I}_{\tau-1}^{**}$ under perfect foresight about future extraction rates ($\varepsilon_t^e = \varepsilon_t^{**},\ t = \tau+1, \tau+2, \ldots$).*

The following theorem provides the characterization of the sequence of extraction rates in every intertemporal voting equilibrium.

**Theorem B.3.** *In every intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ the sequence of extraction rates $\mathbb{E}^{**}$ is constant over time and given by*

$$\mathbb{E}^{**} = \mathbb{E}^{\varepsilon^{**}} = \{\varepsilon^{**}, \varepsilon^{**}, \ldots\}, \tag{B.83}$$

*where $\varepsilon^{**}$ is defined by (B.82).*

*Proof.* The sequence of extraction rates in every intertemporal voting equilibrium is the sequence of time $\tau$ equilibrium extraction rates. It follows from Theorem B.2 that every equilibrium extraction rate is constant and given by (B.82). □

The answer to the question about the existence and uniqueness of an intertemporal voting equilibrium is provided by the following theorem. It states that if the initial state is such that the whole capital stock belongs to the most patient agents, then an intertemporal voting equilibrium exists and is unique.

**Theorem B.4.** *Suppose that the initial state $\mathcal{I}_{-1}$ is such that $\hat{s}^j_{-1} = 0 \quad (j \notin J)$. Then there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ starting from $\mathcal{I}_{-1}$. The equilibrium sequence of extraction rates $\mathbb{E}^{**}$ is constant over time and defined by (B.83), and $\mathcal{E}_0^{**}$ is the unique competitive $\mathbb{E}^{**}$-equilibrium starting from $\mathcal{I}_{-1}$, as described in Proposition B.1.*

*Proof.* It follows from Proposition B.1 and Theorem B.3. □

## B.6 Balanced-growth voting equilibrium

**Definition.** *An intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ starting from $\mathcal{I}_{-1}$ is called a balanced-growth voting equilibrium if $\mathcal{E}_0^{**}$ is a balanced-growth $\mathbb{E}^{\varepsilon^{**}}$-equilibrium starting from $\mathcal{I}_{-1}$, where $\varepsilon^{**}$ is defined by (B.82).*

The following theorem maintains that if at the initial instant the whole capital stock belongs to the most patient agents, then the intertemporal voting equilibrium converges in some sense to a balanced-growth voting equilibrium.

**Theorem B.5.** *Suppose that the initial state $\mathcal{I}_{-1}$ is such that $\hat{s}^j_{-1} = 0 \quad (j \notin J)$. The unique intertemporal voting equilibrium starting from $\mathcal{I}_{-1}$ satisfies the following asymptotic properties:*

$$\lim_{t\to\infty} 1 + r_t^{**} = 1 + r^{**} = \frac{1+\gamma^{**}}{\beta_1}, \tag{B.84}$$

$$\lim_{t\to\infty} \frac{k_{t+1}^{**}}{k_t^{**}} = \lim_{t\to\infty} \frac{w_{t+1}^{**}}{w_t^{**}} = \lim_{t\to\infty} \frac{v_{t+1}^{**}}{v_t^{**}} = 1 + \gamma^{**}, \tag{B.85}$$

$$\lim_{t\to\infty} \frac{s_{t+1}^{j**}}{s_t^{j**}} = 1 + \gamma^{**} \quad (j \in J), \tag{B.86}$$

$$\lim_{t\to\infty} \frac{c_{t+1}^{j**}}{c_t^{j**}} = 1 + \gamma^{**}, \quad j = 1, \ldots, L, \tag{B.87}$$

$$\lim_{t\to\infty} \frac{q_{t+1}^{**}}{q_t^{**}} = 1 + \pi^{**}, \tag{B.88}$$

*where*

$$1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \left(\beta_{med}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \tag{B.89}$$

*and*

$$1 + \pi^{**} = \frac{1+\gamma^{**}}{\beta_{med}}. \tag{B.90}$$

*Proof.* It follows from Proposition B.5 and Theorem B.4. □

## B.7 Generalized intertemporal voting equilibria

Our definition of an intertemporal voting equilibrium is given under the assumption of uniqueness of a competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium for any $\varepsilon_\tau \in (0,1)$. This assumption is crucial in the statement of Theorem B.2 about the constant equilibrium extraction rate. Moreover, we obtained the existence and uniqueness of an intertemporal voting equilibrium (Theorem B.4) only for the case in which the underlying competitive equilibria are unique. Thus to guarantee the mere existence of an intertemporal voting equilibrium, we have to prove the uniqueness of a competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$ for any $\varepsilon_\tau \in (0,1)$, which is not an easy task.

Let us discuss the general case in which the competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$ is not necessarily unique. The difficulty here is that we cannot unambiguously define agents' indirect utility functions and obtain from them agents' preferred values of extraction rates. However, if we apply the technique proposed by Borissov et al. (2014b), we can get around this difficulty.

Namely, let us impose an additional assumption on the beliefs of agents. Assume that agents simply act as if a competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium is unique, and do not take into account the possible multiplicity of equilibria.

Formally, let

$$\mathbb{E}^{**} = \mathbb{E}_0^{**} = \{\varepsilon_t^{**}\}_{t=0}^{\infty},$$

and

$$e_t^{**} = e_t(\mathbb{E}^{**}), \qquad R_t^{**} = R_t(\mathbb{E}^{**}), \quad t = 0, 1, \dots.$$

Consider a competitive $\mathbb{E}_0^{**}$-equilibrium

$$\mathcal{E}_0^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\dots}$$

starting from $\mathcal{I}_{-1} = \left\{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\right\}$. Let also for $\tau = 1, 2, \dots$,

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\dots}$$

be the corresponding tail of $\mathcal{E}_0^{**}$.

Suppose that the economy has settled on $\mathcal{E}_0^{**}$. At each time $\tau$, when the economy is in the state $\mathcal{I}_{\tau-1}^{**} = \{(s_{\tau-1}^{j**})_{j=1}^L, R_{\tau-1}^{**}\}$, agents are asked to vote on the time $\tau$ extraction rate. To do this, agents' indirect utility functions should be unambiguously specified. Originally this was done under the assumption of uniqueness of the competitive $\mathbb{E}_\tau^{**}$-equilibrium starting from $\mathcal{I}_{\tau-1}^{**}$. Now let us instead assume that *when voting on the time $\tau$ extraction rate, all agents believe that if $\varepsilon_\tau^{**}$ is replaced by the other extraction rate $\varepsilon_\tau$, then the economy will settle on the path $\tilde{\mathcal{E}}_\tau(\varepsilon_\tau)$, which is linked with the "initial" equilibrium $\mathcal{E}_\tau^{**}$ in the way described in Lemma B.9.*

Recall that under the uniqueness assumption, the interpretation of Lemma B.9 is simple. After changing the time $\tau$ extraction rate from $\varepsilon_\tau^{**}$ to $\varepsilon_\tau$, the unique competitive $\mathbb{E}^{**}$-equilibrium also changes, and becomes the unique competitive $\mathbb{E}_\tau$-equilibrium, described in Lemma B.9. Here the interpretation is slightly different. After changing the time $\tau$ extraction rate, the competitive $\mathbb{E}^{**}$-equilibrium can change unpredictably, and the economy can in principle settle on one of multiple $\mathbb{E}_\tau$-equilibria. Under our assumption about agents' beliefs, agents ignore the possible multiplicity of equilibria and believe that after the change of the time $\tau$ extraction rate, the economy settles on the path $\tilde{\mathcal{E}}_\tau(\varepsilon_\tau)$, which is described in Lemma B.9.

Under this additional assumption, agents' indirect utility functions, which represent their preferences over the time $\tau$ extraction rate, can be defined unambiguously as follows:

$$\mathcal{U}_\tau^j(\varepsilon_\tau) = \ln \tilde{c}_\tau^j(\varepsilon_\tau) + \beta_j \ln \tilde{c}_{\tau+1}^j(\varepsilon_\tau) + \dots, \quad j = 1, \dots, L,$$

where the sequence $\{\tilde{c}_\tau^j(\varepsilon_\tau), \tilde{c}_{\tau+1}^j(\varepsilon_\tau), \dots\}$ is constructed according to (B.70)–(B.71).

**Definition.** *If for each $t = 0, 1, \dots$ there is a Condorcet winner in voting on $\varepsilon_t$ described above, and it coincides with $\varepsilon_t^{**}$, then we call a couple $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ a generalized intertemporal voting equilibrium starting from $\mathcal{I}_{-1}$.*

Clearly, any intertemporal voting equilibrium is a generalized intertemporal voting equilibrium. Moreover, any generalized intertemporal voting equilibrium starting from the initial state where the whole capital stock belongs to the most patient agents is an intertemporal voting equilibrium. Under the additional assumption about agents' beliefs, there always exists a generalized intertemporal voting equilibrium starting from an arbitrary initial state.

**Theorem B.6.** *For any non-degenerate initial state there exists a generalized intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ starting from this state. The equilibrium sequence of extraction rates is constant over time and given by (B.83).*

*Proof.* It is sufficient to repeat the argument used in the proof of Theorem B.3, and refer to Theorem B.1. □

Furthermore, every generalized intertemporal equilibrium converges in some sense to a balanced-growth voting equilibrium.

**Theorem B.7.** *Every generalized intertemporal equilibrium starting from an arbitrary initial state satisfies asymptotic properties (B.84)–(B.88), where $\gamma^{**}$ and $\pi^{**}$ are given by (B.89) and (B.90) respectively.*

*Proof.* It follows from Proposition B.5 and Theorem B.6. □

# C Appendix 3. Private property regime with capital taxation

Consider a competitive equilibrium in the private property regime, and assume in addition that the capital income paid by competitive firms to the capital holders is taxed at some constant rate $\theta$, and the revenue is lump-sum redistributed among all agents.

We define a competitive equilibrium with capital income tax,

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots},$$

similarly to the competitive equilibrium without tax. The definition of competitive equilibrium with capital income tax repeats Definition A.1, except for the following changes:

1′. *For each $j = 1, \dots, L$, the sequence $\{c_t^{j*}, s_t^{j*}\}_{t=0}^\infty$ is a solution to the following utility maximization problem:*

$$\max \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j,$$
$$\text{s. t.} \quad c_t^j + s_t^j \le (1 + r_t)(1 - \theta)\, s_{t-1}^j + w_t + \theta (1 + r_t)\, k_t \quad t = 0, 1, \dots,$$
$$s_t^j \ge 0, \quad t = 0, 1, \dots$$

*at* $r_t = r_t^*$, $w_t = w_t^*$, $k_t = k_t^*$, *and* $s_{-1}^j = \frac{q_0^*}{(1+r_0^*)(1-\theta)}\hat{R}_{-1}^j + \hat{k}_0^j$;

5′. *The Hotelling rule takes the form*

$$q_{t+1}^* = (1 + r_{t+1}^*)(1-\theta)q_t^*, \quad t = 0, 1, \ldots;$$

7′. *Total agents' savings are equal to the investment into physical capital and natural resources:*

$$\sum_{j=1}^{L} s_t^{j*} = \frac{q_{t+1}^*}{(1 + r_{t+1}^*)(1-\theta)} R_t^* + Lk_{t+1}^*, \quad t = 0, 1, \ldots.$$

Taking into account the new form of the Hotelling rule, we can define a balanced-growth equilibrium with capital taxation along the lines of Definition A.3. Slightly modifying the arguments from the proof of Propositions A.3 and A.4, we can provide a characterization of a balanced-growth equilibrium with capital taxation.

**Proposition C.1.** *For every balanced-growth equilibrium with capital taxation,*

$$1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}, \tag{C.1}$$

$$(1 + r^*)(1-\theta) = \frac{1+\gamma^*}{\beta_1}, \tag{C.2}$$

$$\varepsilon^* = 1 - \beta_1. \tag{C.3}$$

*Proof.* A balanced-growth equilibrium with capital taxation

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\ldots}$$

is a competitive equilibrium with capital taxation in which real variables grow at a constant rate $\gamma^*$, while the interest rate $r^*$ and extraction rate $\varepsilon^*$ are constant over time. In a competitive equilibria with capital taxation the post-tax interest rate received by agents is the pre-tax gross interest rate $(1+r^*)$ multiplied by $(1-\theta)$. Repeating an argument by Becker (1980, 2006), we obtain that a balanced-growth equilibrium with capital taxation is characterized as follows:

$$s_{t-1}^{j*} = 0 \quad (j \notin J), \quad t = 0, 1, \ldots,$$

$$1 + \gamma^* = \beta_1(1 + r^*)(1-\theta). \tag{C.4}$$

Moreover, since in a balanced-growth equilibrium with capital taxation the extraction rate is constant,

$$1 = \frac{1 + r_{t+1}^*}{1 + r_t^*} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1 - 1}\left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3} = (1+\lambda)(1+\gamma^*)^{\alpha_1-1}(1-\varepsilon^*)^{\alpha_3},$$

and we get

$$(1+\gamma^*)^{1-\alpha_1} = (1+\lambda)(1-\varepsilon^*)^{\alpha_3}. \tag{C.5}$$

We also have

$$(1 + r^*)(1-\theta) = \frac{q_{t+1}^*}{q_t^*} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1}\left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3 - 1} = (1+\lambda)(1+\gamma^*)^{\alpha_1}(1-\varepsilon^*)^{\alpha_3-1},$$

and it follows that

$$(1 + r^*)(1 - \theta) = \frac{1 + \gamma^*}{1 - \varepsilon^*}. \tag{C.6}$$

Comparing (C.4) and (C.6), we immediately obtain (C.3). Now (C.2) follows from (C.3) and (C.6), while (C.1) follows from (C.3) and (C.5). □

It follows that the long-run growth rate, the post-tax interest rate and the extraction rate in the model with capital income tax are the same as in a model without capital tax.

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