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JEL Classification: D91, O41, H41, D72

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Public Goods, Voting, and Growth*

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We study a parametric politico-economic model of economic growth with productive public goods and public consumption goods. The provision of public goods is funded by a proportional tax on consumers' income. Agents are heterogeneous in their initial capital endowments, discount factors and the relative weights of public consumption in overall private utility. They vote on the shares of public goods in GDP. We propose a definition of voting equilibrium, prove the existence and provide a characterization of voting equilibria, and obtain a closed-form solution for the voting outcomes.

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1 Introduction

Public intervention is one of the key factors explaining differences in economic growth patterns among countries in the world. The role of governments in the growth process has been at the heart of many theoretical and empirical contributions.

Public expenditures on goods and services are traditionally classified as productivity-enhancing or utlity-enhancing. Barro (1990), Barro and Sala-i-Martin (1992), Glomm and Ravikumar (1994a) and many others study the optimal level of government expenditures when they take the form of public production services (see de Haan and Romp, 2007, for a recent survey of empirical literature and Irmen and Kuehnel (2009) for a survey of theoretical literature). Bianconi and Turnovsky (1997), Devereux and Wen (1998) and others study the more conventional case in which government expenditures take the form of utility-enhancing public services that provide direct utility to households. Some analyses have included both aspects of public spending (see, e.g., Baier and Glomm (2011), Baxter and King (1993), Chang (1999), Chen (2006), Marrero (2010), Economides *et al* (2011)).

There are two alternative approaches to deal with public intervention in the economy: either a social planner is assumed to look for an optimal solution and policy instruments are then designed to decentralize this solution or policy decisions are assumed to be the outcome of a political process inside which policy makers and/or voters interact. As far as economic growth is concerned, both efficiency issues and equity issues can be dealt with both approaches.

In the vast majority of papers on economic growth with public intervention the shares of government expenditures are either exogenous or chosen by a benevolent planner. Papers in this strand of the literature usually analyze the growth- and/or welfare-maximizing size and composition of government expenditures and thus do not reflect fundamental characteristics of the process of collective decision-making in democratic societies. They cannot explain substantial variations among countries in their tax and expenditure policies, even among developed democracies sharing similar economic and political regimes.

The political-economy literature describes collective choice mechanisms and explains how fundamentals (preferences and technologies) together with political institutions determine political outcomes. The political-economy approach can help us understand the way in which cultural factors and economic conditions may combine to influence the size and composition of government expenditures.

One difficulty when applying the tools of politico-economic analysis in neoclassical growth models with rational optimizing agents is to account for voters expectations of current and future equilibrium prices and of future equilibrium policies (see e.g., Krusell et al. (1997) and Krusell and Ríos-Rull (1999) for a discussion of the problems in developing a dynamic version of Meltzer and Richard (1981) median voter model of the size of government).

Alesina and Rodrik (1994) ignore this difficulty. They consider a growth model in the spirit of Barro (1990), focusing on how income inequality affects policy and growth. A critical feature of their model is that taxes are voted on in period zero only and then are constant over time. These solutions are typically time-inconsistent. Kaas (2003) overcomes this difficulty in a framework of an overlapping generations model. Creedy *et al* (2011) consider an overlapping generations model where the difficulties related to expectations of future equilibrium policies do not arise. Darbya *et al* (2004) propose an infinite horizon model with no private capital, and examine the dynamic interaction of public expenditure decision and electoral turnover in an endogenous growth model. They assume a partisan-type political economy set-up in which two political parties alternate in power and focus on the link between political uncertainty in electoral outcomes in democracies and economic growth.

Our paper is most closely related to Glomm and Ravikumar (1994b) and Koulovatianos and Mirman (2004, 2005), who consider infinite-horizon economies where public sector investments and public consumption goods are financed by income taxes. They endogenize the provision of public goods through majority voting. In their models, households can differ only with respect to their initial endowments of private capital. However, this heterogeneity does not lead to any disagreement in voting. All agents vote unanimously.

Park and Philippopoulos (2003) set up a model where households also can differ only with respect to initial capital endowments. They assume that the government taxes capital income to finance transfer payments, public consumption services and public production services. In their model, voting outcomes are not unanimous, but eventually they restrict their attention to symmetric equilibria.

In this paper, we propose a parametric politico-economic model of economic growth with productive public goods that increases private production possibilities and public consumption goods that contributes to private utility. The government levies a proportional tax on the consumers income, which funds the provision of public goods.

A distinctive feature of our model is that the agents are heterogeneous not only in their initial endowments of private capital, but also in their discount factors and preferences for public consumption goods. A Ramsey-type model where consumers have different discount factors was introduced by Becker (1980). In that model, the long-run capital intensity is determined by the discount factor of the most patient consumer. It is reasonable to expect that in an endogenous growth model it is the long-run rate of growth that is determined by the discount factor of the most patient consumer. This is the case in Barro's model with heterogeneous agents, but only if the tax rates and the size of government expenditures are given exogenously. If agents vote on the size of government and the composition of expenditures, it is reasonable to expect that the discount factors of agents other than the most patient ones and the preferences for public consumption play their roles.

There is a rich literature on models of economic growth with consumers having different rates of impatience (for a survey, see Becker (2006)). However, most existing papers on this topic ignore public sector. Few exceptions include papers by Sarte (1997), Sorger (2002), Li and Sarte (2004) and Bosi and Seegmuller (2010), who study the impact of a progressive income tax on the long-run growth and distribution, but do not discuss voting.

In our model, agents vote on the shares of productive public goods and public consumption goods in GDP. We propose a definition of voting equilibrium, prove the existence of voting equilibria, provide their characterization and obtain a closed-form solution for the voting outcomes, which are shown to be constant over time. Finally, we undertake comparative static analysis of the shares of public goods in GDP and of the rate of balanced growth with respect to the discount factors and the preferences for public consumption.

Similar to Glomm and Ravikumar (1994a,b) and Koulovatianos and Mirman (2004), we assume logarithmic preferences, Cobb-Douglas technology and 100 per cent depreciation of capital. This assumption helps us obtain close-form solutions for voting equilibria and provides the following important simplification: at each time the outcomes of voting do not depend on the current state of the economy and on expectations about future voting outcomes. This simplification eliminates a strategic motive to influence the outcomes of future votes, hence they can be taken as given[1].

We proceed as follows. We first analyze competitive equilibria under given public policies. Then we endogenize the public policies by letting each agent at the beginning of each period vote on the shares of public goods in GDP in this period under some expectations about future shares.

First we carry out the analysis under the assumption that for any public policy the competitive equilibrium is unique. We prove that this is the case if all capital is owned by the most patient consumers (in particular, if the

[1] Models where voters can ignore the effect of their choices on future political environment due to logarithmic preferences are not untypical in the literature (see e.g., Glomm and Ravikumar (1994), Zhang et al. (2003), Gradstein and Kaganovich (2004), Koulovatianos and Mirman (2004), Creedy *et al* (2011)).

discount factors are the same for all agents).

At each time agents vote on two shares and hence the policy space is bidimensional. Therefore, generically a Condorcet winner fails to exist. To overcome this difficulty, following Kramer (1972) and Shepsle (1979), we assume that agents vote separately on the two shares within the same period. They cast their vote on each share assuming that the other share has been settled. A solution is consistent if the pair of shares obtained through that procedure is self-supporting in a Nash-like manner[2].

We prove that if for any public policy a competitive equilibrium is unique, then at each time there is a unique outcome of voting on the two shares. Moreover, we obtain explicit expressions for this outcome, which does not depend on the initial distribution of private capital and expectations. It follows that if at each time the shares of public goods in GDP are determined by voting, then they are constant over time. It should be noted that the winning shares are not necessarily those preferred by voters sharing the median values of the discount factor and the parameter indicating preferences for public consumption.

At first glance it would seem impossible to propose a consistent voting procedure if the uniqueness of a competitive equilibrium cannot be guaranteed for some policies. It turns out that this is not the case. We show that under some additional assumption about agents' beliefs it is possible to generalize the voting procedure in a consistent way to the case where the uniqueness of competitive equilibria is not ensured and that the outcome of this generalized voting procedure is the same as in the case of uniqueness.

The structure of the paper is as follows. Section 2 presents the main building blocks describing the production technology, the government spending and the agents' preferences. Section 3 provides a preliminary analysis of competitive equilibria assuming that the shares of public goods in GDP are given. It studies the existence and uniqueness of an intertemporal equilibrium and the key characteristics of a balanced growth equilibrium for given shares. Section 4 endogenizes the shares of public goods through a voting procedure and describes the outcome of voting under the assumption that for any public policy the competitive equilibrium is unique. Finally section 5 extends the analysis to the case where the uniqueness is not ensured. Section 6 concludes.

[2]There is another approach widely used in the literature on multidimensional voting, the Stackelberg one, which assumes that agents vote sequentially on each dimension with an exogenous ordering of the dimensions (see, e.g., De Donder *et al*, 2012). It can be shown that in our model the Stackelberg approach results in approximately the same picture as the Kramer-Shepsle one.

2 Main building blocks of the model

2.1 Production sector

The aggregate output (GDP) at each time t, Y_t, is given by a production function

$$Y_t = q(g_t)F(K_t, L),$$

where K_t is the capital stock, g_t is the time t per capita quantity of productive public goods and L is the input of labor, which is assumed to be equal to the number of agents (each agent supplies one unit of labor) and constant over time. Capital fully depreciates during one time period. We assume that the production function is Cobb-Douglas:

$$F(K, L) = K^{\alpha} L^{1-\alpha}, \qquad 0 < \alpha < 1,$$

and that

$$q(g) = g^{1-\alpha}.$$

Thus,

$$Y_t = g_t^{1-\alpha} K_t^{\alpha} L^{1-\alpha}.$$

Per capita GDP, $y_t = Y_t/L$, can be written as

$$y_t = q(g_t) f(k_t) = g_t^{1-\alpha} k_t^{\alpha},$$

where $k_t = K_t/L$ and $f(k_t) = k_t^{\alpha}$. The wage rate, w_t, and the gross interest rate, $1 + r_r$, are respectively the marginal products of labor and capital:

$$w_t = q(g_t)(f(k_t) - f'(k_t)k_t) \ (= (1-\alpha)(g_t)^{1-\alpha}(k_t)^{\alpha}),$$

$$1 + r_t = q(g_t) f'(k_t) \ (= \alpha(g_t/k_t)^{1-\alpha}).$$

In this paper we assume constant returns to scale to the reproducible factors[3].

2.2 Government

The government provides utility-enhancing public consumption goods and productive public goods. They are financed by the use of a proportional income tax. The government runs a balanced budget. Therefore, if the per

[3] It is noteworthy that most of our results concerning voting equilibria have their counterparts in a model with decreasing returns to scale (with somewhat more tedious formulas).

capita level of public consumption goods provision at time t is h_t and the per capita level of productive public good provision at time $t+1$ is g_{t+1}, then the tax rate is equal to $\theta_t + \lambda_t$, where

$$\theta_t := \frac{h_t L}{Y_t}$$

is the share of the public consumption goods in GDP and

$$\lambda_t := \frac{g_{t+1} L}{Y_t}$$

is the share of the productive public goods. We assume that the part of taxes collected in period t and used for the productive public goods is spent in period $t+1$.

2.3 Consumers

There are $L > 1$ agents indexed by $j = 1, \ldots, L$. L is assumed to be odd. Agent j is endowed with one unit of labor, derives utility from personal and public consumption, and discounts future utilities by the factor β_j. Her instantaneous utility at time t is equal to $\ln c_t^j + \delta_j \ln h_t$, where c_t^j is her personal consumption and $\delta_j \geq 0$. We assume that $1 > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_L > 0$. By J we denote the set of agents with the highest discount factor:

$$J = \{j \mid \beta_j = \beta_1\}.$$

Given $s_{\tau-1}^{j*}$ and sequences $\{w_\tau,\ w_{\tau+1},\ w_{\tau+2}, \ldots\}$, $\{r_\tau,\ r_{\tau+1},\ r_{\tau+2}, \ldots\}$, $\{h_\tau,\ h_{\tau+1},\ h_{\tau+2}, \ldots\}$, $\{\lambda_\tau,\ \lambda_{\tau+1},\ \lambda_{\tau+2}, \ldots\}$ and $\{\theta_\tau,\ \theta_{\tau+1},\ \theta_{\tau+2}, \ldots\}$, at time τ consumer j maximizes

$$\ln c_\tau^j + \delta_j \ln h_\tau + \beta_j (\ln c_{\tau+1}^j + \delta_j \ln h_{\tau+1}) + \beta_j^2 (\ln c_{\tau+2}^j + \delta_j \ln h_{\tau+2}) + \ldots \quad (1)$$

subject to the following budget constraints:

$$c_t^j + s_t^j = (1 - \theta_t - \lambda_t)[(1 + r_t)s_{t-1}^j + w_t],\ t = \tau, \tau+1, ...,$$

$$s_t^j \geq 0,\ t = \tau, \tau+1, ...$$

where $s_{\tau-1}^j = s_{\tau-1}^{j*}$. Note the no-borrowing constraints in our model, $s_t^j \geq 0$, $t = \tau, \tau+1, ...$, which implies that future wage income cannot be discounted to the present.

Since the sequence $\{h_\tau,\ h_{\tau+1},\ h_{\tau+2}, \ldots\}$ is taken as given by consumer j, maximizing (1) is equivalent to maximizing the utility obtained from personal consumption only, which is equal to

$$\ln c_\tau^j + \beta_j \ln c_{\tau+1}^j + \beta_j^2 \ln c_{\tau+2}^j + \ldots .$$

3 Preliminary analysis: equilibria at given shares of public goods in GDP

3.1 Competitive equilibria at given shares of public goods in GDP

Suppose that the economy at time τ is in a non-generate state $\mathcal{I}^*_\tau = \{(s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau\}$[4] and that the shares of utility-enhancing and productive public goods in GDP, θ_t and λ_t, $t = \tau, \tau+1, \ldots$, are given. It follows that the tax rate at time $t = \tau, \tau+1, \ldots$ is $\theta_t + \lambda_t$. We assume that

$$\liminf_{t\to\infty} \lambda_t > 0 \text{ and } \limsup_{t\to\infty} (\lambda_t + \theta_t) < 1. \tag{2}$$

Let $\Theta_\tau = \{\theta_t\}^\infty_{t=\tau}$ and $\Lambda_\tau = \{\lambda_t\}^\infty_{t=\tau}$. Further, let $1 + r^*_\tau = q(g^*_\tau)f'(k^*_\tau)$ and $w^*_\tau = q(g^*_\tau)(f(k^*_\tau) - f'(k^*_\tau)k^*_\tau)$ be the pre-tax gross interest and wage rates at time τ.

Definition. *A sequence*

$$\mathcal{E}^* = \{(c^{j*}_t)^L_{j=1},\ (s^{j*}_t)^L_{j=1},\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t\}_{t=\tau,\tau+1,\tau+2,\ldots}$$

is called a competitive Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}^*_\tau$ if

1) *for each $j = 1, \ldots, L$, the sequence $\{c^{j*}_t, s^{j*}_t\}_{t=\tau,\tau+1,\tau+2,\ldots}$ is a solution to*

$$\begin{gathered}\max\{\ln c^j_\tau + \beta_j \ln c^j_{\tau+1} + \beta_j^2 \ln c^j_{\tau+2} + \ldots\},\\ c^j_t + s^j_t = (1 - \theta_t - \lambda_t)[(1 + r^*_t)s^j_{t-1} + w^*_t],\ t = \tau, \tau+1, ...,\\ s^j_t \geq 0,\ t = \tau, \tau+1, ...,\ (\text{where } s^j_{\tau-1} = s^{j*}_{\tau-1});\end{gathered} \tag{3}$$

2) $k^*_{t+1}L = \sum_{j=1}^L s^{j*}_t,\ t = \tau, \tau+1, ...;$

3) $1 + r^*_t = q(g^*_t)f'(k^*_t)\ (= \alpha(g^*_t/k^*_t)^{1-\alpha}),\ t = \tau+1, \tau+2, ...;$

4) $w^*_t = q(g^*_t)(f(k^*_t) - f'(k^*_t)k^*_t)\ (= (1-\alpha)(g^*_t)^{1-\alpha}(k^*_t)^\alpha),\ t = \tau+1, \tau+2, ...;$

5) $g^*_{t+1} = \lambda_t q(g^*_t)f(k^*_t)(= \lambda_t(g^*_t)^{1-\alpha}(k^*_t)^\alpha)),\ t = \tau, \tau+1, ...;$

6) $h^*_t = \theta_t q(g^*_t)f(k^*_t)(= \lambda_t(g^*_t)^{1-\alpha}(k^*_t)^\alpha)),\ t = \tau, \tau+1, ...$

The following proposition is a modification of the extended existence theorem for the Ramsey model with heterogeneous agents proved in Becker, Boyd III and Foias (1991) and can be proved in a similar way.

Proposition 1. *For any non-degenerate state of the economy at time τ, $\mathcal{I}^*_\tau = \{(s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau\}$, there exists a competitive Θ_τ-Λ_τ-equilibrium starting from $\mathcal{I}^*_\tau$.*

[4] A tuple $\{(s^j)^L_{j=1}, k, g\}$ is called a *non-degenerate state* of the economy if $s^j \geq 0,\ j = 1, \ldots, L,\ \sum_{j=1}^L s^j = kL > 0$ and $g > 0$.

The question arises of whether the competitive Θ-Λ-equilibrium is unique. Because of the Cobb-Douglas parametric specification of our model, it is reasonable to conjecture that the answer to this question is yes. However, we have no proof of this conjecture. At the same time, the following proposition says that if the initial state is such that all capital is owned by the most patient consumers, then the answer to the above question is positive.

Proposition 2. *For any non-degenerate state* $\mathcal{I}_\tau^* = \{(s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*\}$ *satisfying*

$$k_\tau^* L = \sum_{j\in J} s_{\tau-1}^{j*}\ (i.e.\ s_{\tau-1}^{j*} = 0,\ j \notin J),$$

the Θ_τ-Λ_τ*-equilibrium*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=\tau,\tau+1,\ldots}$$

starting from $\mathcal{I}_\tau^*$ *is unique and determined as follows* $(t = \tau, \tau+1, \ldots)$:

$$k_{t+1}^* = \beta_1(1-\theta_t-\lambda_t)\alpha q(g_t^*) f(k_t^*),$$

$$s_t^{j*} = \beta_1(1-\theta_t-\lambda_t)(1+r_t^*)s_{t-1}^{j*},$$
$$c_t^{j*} = (1-\theta_t-\lambda_t)[(1-\beta_1)(1+r_t^*)s_{t-1}^{j*} + w_t^*],\ j\in J,$$

$$s_t^{j*} = 0,\ c_t^{j*} = (1-\theta_t-\lambda_t)w_t^*,\ j \notin J.$$
$$1 + r_{t+1}^* = q(g_{t+1}^*)f'(k_{t+1}^*),\ w_{t+1}^* = q(g_{t+1}^*)(f(k_{t+1}^*) - f'(k_{t+1}^*)k_{t+1}^*),$$
$$g_{t+1}^* = \lambda_t q(g_t^*)f(k_t^*),\ h_t^* = \theta_t q(g_t^*)f(k_t^*).$$

Proof. See Appendix 1.

In his seminal article on optimal capital accumulation, Ramsey (1928) conjectured that in a model with households differentiated by their rates of time preference eventually the most patient households own the entire capital stock of the economy. The literature on Ramsey's conjecture is comprehensively surveyed in Becker (2006). The following proposition maintains that in our model this conjecture is true.

Proposition 3. *For any competitive* Θ_τ-Λ_τ*-equilibrium*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=\tau,\tau+1,\ldots}$$

there is T *such that*

$$k_t^* L = \sum_{j\in J} s_{t-1}^{j*}\ (i.e.\ s_{t-1}^{j*} = 0,\ j \notin J),\ t = T, T+1, \ldots.$$

Proof. See Appendix 1.

In what follows, if some sequences $\Theta = \{\theta_t\}_{t=0}^{\infty}$ and $\Lambda = \{\lambda_t\}_{t=0}^{\infty}$ are taken as given, then for all $\tau = 0, 1, \ldots,$ we denote the tails of these sequences starting at time τ as Θ_τ and Λ_τ: $\Theta_\tau := \{\theta_t\}_{t=\tau}^{\infty}$ and $\Lambda_\tau := \{\lambda_t\}_{t=\tau}^{\infty}$. In particular, $\Theta_0 = \Theta$ and $\Lambda_0 = \Lambda$.

It should be noted that if

$$\{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

is a competitive Θ_0-Λ_0-equilibrium starting from some initial state $\{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$, then for each $\tau = 1, 2, \ldots,$ its tail

$$\{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=\tau,\tau+1,\ldots}$$

is a competitive Θ_τ-Λ_τ-equilibrium starting from $\{(s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*\}$.

3.2 Balanced-growth equilibria at given shares of public goods

Let $\theta > 0$ and $\lambda > 0$ be such that $\theta + \lambda < 1$ and let $\Theta = \{\theta, \theta, \theta, \ldots\}$ and $\Lambda = \{\lambda, \lambda, \lambda, \ldots\}$. A competitive Θ-Λ-equilibrium

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

starting from $\mathcal{I}_0^* = \{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ is called a *balanced growth* θ-λ-*equilibrium* if there is an equilibrium rate of balanced growth, γ^*, such that for $t = 0, 1, \ldots,$

$$k_{t+1}^* = (1+\gamma^*)k_t^*, \tag{4}$$

$$g_{t+1}^* = (1+\gamma^*)g_t^*, \tag{5}$$

$$w_{t+1}^* = (1+\gamma^*)w_t^*, \tag{6}$$

$$h_{t+1}^* = (1+\gamma^*)h_t^*, \tag{7}$$

$$s_t^{j*} = (1+\gamma^*)s_{t-1}^{j*},\ j = 1, \ldots, L, \tag{8}$$

$$c_t^{j*} = (1+\gamma^*)c_{t-1}^{j*},\ j = 1, \ldots, L. \tag{9}$$

It is clear that in a balanced growth θ-λ-equilibrium

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

starting from $\mathcal{I}_0^*$, the interest rate r_t^* is constant over time:

$$1 + r_t^* = \alpha \left(\frac{k_t^*}{g_t^*} \right)^{\alpha - 1} = \alpha \left(\frac{k_0^*}{g_0^*} \right)^{\alpha - 1} = 1 + r_0^*, \ t = 0, 1, \ldots . \tag{10}$$

The following proposition is an adaptation of a well-known result by Becker (1980) to our model. It maintains that in a balanced growth θ-λ-equilibrium all capital is owned by the most patient consumers. It also says that the rate of balanced growth, γ^*, is completely determined by the parameters of the production function, the two shares of public goods in GDP, θ and λ, and the discount factor of the most patient consumer, β_1.

Proposition 4. 1) *Let*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L, \ (s_t^{j*})_{j=1}^L, \ k_{t+1}^*, \ r_{t+1}^*, \ w_{t+1}^*, \ g_{t+1}^*, \ h_t^*\}_{t=0,1,\ldots}$$

be a balanced growth θ-λ-equilibrium starting from $\mathcal{I}_0^*$ and γ^* be the corresponding equilibrium rate of balanced growth. Then

$$1 + \gamma^* = \beta_1(1 - \theta - \lambda)(1 + r_0^*) = \lambda^{1-\alpha}(1 - \theta - \lambda)^\alpha (\alpha\beta_1)^\alpha, \tag{11}$$

and for $t = 0, 1, \ldots,$

$$\frac{k_t^*}{g_t^*} = \frac{\alpha\beta_1(1 - \theta - \lambda)}{\lambda}, \tag{12}$$

$$k_t^* L = \sum_{j \in J} s_{t-1}^{j*} \ (i.e. \ s_{t-1}^{j*} = 0, \ j \notin J), \tag{13}$$

2) *Let $k_0^* > 0$, $g_0^* > 0$ and $(s_{-1}^{j*})_{j=1}^L$ $(s_{-1}^{j*} \geq 0, \ j = 1, \ldots, L)$ be such that*

$$k_0^* L = \sum_{j \in J} s_{-1}^{j*} \ (i.e. \ s_{\tau-1}^{j*} = 0, \ j \notin J) \tag{14}$$

and

$$\frac{k_0^*}{g_0^*} = \frac{\alpha\beta_1(1 - \theta - \lambda)}{\lambda}. \tag{15}$$

Then there is a balanced growth θ-λ-equilibrium

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L, \ (s_t^{j*})_{j=1}^L, \ k_{t+1}^*, \ r_{t+1}^*, \ w_{t+1}^*, \ g_{t+1}^*, \ h_t^*\}_{t=0,1,\ldots}$$

starting from $\{(s_{-1}^{j*})_{j=1}^L, \ k_0^*, \ g_0^*\}$, which is completely determined by (4)-(10) with γ^* given by (11).

Proof. 1) Let

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

be a balanced growth θ-λ-equilibrium starting from $\mathcal{I}_0^*$ and γ^* be the corresponding equilibrium rate of balanced growth. Repeating a well-known argument (Becker (1980, 2006)), we find that

$$1+\gamma^* = \beta_1(1-\theta-\lambda)(1+r_0^*), \tag{16}$$

and that $\mathcal{E}^*$ satisfies (13).

At the same time,

$$(1+\gamma^*)g_t^* = \lambda(g_t^*)^{1-\alpha}(k_t^*)^\alpha,\ t=0,1,\ldots,$$

and hence

$$1+\gamma^* = \lambda\left(\frac{k_t^*}{g_t^*}\right)^\alpha,\ t=0,1,\ldots.$$

Taking account of (10) and (16), we get

$$\lambda\left(\frac{k_t^*}{g_t^*}\right)^\alpha = \beta_1(1-\theta-\lambda)\alpha\left(\frac{k_t^*}{g_t^*}\right)^{\alpha-1},$$

which implies (12).

2) Suppose that $k_0^* > 0$, $g_0^* > 0$ and $(s_{-1}^{j*})_{j=1}^L$ ($s_{-1}^{j*} \geq 0,\ \ j=1,\ldots,L$) satisfy (14) and (15) and that

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

is determined by (4)-(10). Taking into account the above argument, it is not difficult to check that $\mathcal{E}^*$ is a competitive Θ-Λ-equilibrium starting from $\{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ with $\Theta = \{\theta,\theta,\theta,\ldots\}$ and $\Lambda = \{\lambda,\lambda,\lambda,\ldots\}$. □

It follows from this proposition that the rate of balanced growth, γ^*, is increasing in β_1 and does not depend on $\beta_j \notin J$. It is clearly decreasing in the share of public consumption goods in GDP, θ. As for the dependence of the rate of balanced growth on the share of productive public goods, λ, it has an inverted U-shaped form, but we shall see that voting leads to shares lying in a range where this dependence is increasing.

The following proposition maintains that if the shares of public goods are constant over time ($\Theta = \{\theta,\theta,\theta,\ldots\}$ and $\Lambda = \{\lambda,\lambda,\lambda,\ldots\}$) and at the initial state all capital is owned by the most patient consumers, then the unique competitive Θ-Λ-equilibrium settles on a balanced growth equilibrium in the first time period.

Proposition 5. *Suppose that* $\Theta = \{\theta, \theta, \theta, \ldots\}$ *and* $\Lambda = \{\lambda, \lambda, \lambda, \ldots\}$. *Then for any competitive Θ-Λ-equilibrium*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots},$$

there exists T *such that* $\mathcal{E}^*$ *satisfies (4)-(10) and (12)-(13) for* $t > T$, *where* γ^* *is given by (11).*

Moreover, if the initial state $\mathcal{I}_0^* = \{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ *is such that* $k_\tau^* L = \sum_{j \in J} s_{\tau-1}^{j*}$ *(i.e.* $s_{\tau-1}^{j*} = 0,\ j \notin J$*), then the unique competitive Θ-Λ-equilibrium*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

starting from $\mathcal{I}_0^*$ *satisfies (4)-(10) and (12)-(13) for* $t = 1, 2, \ldots$.

Proof. It follows from Proposition 2 and Proposition 3. □

4 Voting equilibria

4.1 Definitions

4.1.1 Time τ voting equilibrium

Suppose that at time τ the economy is in a non-degenerate state $\mathcal{I}_\tau^* = \{(s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*\}$ and the agents are asked to vote on the time τ shares of public goods in GDP, θ_τ and λ_τ. We assume that when voting on θ_τ and λ_τ, they have some expectations about future shares of public goods in GDP, $\Theta_{\tau+1}^e = \{\theta_t^e\}_{t=\tau+1}^\infty$ and $\Lambda_{\tau+1}^e = \{\lambda_t^e\}_{t=\tau+1}^\infty$, such that $\liminf_{t\to\infty} \lambda_t^e > 0$ and $\limsup_{t\to\infty}(\lambda_t^e + \theta_t^e) < 1$.

To describe the voting outcome, it is necessary to specify indirect utility functions of agents that represent their policy preferences when they vote. This can easily be done if for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences of tax rates Θ_τ and Λ_τ given by

$$\Theta_\tau = \{\theta_\tau, \theta_{\tau+1}^e, \theta_{\tau+2}^e, \ldots\},\ \Lambda_\tau = \{\lambda_\tau, \lambda_{\tau+1}^e, \lambda_{\tau+2}^e, \ldots\}, \quad (17)$$

there is a unique competitive Θ_τ-Λ_τ-equilibrium path

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=\tau,\tau+1,\ldots}$$

starting from $\mathcal{I}_\tau^*$. In this case, the indirect utility function representing policy preferences of agent j over θ_τ and λ_τ is given by the utility she receives on that path:

$$V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*) := \ln c_\tau^{j*} + \delta_j \ln h_\tau^* + \beta_j(\ln c_{\tau+1}^{j*} + \delta_j \ln h_{\tau+1}^*) + \beta_j^2(\ln c_{\tau+2}^{j*} + \delta_j \ln h_{\tau+2}^*) + \dots \quad (18)$$

Since voting on θ_τ and λ_τ is bidimensional, there can be no Condorcet winner if the agents vote on θ_τ and λ_τ simultaneously. Because of this, following Kramer (1972) and Shepsle (1979), we assume that the agents vote on the time τ shares of public goods in GDP, θ_τ and λ_τ, separately: when voting on θ_τ, each agent j takes λ_τ as given and seeks to maximize $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)$ over $\theta_\tau \in (0,1)$, and when voting on λ_τ, each agent j takes θ_τ as given and seeks to maximize $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)$ over $\lambda_\tau \in (0,1)$.

Definition. *Suppose that for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and the sequences of shares of public goods in GDP, Θ_τ and Λ_τ, given by (17), there is a unique Θ_τ-Λ_τ-equilibrium path starting from $\mathcal{I}_\tau^*$. Suppose also that $\theta_\tau^* > 0$ and $\lambda_\tau^* > 0$ are the Condorcet winners in voting on θ_τ at $\lambda_\tau = \lambda_\tau^*$ and on λ_τ at $\theta_\tau = \theta_\tau^*$ and that $\theta_\tau^* + \lambda_\tau^* < 1$. Then we say that the couple $(\theta_\tau^*, \lambda_\tau^*)$ is a time τ voting equilibrium.*

It should be noticed that in this definition it is not presupposed that the expected shares of public goods in GDP, θ_t^e and λ_t^e, are time t voting equilibria for $t > \tau$. Moreover, the expectations are not assumed to be correct. By contrast, in the next definition, the shares of public goods in GDP are time t voting equilibria for each time t and the agents have perfect foresight about future shares of public goods.

4.1.2 Intertemporal voting equilibrium

Let

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\dots}$$

be a competitive Θ^*-Λ^*-equilibrium starting from a non-degenerate initial state $\mathcal{I}_0^* = \{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ for some sequences of shares of public goods in GDP, $\Theta^* = \{\theta_t^*\}_{t=0}^\infty$ and $\Lambda^* = \{\lambda_t^*\}_{t=0}^\infty$.

Definition. *If, for each $\tau = 0, 1, ...,$ the couple $(\theta_\tau^*, \lambda_\tau^*)$ is a time τ voting equilibrium at $\theta_t^e = \theta_t^*,\ t = \tau+1, \tau+2, \dots,$ and $\lambda_t^e = \lambda_t^*,\ t = \tau+1, \tau+2, \dots,$*

then we say that the triple $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ *is an intertemporal voting equilibrium starting from* $\mathcal{I}_0^*$.

In this definition, it is presupposed that for each $\tau = 0, 1, ...$, for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences of shares of public goods in GDP, Θ_τ and Λ_τ, given by

$$\Theta_\tau = \{\theta_\tau, \theta^*_{\tau+1}, \theta^*_{\tau+2}, \dots\},\ \Lambda_\tau = \{\lambda_\tau, \lambda^*_{\tau+1}, \lambda^*_{\tau+2}, \dots\},$$

there is a unique competitive Θ_τ-Λ_τ-equilibrium path starting from $\{(s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau\}$.

4.2 Two important lemmas

In this subsection we make important comparative dynamics for competitive equilibria, which will help us to describe voting equilibria. Let $\Theta^\circ_\tau = \{\theta^\circ_t\}^\infty_{t=\tau}$ and $\Lambda^\circ_\tau = \{\lambda^\circ_t\}^\infty_{t=\tau}$ be sequences of the shares of utility-enhancing and productive public goods in GDP and

$$\mathcal{E}^*_\tau = \{(c^{j*}_t)^L_{j=1},\ (s^{j*}_t)^L_{j=1},\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t\}_{t=\tau,\tau+1,\dots}$$

be a competitive Θ°_τ-Λ°_τ-equilibrium starting from an initial state $\{(s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau\}$. Let further $\theta_\tau > 0$ and $\lambda_\tau > 0$ be such that $\theta_\tau + \lambda_\tau < 1$ and let

$$\nu_\tau := \frac{1 - \lambda_\tau - \theta_\tau}{1 - \lambda^\circ_\tau - \theta^\circ_\tau}. \tag{19}$$

Consider the sequence

$$\mathbf{E}(\theta_\tau, \lambda_\tau, \Theta^\circ_\tau, \Lambda^\circ_\tau, \mathcal{E}^*_\tau) = \{(\tilde{c}^j_t)^L_{j=1},\ (\tilde{s}^j_t)^L_{j=1},\ \tilde{k}_{t+1}, \tilde{r}_{t+1},\ \tilde{w}_{t+1},\ \tilde{g}_{t+1},\ \tilde{h}_t\}_{t=\tau,\tau+1,\dots}$$

defined by the following formulas[5]:

$$\tilde{h}_\tau = h_\tau(\theta_\tau, \lambda_\tau, \Theta^\circ_\tau, \Lambda^\circ_\tau, \mathcal{E}^*_\tau) := (\theta_\tau/\theta^\circ_\tau)h^*_\tau,$$

$$\tilde{g}_{\tau+1} = g_{\tau+1}(\theta_\tau, \lambda_\tau, \Theta^\circ_\tau, \Lambda^\circ_\tau, \mathcal{E}^*_\tau) := (\lambda_\tau/\lambda^\circ_\tau)g^*_{\tau+1},$$

[5] To be more precise, these formulas define the map $\mathbf{E}_\tau(\cdot,\cdot,\cdot,\cdot,\cdot)$ that takes each tuple $(\theta_\tau, \lambda_\tau, \Theta^\circ_\tau, \Lambda^\circ_\tau, \mathcal{E}^*_\tau)$ that consists of numbers $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$, sequences of the shares of public goods in GDP, $\Theta^\circ_\tau = \{\theta^\circ_t\}^\infty_{t=\tau}$ and $\Lambda^\circ_\tau = \{\lambda^\circ_t\}^\infty_{t=\tau}$, and a competitive Θ°_τ-Λ°_τ-equilibrium starting from an initial state $\{(s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau\}$, $\mathcal{E}^*_\tau = \{(c^{j*}_t)^L_{j=1},\ (s^{j*}_t)^L_{j=1},\ k^*_{t+1},\ r^*_{t+1},\ w^*_{t+1},\ g^*_{t+1},\ h^*_t\}_{t=\tau,\tau+1,\tau+2,\dots}$, to the sequence $\mathbf{E}(\theta_\tau, \lambda_\tau, \Theta^\circ_\tau, \Lambda^\circ_\tau, \mathcal{E}^*_\tau)$.

$$\tilde{c}_\tau^j = c_\tau^j(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau c_\tau^{j*},$$
$$\tilde{s}_\tau^j = s_\tau^j(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau s_\tau^{j*},$$
$$\tilde{k}_{\tau+1} = k_{\tau+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau k_{\tau+1}^*,$$
$$\tilde{w}_{\tau+1} = w_{\tau+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} w_{\tau+1}^*,$$
$$\tilde{r}_{\tau+1} = r_{\tau+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^{\alpha-1}(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}(1 + r_{\tau+1}^*) - 1,$$
$$\tilde{h}_t = h_t(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} h_t^*, \ t = \tau+1, \tau+2, ...,$$
$$\tilde{g}_{t+1} = g_{t+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} g_{t+1}^*, \ t = \tau+1, \tau+2, ...,$$

$$\tilde{c}_t^j = c_t^j(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} c_t^{j*}, \quad t = \tau+1, \tau+2, ..., \ j = 1, ..., L,$$

$$\tilde{s}_t^j = s_t^j(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} s_t^{j*}, \quad t = \tau+1, \tau+2, ..., \ j = 1, ..., L,$$

$$\tilde{k}_{t+1} = k_{t+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} k_{t+1}^*, \ t = \tau+1, \tau+2, ...,$$
$$\tilde{w}_{t+1} = w_{t+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha} w_{t+1}^*, \ t = \tau+1, \tau+1, ...,$$
$$\tilde{r}_{t+1} = r_{t+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*) := r_{t+1}^*, \ t = \tau+1, \tau+2, ...$$

The sequence $\mathbf{E}(\theta_\tau, \lambda_\tau, \Theta_\tau^\circ, \Lambda_\tau^\circ, \mathcal{E}_\tau^*)$ shows what will be the dynamics of the economy if we change the shares of utility-enhancing and productive public goods at time τ, θ_τ° to θ_τ and λ_τ° to λ_τ, (leaving these shares intact for $t > \tau$), but the agents keep their savings rates unchanged. The above formulas look somewhat cumbersome, but in fact they are very simple and intuitive.

Suppose, for example, that we increase both shares, i.e. that $\theta_\tau > \theta_\tau^\circ$ and $\lambda_\tau > \lambda_\tau^\circ$. This leads to a decrease in the share of the private sector of the economy at time τ, which changes by a factor of ν_τ, and hence to a proportional decrease in the after-tax wealth of each agent. Then the short-run impact of the change on the economy is as follows: the per capita provision of public consumption goods at time τ will increase and become $(\theta_\tau/\theta_\tau^\circ)h_\tau^*$ and the per capita provision of productive public goods at time $\tau+1$ will increase and become $(\lambda_\tau/\lambda_\tau^\circ)g_{\tau+1}^*$; the time τ consumption and savings of each agent j will decrease and become $\nu_\tau c_\tau^{j*}$ and $\nu_\tau s_\tau^{j*}$ respectively, hence the time $\tau+1$ per capita stock of private capital will also decrease and become $\nu_\tau k_{\tau+1}^*$.

The time $\tau+1$ per capita output and wage rate will either increase or decrease, they will change by a factor of $\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}$ and become

$\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}(k_{\tau+1}^*)^\alpha(g_{\tau+1}^*)^{1-\alpha}$ and $\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}w_{\tau+1}^*$ respectively. As for the time $\tau+1$ gross interest rate, it will clearly be $\nu_\tau^{\alpha-1}(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}(1+r_{\tau+1}^*)$.

We don't change the shares of utility-enhancing and productive goods in GDP at time $\tau+1$. Therefore, the time $\tau+1$ per capita provision of public consumption goods and the time $\tau+2$ per capita productive public expenditures will change by the same factor as the time $\tau+1$ per capita output; they will respectively become $\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}h_{\tau+1}^*$ and $\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}g_{\tau+2}^*$.

Therewith, the share of each agent in national income will not change. Therefore, the time $\tau+1$ savings of each agent j will change by the same factor as the time $\tau+1$ per capita output, they will be $\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}s_{\tau+1}^{j*}$. It is easy to note that from time $\tau+2$ onward all variables except the interest rate will change by a factor $\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}$; as for the interest rate, it will be the same as before the change in the time τ policy.

The following lemma maintains that the sequence $\mathbf{E}(\theta_\tau,\lambda_\tau,\Theta_\tau^\circ,\Lambda_\tau^\circ,\mathcal{E}_\tau^*)$ is a post-change competitive equilibrium.

Lemma 1. *The sequence* $\mathbf{E}(\theta_\tau,\lambda_\tau,\Theta_\tau^\circ,\Lambda_\tau^\circ,\mathcal{E}_\tau^*)$ *is a competitive* Θ'_τ-Λ'_τ-*equilibrium starting from* $\{(s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*\}$ *at*

$$\Lambda'_\tau=\{\lambda_\tau,\lambda_{\tau+1}^\circ,\ldots\},\ \Theta'_\tau=\{\theta_\tau,\theta_{\tau+1}^\circ,\ldots\}.$$

Proof. See Appendix 2.

Let us now compare the utility each agent j obtains on path $\mathbf{E}(\theta_\tau,\lambda_\tau,\Theta_\tau^\circ,\Lambda_\tau^\circ,\mathcal{E}_\tau^*)$,

$$\tilde{U}^j:=\ln\tilde{c}_\tau^j+\delta_j\ln\tilde{h}_\tau+\beta_j(\ln\tilde{c}_{\tau+1}^j+\delta_j\ln\tilde{h}_{\tau+1})+\beta_j^2(\ln\tilde{c}_{\tau+2}^j+\delta_j\ln\tilde{h}_{\tau+2})+\ldots,$$

with the utility she obtains on path $\mathcal{E}_\tau^*$,

$$U^{j*}:=\ln c_\tau^{j*}+\delta_j\ln h_\tau^*+\beta_j(\ln c_{\tau+1}^{j*}+\delta_j\ln h_{\tau+1}^*)+\beta_j^2(\ln c_{\tau+2}^{j*}+\delta_j\ln h_{\tau+2}^*)+\ldots.$$

Since $\tilde{c}_\tau^j=\nu_\tau c_\tau^{j*}$ and $\tilde{h}_\tau=(\theta_\tau/\theta_\tau^\circ)h_\tau^*$, we have

$$(\ln\tilde{c}_\tau^j+\delta_j\ln\tilde{h}_\tau)-(\ln c_\tau^{j*}+\delta_j\ln h_\tau^*)=\ln\nu_\tau+\delta_j\ln(\frac{\theta_\tau}{\theta_\tau^\circ}).$$

This is the short-run effect of the time τ change in policy on the utility of agent j. As for the long-run effect, it follows from the equalities $\tilde{h}_t=\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}h_t^*,\ t=\tau+1,\tau+2,...,$ and $\tilde{c}_t^j=\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}c_t^{j*},\ t=\tau+1,\tau+2,...,$ which imply

$$(\ln\tilde{c}_t^j+\delta_j\ln\tilde{h}_t)-(\ln c_t^{j*}+\delta_j\ln h_t^*)=\ln[\nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha},\ t=\tau+1,\tau+2,\ldots.$$

Therefore,

$$\begin{aligned}
\tilde{U}^j - U^{j*} &= \ln(1-\lambda_\tau-\theta_\tau) + \delta_j \ln\theta_\tau - (\ln(1-\lambda_\tau^\circ-\theta_\tau^\circ) + \delta_j \ln\theta_\tau^\circ) \\
&\quad + \beta_j(\ln[(1-\lambda_\tau-\theta_\tau)^\alpha \lambda_\tau^{1-\alpha}] + \delta_j \ln[(1-\lambda_\tau-\theta_\tau)^\alpha \lambda_\tau^{1-\alpha}]) \\
&\quad - \beta_j(\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}] + \delta_j \ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}]) \\
&\quad + \beta_j^2(\ln[(1-\lambda_\tau-\theta_\tau)^\alpha \lambda_\tau^{1-\alpha}] + \delta_j \ln[(1-\lambda_\tau-\theta_\tau)^\alpha \lambda_\tau^{1-\alpha}]) \\
&\quad - \beta_j^2(\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}] + \delta_j \ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}]) + \ldots \\
&= \delta_j \ln\theta_\tau + \frac{1-\beta_j+(1+\delta_j)\alpha\beta_j}{1-\beta_j}\ln(1-\lambda_\tau-\theta_\tau) + \frac{(1+\delta_j)(1-\alpha)\beta_j}{1-\beta_j}\ln\lambda_\tau - \Gamma_\tau^j,
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_\tau^j &:= \ln(1-\lambda_\tau^\circ-\theta_\tau^\circ) + \delta_j \ln\theta_\tau^\circ \\
&\quad + \beta_j(\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}] + \delta_j \ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}]) \\
&\quad + \beta_j^2(\ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}] + \delta_j \ln[(1-\lambda_\tau^\circ-\theta_\tau^\circ)^\alpha (\lambda_\tau^\circ)^{1-\alpha}]) + \ldots . \qquad (20)
\end{aligned}$$

Thus, we have proved the following lemma.

Lemma 2.

$$\begin{aligned}
\tilde{U}^j - U^{j*} &= \delta_j \ln\theta_\tau + \frac{1-\beta_j+(1+\delta_j)\alpha\beta_j}{1-\beta_j}\ln(1-\lambda_\tau-\theta_\tau) \\
&\quad + \frac{(1+\delta_j)(1-\alpha)\beta_j}{1-\beta_j}\ln\lambda_\tau - \Gamma_\tau^j. \;\square
\end{aligned}$$

4.3 Time τ voting equilibria

Suppose that a non-degenerate state of the economy at time τ, $\mathcal{I}_\tau^* = \{(s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*\}$, and expectations about future shares of public goods in GDP, $\Theta_{\tau+1}^e = \{\theta_t^e\}_{t=\tau+1}^\infty$ and $\Lambda_{\tau+1}^e = \{\lambda_t^e\}_{t=\tau+1}^\infty$, are such that a time τ voting equilibrium exists. It seems reasonable to conjecture that it depends on these expectations. However, we show in this section that this is not the case and that the time τ voting equilibrium is fully determined by exogenous parameters of the model. Moreover, we provide explicit expressions for it.

Taking into account Lemma 1 and Lemma 2, we can describe the outcome of voting on θ_τ and λ_τ. Let $\theta_\tau^\circ > 0$ and $\lambda_\tau^\circ > 0$ be chosen arbitrary in such a way that $\theta_\tau^\circ + \lambda_\tau^\circ < 1$. By Lemma 2, for each j we have

$$V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*) = V_\tau^j(\theta_\tau^\circ, \lambda_\tau^\circ, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*) - \Gamma_\tau^j + \delta_j \ln \theta_\tau + \frac{1-\beta_j + (1+\delta_j)\alpha\beta_j}{1-\beta_j} \ln(1-\lambda_\tau - \theta_\tau) + \frac{(1+\delta_j)(1-\alpha)\beta_j}{1-\beta_j} \ln \lambda_\tau,$$

where Γ_τ^j is defined by (20).

When voting on θ_τ, each agent j maximizes $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)$ over $\theta_\tau \in (0,1)$. To do this, it is sufficient to solve the following equation:

$$\frac{\partial V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)}{\partial \theta_\tau} = 0. \tag{21}$$

We have

$$\frac{\partial V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)}{\partial \theta_\tau} = \frac{\delta_j}{\theta_\tau} - \frac{1-\beta_j + (1+\delta_j)\alpha\beta_j}{(1-\beta_j)(1-\lambda_\tau - \theta_\tau)}.$$

Therefore, (21) can be rewritten as

$$\delta_j(1-\beta_j)(1-\lambda_\tau-\theta_\tau) = (1-\beta_j+(1+\delta_j)\alpha\beta_j)\theta_\tau.$$

Thence the most-preferred value of θ_τ for agent j is equal to

$$\theta_\tau = \frac{\delta_j(1-\beta_j)}{(1+\delta_j)(1-\beta_j+\alpha\beta_j)}(1-\lambda_\tau) = \kappa_j(1-\lambda_\tau),$$

where

$$\kappa_j := \frac{\delta_j(1-\beta_j)}{(1+\delta_j)(1-\beta_j+\alpha\beta_j)}.$$

Note that $0 \le \kappa_j < 1$ ($\kappa_j = 0 \Leftrightarrow \delta_j = 0$) and that κ_j is increasing in δ_j and decreasing in β_j. The median voter theorem applies to voting on θ_τ at a given λ_τ and the outcome of voting is

$$\theta_\tau = \kappa_{med}(1-\lambda_\tau),$$

where κ_{med} is the median value of κ_j, $j = 1, \ldots, L$.

When voting on λ_τ, agent j maximizes $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)$ over $\lambda_\tau \in (0,1)$. To do this, it is sufficient to solve the following equation

$$\frac{\partial V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^e, \Lambda_{\tau+1}^e, \mathcal{I}_\tau^*)}{\partial \lambda_\tau} = 0. \tag{22}$$

We have

$$\frac{\partial V_\tau^j(\theta_\tau, \lambda_\tau, \Theta^e_{\tau+1}, \Lambda^e_{\tau+1}, \mathcal{I}^*_\tau)}{\partial \lambda_\tau} = \frac{(1+\delta_j)(1-\alpha)\beta_j}{(1-\beta_j)\lambda_\tau} - \frac{1-\beta_j+(1+\delta_j)\alpha\beta_j}{(1-\beta_j)(1-\lambda_\tau-\theta_\tau)}.$$

Therefore, (22) can be rewritten as

$$(1+\delta_j)(1-\alpha)\beta_j(1-\lambda_\tau-\theta_\tau) = (1-\beta_j+(1+\delta_j)\alpha\beta_j)\lambda_\tau.$$

Hence the most-preferred value of λ_τ for agent j is

$$\lambda_\tau = \frac{(1+\delta_j)(1-\alpha)\beta_j}{1+\delta_j\beta_j}(1-\theta_\tau) = \chi_j(1-\theta_\tau),$$

where

$$\chi_j := \frac{(1+\delta_j)(1-\alpha)\beta_j}{1+\delta_j\beta_j}.$$

It is obvious that $0 < \chi_j < 1-\alpha$ and that χ_j is increasing in both δ_j and β_j. The median voter theorem applies to voting on λ_τ at a given θ_τ and the outcome of voting is

$$\lambda_\tau = \chi_{med}(1-\theta_\tau),$$

where χ_{med} is the median value of $\chi_j,\ j=1,\dots,L$.

Therefore, the pair $(\lambda^*_\tau, \theta^*_\tau)$ is a time τ voting equilibrium if and only if it is a solution to the following system of two equations:

$$\theta = \kappa_{med}(1-\lambda),\ \lambda = \chi_{med}(1-\theta)$$

or, equivalently, if and only if $\theta^*_\tau = \theta^*,\ \lambda^*_\tau = \lambda^*$, where

$$\theta^* := \frac{\kappa_{med} - \kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}},\ \lambda^* := \frac{\chi_{med} - \kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}}.$$

Thus, we have proved the following theorem.

Theorem 1. *Suppose that a non-degenerate state of the economy at time* τ, $\mathcal{I}^*_\tau = \{(s^{j*}_{\tau-1})^L_{j=1},\ k^*_\tau,\ g^*_\tau\}$, *and sequences of expected shares of public goods in GDP,* $\Theta^e_{\tau+1} = \{\theta^e_t\}^\infty_{t=\tau+1}$ *and* $\Lambda^e_{\tau+1} = \{\lambda^e_t\}^\infty_{t=\tau+1}$, *are given. Suppose also that, for any* $\theta_\tau > 0$ *and* $\lambda_\tau > 0$ *such that* $\theta_\tau + \lambda_\tau < 1$ *and for the sequences* Θ_τ *and* Λ_τ *given by (17), there is a unique competitive* Θ_τ-Λ_τ*-equilibrium starting from* $\mathcal{I}^*_\tau$. *Then there is a unique time* τ *voting equilibrium* $(\theta^*_\tau, \lambda^*_\tau)$, *which is given by*

$$\theta^*_\tau = \theta^*,\ \lambda^*_\tau = \lambda^*. \ \square$$

It should be noticed that κ_{med} and χ_{med} are not necessarily determined by the median values of $\beta_j,\ j = 1, \ldots, L$, and $\delta_j,\ j = 1, \ldots, L$, β_{med} and δ_{med}.

However, if all agents share the same value of discount factor ($\beta_1 = \beta_2 = \ldots = \beta_L$) or the same value of the weight given to public consumption goods relative to private consumption ($\delta_1 = \delta_2 = \ldots = \delta_L$), then

$$\kappa_{med} = \frac{\delta_{med}(1-\beta_{med})}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})},\ \chi_{med} = \frac{(1+\delta_{med})(1-\alpha)\beta_{med}}{1+\delta_{med}\beta_{med}}. \tag{23}$$

Proposition 6. *If the equalities in (23) hold, then*

$$\theta^* = \frac{\delta_{med}(1-\beta_{med})}{1+\delta_{med}},\ \lambda^* = (1-\alpha)\beta_{med}.$$

Proof is in Appendix 3.

4.4 Intertemporal voting equilibria

The following theorem maintains that on any intertemporal voting equilibrium the shares of public goods in GDP are constant over time. It follows directly from Theorem 1.

Theorem 2. *For any intertemporal voting equilibrium* $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$*, the sequences of the shares of public goods in GDP,* Θ^* *and* Λ^**, are constant over time and determined as follows:*

$$\Theta^* = \{\theta^*, \theta^*, \ldots\},\ \Lambda^* = \{\lambda^*, \lambda^*, \ldots\}. \tag{24}$$

Theorem 2 describes the structure of intertemporal voting equilibria, but leaves the question about their existence unanswered. However, if we take into account Proposition 2, we can formulate the following theorem, which maintains that a unique intertemporal voting equilibrium exists if in the initial state all capital is owned by the most patient consumers.

Theorem 3. *Suppose that the initial state* $\mathcal{I}_0^* = \{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ *is such that*

$$k_0^* L = \sum_{j\in J} s_{-1}^{j*},\ (i.e.\ s_{-1}^{j*} = 0,\ j \notin J).$$

Then there is a unique intertemporal voting equilibrium $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ *starting from* $\mathcal{I}_0^*$*, which is constructed as follows:* Θ^* *and* Λ^* *are determined by (24)* *and*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

is determined by the following relationships $(t = 0, 1, \ldots)$:

$$k_{t+1}^* = \beta_1(1 - \theta^* - \lambda^*)\alpha q(g_t^*) f(k_t^*),$$

$$s_t^{j*} = \beta_1(1 - \theta^* - \lambda^*)(1 + r_t^*) s_{t-1}^{j*},$$
$$c_t^{j*} = (1 - \theta^* - \lambda^*)[(1 - \beta_1)(1 + r_t^*) s_{t-1}^{j*} + w_t^*],\ j \in J,$$

$$s_t^{j*} = 0,\ c_t^{j*} = (1 - \theta^* - \lambda^*) w_t^*, j \notin J,$$

$$1 + r_{t+1}^* = q(g_{t+1}^*) f'(k_{t+1}^*),\ w_{t+1}^* = q(g_{t+1}^*)(f(k_{t+1}^*) - f'(k_{t+1}^*) k_{t+1}^*),$$

$$g_{t+1}^* = \lambda^* q(g_t^*) f(k_t^*),\ h_t^* = \theta^* q(g_t^*) f(k_t^*),$$

where $1 + r_0^* = q(g_0^*) f'(k_0^*)$ *and* $w_0^* = q(g_0^*)(f(k_0^*) - f'(k_0^*) k_0^*)$.

Corollary. *If the discount factors of all consumers are the same, i.e. if* $\beta_1 = \ldots = \beta_L$*, then for any non-degenerate initial state there exists a unique itertemporal voting equilibrium starting from this initial state.*

4.5 Balanced growth voting equilibria

Let us move on balanced growth voting equilibria.

Definition. *An intertemporal voting equilibrium* $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ *starting from* $\{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ *is called a balanced growth voting equilibrium if*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,2,\ldots}$$

is a balanced growth θ^**-*λ^**-equilibrium starting from* $\{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$*. The equilibrium rate of growth corresponding to* $\mathcal{E}^*$ *is called the voting equilibrium rate of balanced growth.*

The following theorem describes the structure of balanced growth voting equilibria.

Theorem 4. 1) *Let* $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ *be a balanced growth voting equilibrium starting from* $\{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$,

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots},$$

and let γ^* be the corresponding voting equilibrium rate of balanced growth. Then

$$1+\gamma^* = (\lambda^*)^{1-\alpha}(1-\theta^*-\lambda^*)^{\alpha}(\alpha\beta_1)^{\alpha}, \tag{25}$$

and for $t = 0, 1, \ldots,$

$$\frac{k_t^*}{g_t^*} = \frac{\alpha\beta_1(1-\theta^*-\lambda^*)}{\lambda^*}, \tag{26}$$

$$k_t^* L = \sum_{j\in J} s_{t-1}^{j*} \ (i.e.\ s_{t-1}^{j*} = 0,\ j \notin J). \tag{27}$$

2) *Let $k_0^* > 0$, $g_0^* > 0$ and $(s_{-1}^{j*})_{j=1}^{L}$ ($s_{-1}^{j*} \geq 0$, $j = 1, \ldots, L$) be such that*

$$k_0^* L = \sum_{j\in J} s_{-1}^{j*} \ (i.e.\ s_{\tau-1}^{j*} = 0,\ j \notin J)$$

and

$$\frac{k_0^*}{g_0^*} = \frac{\alpha\beta_1(1-\theta^*-\lambda^*)}{\lambda^*}.$$

Then there is a balanced growth voting equilibrium $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ starting from $\{(s_{-1}^{j*})_{j=1}^{L},\ k_0^*,\ g_0^*\}$, which is constructed as follows: Θ^* and Λ^* are determined by (24) and

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^{L},\ (s_t^{j*})_{j=1}^{L},\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

is determined by (4)-(10) with γ^* given by (25).

Proof. It follows from Proposition 4, Theorem 2 and Theorem 3. □

The following proposition maintains that if at the initial state all capital is owned by the most patient consumers, then the unique intertemporal voting equilibrium settles on a balanced growth voting equilibrium at the first time.

Proposition 7. *Suppose that the initial state $\mathcal{I}_0^* = \{(s_{-1}^{j*})_{j=1}^{L},\ k_0^*,\ g_0^*\}$ is such that $k_0^* L = \sum_{j\in J} s_{-1}^{j*}$ (i.e. $s_{-1}^{j*} = 0$, $j \notin J$). Then the unique intertemporal voting equilibrium $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ starting from $\mathcal{I}_0^*$ is such that*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^{L},\ (s_t^{j*})_{j=1}^{L},\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}$$

satisfies (4)-(10) and (26)-(27) for $t = 1, 2, \ldots,$ with γ^* given by (25).

Proof. It follows from Theorem 2 and Proposition 5. □

4.6 Comparative statics

4.6.1 Comparative statics of the shares of public goods in GDP

To describe the dependence of the shares of public goods in GDP on the discount factors and the weights given to public consumption relative to private consumption, note that λ^* is increasing in χ_{med}, and since $0 < \chi_{med} < 1 - \alpha$, it is decreasing in κ_{med}. As for θ^*, it is increasing in κ_{med} and, taking account of the inequalities $0 < \kappa_{med} < 1$, decreasing in χ_{med}.

Therefore, *if all β_j, $j = 1, \ldots, L$, increase, then all κ_j will decrease and all χ_j will increase and hence λ^* will increase and θ^* will decrease.* This does not look counterintuitive: as all the agents become more patient, they prefer a larger share of productive public goods and a smaller share of public consumption goods.

As for the dependence of θ^* and λ^* on the preference parameters that measure how much the agents value public consumption, it is quite ambiguous: *if all δ_j, $j = 1, \ldots, L$, increase, then θ^* and λ^* can increase, decrease or remain unchanged.*

To show this, first consider a configuration where for some k, $\kappa_{med} = \kappa_k$; $\kappa_{med} \neq \kappa_j$ $j \neq k$; $\chi_{med} \neq \chi_k$. In this configuration, *increases in the weights given to public consumption relative to private consumption can result in a higher share of public consumption goods in GDP and a lower share of productive public goods.* Indeed, a small increase in δ_k will lead to an increase in $\kappa_{med} = \kappa_k$, but will not change χ_{med}; therefore θ^* will increase and λ^* will decrease. By continuity argument, it is possible to show that even increases in all δ_j, $j = 1, \ldots, L$, can lead to an increase in θ^* and a decrease in λ^*.

Now consider a configuration where for some for some l, $\chi_{med} = \chi_l$; $\chi_{med} \neq \chi_j$, $j \neq l$; $\kappa_{med} \neq \kappa_l$. In this configuration, *increases in the weights given to public consumption relative to private consumption can lead a lower share of public consumption goods in GDP and a higher share of productive public goods*, which seems somewhat unexpected. Indeed, a small increase in δ_l will increase χ_{med}, but will not change κ_{med}; therefore, θ^* will decrease and λ^* will increase. Moreover, by a continuity argument, it is possible to show that increases in all δ_j, $j = 1, \ldots, L$, can lead to a decrease in θ^* and an increase in λ^*.

In the case where the equalities in (23) hold before and after all δ_j, $j = 1, \ldots, L$, increase, these increases will lead to an increase in θ^* and will not change λ^*.

4.6.2 Comparative statics of the voting equilibrium rate of balanced growth

To describe the dependence of γ^* on the the discount factors and the weights the agents give to public consumption relative to private consumption, note that γ^* is decreasing in θ^* for $0 < \theta^* < 1 - \lambda^*$ and increasing in λ^* for $0 < \lambda^* < (1-\alpha)(1-\theta^*)$. Therefore, γ^* is decreasing in κ_{med} for $0 \le \kappa_{med} < 1$ and increasing in χ_{med} for $0 < \chi_{med} < 1-\alpha$. At the same time, $0 \le \kappa_{med} < 1$ and $0 < \chi_{med} < 1 - \alpha$ for any $\beta_j \in (0,1),\ j = 1, \dots, L$, and any $\delta_j \ge 0,\ j = 1, \dots, L$.

It follows that *if all agents become more patient, then the voting equilibrium rate of balanced growth will increase*, because an increase in the patience of all agents will unambiguously lead to a decrease of κ_{med} and an increase of χ_{med}, but will not shift χ_{med} outside the interval $(0, 1-\alpha)$. An increase in the patience of one of the agents will lead to an increase in γ^* only if this agent is the most patient or if this agent determines either κ_{med} or χ_{med}.

Let us show that *the dependence of the voting equilibrium rate of balanced growth on the weights given to public consumption relative to private consumption is ambiguous.*

Indeed, consider a configuration where for some for some l, $\chi_{med} = \chi_l$; $\chi_{med} \neq \chi_j,\ j \neq l$; $\kappa_{med} \neq \kappa_l$. In this configuration, increases in all $\delta_j,\ j = 1, \dots, L$, can lead to a decrease in θ^* and an increase in λ^* (this was noted in the previous subsection) and hence to an increase in γ^*.

At the same, *if all $\delta_j,\ j = 1, \dots, L$, increase and the equalities in (23) hold both before and after the changes, then the voting equilibrium rate of balanced growth, γ^*, will decrease.* This follows from the fact that γ^* is decreasing in θ^* and Proposition 6. Thus, if all agents have the same discount factor or they give the same weight to public consumption relative to private consumption, then increasing in the weight(s) they give to public consumption relative to private consumption will unambiguously lead to a decrease in the rate of growth.

5 Generalized intertemporal voting equilibria

We proved the existence of intertemporal voting equilibria only in the case where initially all capital is owned by the most patient consumers. A natural question arises: do intertemporal voting equilibria exist if less patient consumers also own some fraction of capital in the initial state? To give a positive answer to this question, it is necessary to prove the uniqueness of a competitive Θ-Λ-equilibrium starting from arbitrary given initial states, which is a difficult task. However, we can easily get around the difficulty connected with possible non-uniqueness of competitive Θ-Λ-equilibria. Namely, we can slightly modify the notion of voting equilibrium. Lemma 1 will help us to do this.

Let

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,...}$$

be a competitive Θ^*-Λ^*-equilibrium starting from a non-degenerate initial state $\{(s_{-1}^{j*})_{j=1}^L,\ k_0^*,\ g_0^*\}$ for some sequences of shares of public goods in GDP, $\Theta^* = \{\theta_t^*\}_{t=0}^\infty$ and $\Lambda^* = \{\lambda_t^*\}_{t=0}^\infty$.

Suppose that the economy has settled on this competitive Θ^*-Λ^*-equilibrium and at each time τ, when the economy is at the state $\mathcal{I}_\tau^* = \{(s_{\tau-1}^{j*})_{j=1}^L,\ k_\tau^*,\ g_\tau^*\}$, the agents are asked to vote on the time τ shares of public goods in GDP, θ_τ and λ_τ.

If for any $\theta_\tau > 0$ and $\lambda_\tau > 0$ such that $\theta_\tau + \lambda_\tau < 1$ and for the sequences Θ_τ and Λ_τ given by

$$\Theta_\tau = \{\theta_\tau, \theta_{\tau+1}^*, \theta_{\tau+2}^*, \ldots\},\ \Lambda_\tau = \{\lambda_\tau, \lambda_{\tau+1}^*, \lambda_{\tau+2}^*, \ldots\} \qquad (28)$$

there were a unique competitive Θ_τ-Λ_τ-equilibrium path starting from $\mathcal{I}_\tau^*$, we could specify the indirect utilities of agents in this vote unambiguously by (18). Otherwise, the choice of indirect utility functions representing policy preferences of agents is ambiguous.

However, this ambiguity can be overcome if we make a special assumption about the beliefs of the agents.

Let us assume that, *when voting at time τ, all agents believe that if the shares of public goods in GDP at time τ, θ_τ^* and λ_τ^*, are replaced by some other shares of public goods in GDP, $\theta_\tau > 0$ and $\lambda_\tau > 0$, then the economy will settle on the path* $\mathbf{E}(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*)$, where for each $\tau = 0, 1, ...$,

$$\mathcal{E}_\tau^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=\tau,\tau+1,...}$$

is the tail of path $\mathcal{E}^*$ and $\mathbf{E}(\cdot,\cdot,\cdot,\cdot,\cdot)$ is the map introduced in Subsection 4.2.

This assumption means that the agents ignore possible multiplicity of competitive equilibria. At each time τ when voting, they evaluate different policies as if for any θ_τ and λ_τ, $\mathbf{E}(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*)$ were the only Θ_τ-Λ_τ-equilibrium path starting from $\mathcal{I}_\tau^*$.

Under this assumption, the indirect utility function $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^*, \Lambda_{\tau+1}^*, \mathcal{I}_\tau^*)$ that represents the policy preferences of agent j

over θ_τ and λ_τ when voting can be defined unambiguously as follows:

$$
\begin{aligned}
V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^*, \Lambda_{\tau+1}^*, \mathcal{I}_\tau^*) &:= \ln c_\tau^j(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*) \\
&+ \delta_j \ln h_\tau(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*) + \beta_j(\ln c_{\tau+1}^j(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*) \\
&+ \delta_j \ln h_{\tau+1}(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*)) + \beta_j^2(\ln c_{\tau+2}^j(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*) \\
&+ \delta_j \ln h_{\tau+2}(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*)) + \dots,
\end{aligned}
$$

where $c_t^j(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*)$ and $h_t(\theta_\tau, \lambda_\tau, \Theta_\tau^*, \Lambda_\tau^*, \mathcal{E}_\tau^*)$ are constructed for $t = \tau, \tau+1, \dots$ as described in Subsection 4.2.

As above, we assume that the agents vote on θ_τ and λ_τ separately: when voting on θ_τ, agent j takes λ_τ as given and maximizes $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^*, \Lambda_{\tau+1}^*, \mathcal{I}_\tau^*)$ over $\theta_\tau \in (0,1)$, and when voting on λ_τ she takes θ_τ as given and maximizes $V_\tau^j(\theta_\tau, \lambda_\tau, \Theta_{\tau+1}^*, \Lambda_{\tau+1}^*, \mathcal{I}_\tau^*)$ over $\lambda_\tau \in (0,1)$.

Definition. *If for each $\tau = 0, 1, \dots,$ there are Condorcet winners in the votes on θ_τ and λ_τ described above and they coincide with θ_τ^* and λ_τ^* respectively, the triple $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ is called a generalized intertemporal voting equilibrium starting from $\mathcal{I}_0^*$.*

It is clear that any intertemporal voting equilibrium is a generalized intertemporal voting equilibrium and any generalized intertemporal voting equilibrium starting from the initial state where all capital is owned by the most patient consumers is an intertemporal voting equilibrium.

The following theorem maintains that the shares of public goods in GDP in generalized voting intertemporal equilibria are the same as in intertemporal voting equilibria. At the same time, unlike intertemporal voting equilibria, the existence of generalized intertemporal voting equilibria is assured for any non-degenerate initial state.

Theorem 5. *For any generalized intertemporal voting equilibrium $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$, the sequences of the shares of public goods in GDP, Θ^* and Λ^*, are constant over time and given by (24).*

Moreover, for any non-degenerate initial state there exists a generalized intertemporal voting equilibrium starting from that state.

Proof. To prove the theorem, it is sufficient to repeat the argument used in the proof of Theorem 1 and to refer to Proposition 1. □

The long-run behavior of generalized intertemporal voting equilibria is described by the following theorem. It says that any generalized intertemporal voting equilibrium at some time settles on a balanced growth voting equilibrium[6].

Theorem 6. *Let* $\{\Theta^*, \Lambda^*, \mathcal{E}^*\}$ *be a generalized intertemporal voting equilibrium,*

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L,\ (s_t^{j*})_{j=1}^L,\ k_{t+1}^*,\ r_{t+1}^*,\ w_{t+1}^*,\ g_{t+1}^*,\ h_t^*\}_{t=0,1,\ldots}.$$

Then for $t > T$, $\mathcal{E}^*$ *satisfies (4)-(10) and (26)-(27) with* γ^* *given by (25).*

Proof. It follows from Theorem 2 and Proposition 5. □

6 Conclusion

Since the publication of the seminal paper by Barro (1990), growth effects of public spending have been one of the popular topics in economic literature, which usually analyzes the size of government and composition of different types of government expenditure. In this literature the shares of government expenditure are either exogenous or chosen optimally by a social planner. These assumptions do not reflect the fundamental characteristics of collective decision-making and the distribution of preferences, and models based on these assumptions cannot explain substantial variations among countries in their tax and expenditure policies, even among developed democracies sharing similar economic and political regimes.

In this paper we try to answer the question of why tax and expenditure policies differ among democratic countries. In order to understand how fundamentals (preferences and technologies) together with political institutions determine the tax and expenditure policies, this paper examines the determination of the shares of productive public good and utility-enhancing public consumption goods in GDP under majority voting in the context of a parametric model with infinitely-lived agents that are heterogeneous in their discount factors and preferences for public consumption goods.

In our model, the shares of public goods in GDP are an outcome of a dynamic voting process. We have introduced the notions of voting equilibrium and generalized voting equilibrium, proved the existence of generalized voting equilibria and shown that any generalized intertemporal voting equilibrium at some time settles on a balanced growth voting equilibrium.

Due to two assumptions, log preferences and complete depreciation of the capital stock after each period, we have obtained a closed-form solution for the voting outcomes. These outcomes are constant over time, they are fully

[6] Of course, in a model with decreasing returns to scale sustained growth is impossible and the economy does not settle on a balanced growth path, but converges to a stationary equilibrium.

determined by the the technology and preference parameters: the elasticity of output with respect to physical capital, the discount factors and the parameters that measure how much the agents value public consumption. It is noteworthy that it is not necessarily the median values of the discount factors and the weights given to public consumption relative to private consumption that determine the voting outcome.

We have also undertaken comparative some static analysis of the shares of public goods in GDP and of the long-run rate of growth. This analysis shows that if *all* agents become more patient, then the share of productive public goods will increase, the share of public consumption goods will decrease, and the long-run rate of growth will increase. As for the dependence of the two shares and the long-run rate of growth on preferences for public goods, it is ambiguous: if *all* agents increase the wights they give to public consumption relative to private consumption, then the shares of productive and utility-enhancing public goods in GDP and the long-run rate of growth can either increase or decrease.

Much of the recent literature on optimal public expenditures in growth models assumes specific functional forms and computes optimal public policies [7]. We also assumes specific functional form, which is not entirely satisfactory. Existence and characterization of voting equilibria for reasonably general preferences and technologies could be a topic of future research.

A Appendix 1. Proofs of Proposition 2 and Proposition 3

With no loss of generality we prove Proposition 2 and Proposition 3 for $\tau = 0$. Suppose we are given sequences $\Theta = \{\theta_t\}_{t=0}^{\infty}$ and $\Lambda = \{\lambda_t\}_{t=0}^{\infty}$ satisfying (2) and denote

$$\psi_t := 1 - \theta_t - \lambda_t, \ t = 0, 1, \ldots.$$

It follows from (2) that

$$\liminf_{t \to \infty} \psi_t > 0. \tag{29}$$

Consider a competitive Θ-Λ-equilibrium

$$\mathcal{E}^* = \{(c_t^{j*})_{j=1}^L, \ (s_t^{j*})_{j=1}^L, \ k_{t+1}^*, \ r_{t+1}^*, \ w_{t+1}^*, \ g_{t+1}^*, \ h_t^*\}_{t=0,1,2,\ldots}$$

starting from $\{(s_{-1}^{j*})_{j=1}^L, \ k_0^*, \ g_0^*\}$.

[7] See, however, Glomm and Ravikumar (1999).

For each $j = 1, \ldots, L$, the sequence $\{c_t^{j*}, s_t^{j*}\}_{t=0,1,2,\ldots}$ is a solution to

$$\max\{\ln c_0^j + \beta_j \ln c_1^j + \beta_j^2 \ln c_2^j + \ldots\},$$
$$c_t^j + s_t^j = \psi_t[(1 + r_t^*)s_{t-1}^j + w_t^*],\ t = 0, 1, 2, ...,$$
$$s_t^j \geq 0,\ t = 0, 1, 2, ...,\ (\text{where } s_{-1}^j = s_{-1}^{j*}).$$

Therefore, it satisfies the following firs-order conditions:

$$\beta_j \psi_{t+1}(1 + r_{t+1}^*)c_t^{j*} \leq c_{t+1}^{j*}\ (= \text{ if } s_t^{j*} > 0),\ t = 0, 1, 2, ...$$

Lemma A1. *Let $\beta > 0$ be such that for some t',*

$$k_{t+1}^* > \beta\psi_t(1 + r_t^*)k_t^* = \beta\psi_t \alpha q(g_t^*)f(k_t^*),\ t > t'.$$

If $\beta_j < \beta$, then $s_t^{j*} = 0$ for all sufficiently large t.

Proof. Let us take j such that $\beta_j < \beta$ and show that $s_t^{j*} = 0$ for some $t \geq t'$. To do this, assume the converse. Then, by the first-order conditions, for all $t \geq t'$,

$$\beta_j \psi_t(1 + r_t^*)c_{t-1}^{j*} = c_t^{j*}$$

and hence

$$\frac{c_t^{j*}}{k_{t+1}^*} \leq \frac{\beta_j \psi_t(1 + r_t^*)c_{t-1}^{j*}}{\beta\psi_t(1 + r_t^*)k_t^*} \leq \frac{\beta_j}{\beta}\frac{c_{t-1}^{j*}}{k_t^*}.$$

Since $\beta_j/\beta < 1$, $c_t^{j*}/k_{t+1}^* \to 0$ as $t \to \infty$. Taking account of the evident inequality $k_{t+1}^* \leq \psi_t q(g_t^*)f(k_t^*),\ t = 0, 1, \ldots,$ and (29), we get

$$\frac{c_t^{j*}}{\psi_t w_t^*} = \frac{c_t^{j*}}{\psi_t(1 - \alpha)q(g_t^*)f(k_t^*)} \leq \frac{c_t^{j*}}{\psi_t(1 - \alpha)k_{t+1}^*} \longrightarrow_{t\to\infty} 0.$$

This means that $c_t^{j*} < \psi_t w_t^*$ for all sufficiently large t, which is clearly not optimal for consumer j.

Let us now show that if $s_{t_1}^{j*} = 0$ for $t_1 > t'$, then $s_t^{j*} = 0$ for all $t > t_1$. Assume the converse. Then there are $t_2 \geq t_1$ and $t_3 > t_2 + 1$ such that

$$s_{t_2}^{j*} = 0,\ s_{t_3}^{j*} = 0,\ s_t^{j*} > 0,\ t_2 < t < t_3.$$

Therefore,

$$c_{t_2+1}^{j*} < \psi_{t_2+1}w_{t_2+1}^*,$$
$$c_{t_3}^{j*} > \psi_{t_3}w_{t_3}^*, \tag{30}$$

which is impossible. Indeed, for $t > t'$, we have

$$\frac{\alpha}{1-\alpha}\frac{w^*_{t+1}}{1+r^*_{t+1}} = k^*_{t+1} > \beta\psi_t(1+r^*_t)k^*_t = \frac{\alpha}{1-\alpha}\beta\psi_t w^*_t$$

and hence $\beta(1+r^*_{t+1})\psi_t w^*_t < w^*_{t+1}$. Therefore, taking account of the first-order conditions, we get

$$\begin{aligned} c^{j*}_{t_2+2} = \beta_j\psi_{t_2+2}(1+r^*_{t_2+2})c^{j*}_{t_2+1} &< \beta_j\psi_{t_2+2}(1+r^*_{t_2+2})\psi_{t_2+1}w^*_{t_2+1} \\ &\le \beta_1\psi_{t_2+2}(1+r^*_{t_2+2})\psi_{t_2+1}w^*_{t_2+1} < \psi_{t_2+2}w^*_{t_2+2} \end{aligned}$$

and, repeating the argument,

$$c^{j*}_{t+1} < \psi_{t+1}w^*_{t+1},\ t_2 < t < t_3.$$

This implies $c^{j*}_{t_3} < \psi_{t_3}w^*_{t_3}$, which contradicts (30). □

Lemma A2. $k^*_{t+1} \le \beta_1\psi_t(1+r^*_t)k^*_t = \beta_1\psi_t\alpha q(g^*_t)f(k^*_t),\ t = 0, 1, \ldots.$

Proof. Assume the converse. Then there are t' and $\zeta > 1$ such that

$$k^*_{t'+1} \ge \zeta\beta_1\psi_{t'}(1+r^*_{t'})k^*_{t'} = \zeta\beta_1\psi_{t'}\alpha q(g^*_{t'})f(k^*_{t'}).$$

Let us show that for $t > t'$,

$$k^*_{t+1} \ge \zeta\beta_1\psi_t\alpha q(g^*_t)f(k^*_t). \tag{31}$$

Denote

$$J(t') := \{j \in J \mid s^{j*}_{t'} > 0\}.$$

Since

$$\frac{w^*_t}{(1+r^*_t)k^*_t} = \frac{1-\alpha}{\alpha},\ t = 0, 1, \ldots,$$

we have

$$\sum_{j\in J(t')}(s^{j*}_{t'+1} + c^{j*}_{t'+1}) = \sum_{j\in J(t')}[\psi_{t'+1}(1+r^*_{t'+1})s^{j*}_{t'} + \psi_{t'+1}w^*_{t'+1}] =$$

$$\psi_{t'+1}(1+r^*_{t'+1})k^*_{t'+1}L + |J(t')|\psi_{t'+1}w^*_{t'+1}$$

$$= (L + \frac{1-\alpha}{\alpha}|J(t')|)\psi_{t'+1}(1+r^*_{t'+1})k^*_{t'+1}$$

$$\ge (L + \frac{1-\alpha}{\alpha}|J(t')|)\psi_{t'+1}(1+r^*_{t'+1})\zeta\beta_1\psi_{t'}(1+r^*_{t'})k^*_{t'}$$

$$= \zeta\psi_{t'+1}\beta_1(1+r^*_{t'+1})[\psi_{t'}(1+r^*_{t'})k^*_{t'}L + \frac{1-\alpha}{\alpha}|J(t')|\psi_{t'}(1+r^*_{t'})k^*_{t'}]$$

$$= \zeta\psi_{t'+1}\beta_1(1+r^*_{t'+1})[\psi_{t'}(1+r^*_{t'})\sum_{j=1}^{L} s^{j*}_{t'-1} + |J(t')|\psi_{t'}w^*_{t'}]$$

$$\geq \zeta\psi_{t'+1}\beta_1(1+r^*_{t'+1})\sum_{j\in J(t')}[\psi_{t'}(1+r^*_{t'})s^{j*}_{t'-1} + \psi_{t'}w^*_{t'}]$$

$$= \zeta\psi_{t'+1}\beta_1(1+r^*_{t'+1})\sum_{j\in J(t')}(s^{j*}_{t'} + c^{j*}_{t'}).$$

At the same time, by the first-order conditions,

$$c^{j*}_{t'+1} = \beta_j\psi_{t'+1}(1+r^*_{t'+1})c^{j*}_{t'},\ j\in J(t'),$$

and hence

$$\sum_{j\in J(t')} c^{j*}_{t'+1} = \sum_{j\in J(t')} \beta_j\psi_{t'+1}(1+r^*_{t'+1})c^{j*}_{t'}$$

$$\leq \sum_{j\in J(t')} \beta_1\psi_{t'+1}(1+r^*_{t'+1})c^{j*}_{t'} < \zeta\beta_1\psi_{t'+1}(1+r^*_{t'+1})\sum_{j\in J(t')} c^{j*}_{t'}.$$

It follows that

$$k^*_{t'+2}L = \sum_{j=1}^{L} s^{j*}_{t'+1} \geq \sum_{j\in J(t')} s^{j*}_{t'+1} \geq \zeta\beta_1\psi_{t'+1}(1+r^*_{t'+1})\sum_{j\in J(t')} s^{j*}_{t'}$$

$$= \zeta\beta_1\psi_{t'+1}(1+r^*_{t'+1})k^*_{t'+1}L = \zeta\beta_1\psi_{t'+1}\alpha q(g^*_{t'+1})f(k^*_{t'+1})L.$$

Therefore, $k^*_{t'+2} \geq \zeta\beta_1\psi_{t'+1}\alpha q(g^*_{t'+1})f(k^*_{t'+1})$.

Repeating the argument we infer that (31) holds for all $t > t'$.

By Lemma A1, $s^{j*}_t = 0$ for each j and for all sufficiently large t, which contradicts the evident positivity of k^*_t for all $t = 0, 1, \ldots$. This contradiction proves Lemma A2. □

Lemma A3. $\frac{w^*_{t+1}}{1+r^*_{t+1}} \leq \beta_1\psi_t w^*_t,\ t = 0, 1, \ldots$.

Proof. By Lemma A2, for all $t = 0, 1, \ldots$,

$$\frac{w^*_{t+1}}{1+r^*_{t+1}} = \frac{(1-\alpha)q(g^*_{t+1})f(k^*_{t+1})}{1+r^*_{t+1}} = \frac{(1-\alpha)(1+r^*_{t+1})k^*_{t+1}}{\alpha(1+r^*_{t+1})}$$

$$\leq \frac{(1-\alpha)\beta_1\psi_t(1+r^*_t)k^*_t}{\alpha} = \beta_1\psi_t(1-\alpha)q(g^*_t)f(k^*_t) = \beta_1\psi_t w^*_t.\ \square$$

Lemma A4. $s_{t+1}^{j*} \geq \beta_1 \psi_{t+1}(1+r_{t+1}^*)s_t^{j*},\ j \in J,\ t = -1, 0, 1, \ldots.$
Proof. Let $j \in J$. By the first-order conditions,

$$\beta_1^t c_0^{j*} \leq \frac{c_t^{j*}}{\psi_1(1+r_1^*)\ldots\psi_t(1+r_t^*)},\ t = 1, 2, \ldots,$$

and hence

$$c_0^{j*}(1+\beta_1+\beta_1^2+...) \leq c_0^{j*} + \frac{c_1^{j*}}{\psi_1(1+r_1^*)} + \frac{c_2^{j*}}{\psi_1(1+r_1^*)\psi_2(1+r_2^*)} + \ldots. \quad (32)$$

Also we have

$$c_0^{j*} + s_0^{j*} = \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^*,$$

$$\frac{c_1^{j*}+s_1^{j*}}{\psi_1(1+r_1^*)} = s_0^{j*} + \frac{\psi_1 w_1^*}{\psi_1(1+r_1^*)},$$

$$\frac{c_2^{j*}+s_2^{j*}}{\psi_1(1+r_1^*)\psi_2(1+r_2^*)} = \frac{s_1^{j*}}{\psi_1(1+r_1^*)} + \frac{\psi_2 w_2^*}{\psi_1(1+r_1^*)\psi_2(1+r_2^*)},$$

$$\ldots.$$

Summing these equalities over t, we find that

$$c_0^{j*} + \frac{c_1^{j*}}{\psi_1(1+r_1^*)} + \frac{c_2^{j*}}{\psi_1(1+r_1^*)\psi_2(1+r_2^*)} + \ldots$$
$$= \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^* + \frac{\psi_1 w_1^*}{\psi_1(1+r_1^*)} + \frac{\psi_2 w_2^*}{\psi_1(1+r_1^*)\psi_2(1+r_2^*)} + \ldots. \quad (33)$$

Moreover, taking account of Lemma A3, we obtain

$$\frac{\psi_{t+1} w_{t+1}^*}{\psi_1(1+r_1^*)\ldots\psi_{t+1}(1+r_{t+1}^*)} \leq \frac{\beta_1 \psi_t w_t^*}{\psi_1(1+r_1^*)\ldots\psi_t(1+r_t^*)} \leq \ldots \leq \beta_1^{t+1}\psi_0 w_0,$$

which implies

$$\psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^* + \frac{\psi_1 w_1^*}{\psi_1(1+r_1^*)} + \frac{\psi_2 w_2^*}{\psi_1(1+r_1^*)\psi_2(1+r_2^*)} + \ldots$$
$$\leq \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^*(1+\beta_1+\beta_1^2+...). \quad (34)$$

Combining (32), (33) and (34), we get

$$c_0^{j*}(1+\beta_1+\beta_1^2+...) \leq \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^*(1+\beta_1+\beta_1^2+...)$$

and therefore

$$c_0^{j*} \le \psi_0(1+r_0^*)(1-\beta_1)s_{-1}^{j*} + \psi_0 w_0^*.$$

Thus,

$$\begin{aligned} s_0^{j*} &= \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^* - c_0^{j*} \\ &\ge \psi_0(1+r_0^*)s_{-1}^{j*} + \psi_0 w_0^* - \psi_0(1+r_0^*)(1-\beta_1)s_{-1}^{j*} - \psi_0 w_0^* \\ &= \beta_1\psi_0(1+r_0^*)s_{-1}^{j*}. \end{aligned}$$

This proves the inequality $s_{t+1}^{j*} \ge \beta_1\psi_{t+1}(1+r_{t+1}^*)s_t^{j*}$ for $t=-1$. To prove it for $t=0,1,\ldots,$ it is sufficient to repeat the argument. □

Lemma A5. *If*

$$k_0^* L = \sum_{j\in J} s_{-1}^{j*} \ (i.e.\ s_{-1}^{j*}=0,\ j\notin J),$$

then for all $t=0,1,\ldots,$

$$k_{t+1}^* = \beta_1\psi_t\alpha q(g_t^*)f(k_t^*), \tag{35}$$

$$s_t^{j*} = \beta_1\psi_t(1+r_t^*)s_{t-1}^{j*},\ c_t^{j*} = (1-\beta_1)\psi_t(1+r_t^*)s_{t-1}^{j*} + \psi_t w_t^*,\ j\in J, \tag{36}$$

$$s_t^{j*} = 0,\ c_t^{j*} = \psi_t w_t^*,\ j\notin J. \tag{37}$$

Proof. By Lemma A4, $\beta_1\psi_0(1+r_0^*)k_0^*L = \beta_1\psi_0(1+r_0^*)\sum_{j\in J} s_{-1}^{j*} \le \sum_{j\in J} s_0^{j*} \le k_1^* L$. At the same time, by Lemma A2, $k_1^* \le \beta_1\psi_0(1+r_0^*)k_0^*$. Therefore, $k_1^* = \beta_1\psi_0(1+r_0^*)k_0^*$ and hence $s_0^{j*} = \beta_1\psi_0(1+r_0^*)s_{-1}^{j*},\ j\in J$, and $s_0^{j*}=0,\ j\notin J$. We have proved (35)-(37) for $t=0$. To prove (35)-(37) for $t=1,2,\ldots,$ it is sufficient to repeat the argument. □

Proof of Proposition 2. It follows from Lemma A5. □

Proof of Proposition 3. It follow from the first-order conditions and Lemma A3 that for $j\in J$,

$$\frac{c_{t+1}^{j*}}{w_{t+1}^*} \ge \frac{c_t^{j*}}{w_t^*},\ t=0,1,\ldots,$$

and hence the sequence $\{c_t^{j*}/w_t^*\}_{t=0}^{\infty}$ is non-decreasing. This sequence is bounded, because

$$c_t^{j*} \le L\frac{w_t^*}{1-\alpha},\ t=0,1,\ldots.$$

Hence, it converges. It follows that the sequence $\{\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}}\}_{t=0}^{\infty}$ converges to 1. Since

$$\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}} = \frac{w_{t+1}^*}{\beta_1\psi_{t+1}(1+r_{t+1}^*)w_t^*},\ t=0,1,\ldots,$$

the sequence $\{\frac{w_{t+1}^*}{\beta_1\psi_{t+1}(1+r_{t+1}^*)w_t^*}\}_{t=0}^{\infty}$ converges to 1 as well. It follows that if $\beta < \beta_1$, then

$$k_{t+1}^* = \frac{\alpha}{1-\alpha}\frac{w_{t+1}^*}{1+r_{t+1}^*} > \frac{\alpha}{1-\alpha}\beta\psi_t w_t^* = \beta\psi_t(1+r_t^*)k_t^*,$$

for all sufficiently large t. To complete the proof, it is sufficient to take $\beta < \beta_1$ such that $\beta > \max_{j\notin J}\beta_j$ and to refer to Lemma A1. □

B Appendix 2. Proof of Lemma 1

Proof. Direct calculations show that

$$\tilde{k}_{t+1}L = \sum_{j=1}^{L}\tilde{s}_t^j,\ t=\tau,\tau+1,...$$

and

$$q(\tilde{g}_{t+1})f(\tilde{k}_{t+1}) = \nu_\tau^\alpha(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}q(g_{t+1}^*)f(k_{t+1}^*),\ t=\tau,\tau+1,\ldots. \tag{38}$$

It follows from (38) that

$$\tilde{w}_{t+1} = (1-\alpha)q(\tilde{g}_{t+1})f(\tilde{k}_{t+1}) = q(\tilde{g}_{t+1})(f(\tilde{k}_{t+1}) - f'(\tilde{k}_{t+1})\tilde{k}_{t+1}),$$
$$t=\tau,\tau+1,\ldots,$$

$$\tilde{g}_{t+1} = \lambda_t^\circ q(\tilde{g}_t)f(\tilde{k}_t),\ t=\tau+1,\tau+2,\ldots,$$

and

$$\tilde{h}_t = \theta_t^\circ q(\tilde{g}_t)f(\tilde{k}_t),\ t=\tau+1,\tau+2,\ldots,$$

Also it is clear that $\tilde{g}_{\tau+1} = \lambda_\tau q(\tilde{g}_\tau)f(\tilde{k}_\tau)$ and $\tilde{h}_\tau = \theta_\tau q(\tilde{g}_\tau)f(\tilde{k}_\tau)$.

We have

$$1+\tilde{r}_{\tau+1} = \nu_\tau^{\alpha-1}(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}(1+r_{\tau+1}^*) = \nu_\tau^{\alpha-1}(\lambda_\tau/\lambda_\tau^\circ)^{1-\alpha}q(g_{\tau+1}^*)f'(k_{\tau+1}^*)$$
$$= ((\lambda_\tau/\lambda_\tau^\circ)g_{\tau+1}^*)^{1-\alpha}(\nu_\tau k_{\tau+1}^*)^{\alpha-1} = (\tilde{g}_{\tau+1})^{1-\alpha}(\tilde{k}_{\tau+1})^{\alpha-1} = q(\tilde{g}_{\tau+1})f'(\tilde{k}_{\tau+1}).$$

Also we have $\tilde{g}_{\tau+1}/\tilde{k}_{\tau+1} = g^*_{\tau+1}/k^*_{\tau+1},\ t = \tau+1, \tau+2, \ldots$, and hence

$$1 + \tilde{r}_{t+1} = q(\tilde{g}_{t+1}) f'(\tilde{k}_{t+1}),\ t = \tau+1, \tau+2, \ldots.$$

To complete the proof of the lemma, we need to show that the sequence $\{(\tilde{c}^j_t)^L_{j=1},\ (\tilde{s}^j_t)^L_{j=1}\}_{t=\tau,\tau+1,\tau+2\ldots}$ is a solution to the problem

$$\begin{gathered} \max\{\ln c^j_\tau + \beta_j \ln c^j_{\tau+1} + \beta_j^2 \ln c^j_{\tau+2} + \ldots\}, \\ c^j_\tau + s^j_\tau = (1-\theta_\tau - \lambda_\tau)[(1+r^*_\tau)s^{j*}_{\tau-1} + w^*_\tau], \\ c^j_t + s^j_t = (1-\theta^\circ_t - \lambda^\circ_t)[(1+\tilde{r}_t)s^j_{t-1} + \tilde{w}_t],\ t = \tau+1, \tau+2, ..., \\ s^j_t \geq 0,\ t = \tau, \tau+1, ... \end{gathered} \tag{39}$$

To do this, it is sufficient to show that the following conditions are satisfied:

$$\tilde{c}^j_\tau + \tilde{s}^j_\tau = (1-\theta_\tau - \lambda_\tau)[(1+r^*_\tau)s^{j*}_{\tau-1} + w^*_\tau], \tag{40}$$

$$\tilde{c}^j_t + \tilde{s}^j_t = (1-\theta^\circ_t - \lambda^\circ_t)[(1+\tilde{r}_t)\tilde{s}^j_{t-1} + \tilde{w}_t],\ t = \tau+1, \tau+2, ..., \tag{41}$$

$$\beta_j(1-\theta^e_{t+1} - \lambda^\circ_{t+1})(1+\tilde{r}_{t+1})\tilde{c}^j_t \leq \tilde{c}^j_{t+1}\ (= \text{ if } \tilde{s}^j_t > 0),\ t = \tau, \tau+1, ..., \tag{42}$$

$$\frac{\beta^t_j \tilde{s}^j_{t-1}}{\tilde{c}^j_t} \longrightarrow_{t\to\infty} 0. \tag{43}$$

These relationships follow from the fact that $\{c^{j*}_t, s^{j*}_t\}_{t=\tau,\tau+1,\tau+2,\ldots}$ is a solution to (3) at $\theta_t = \theta^\circ_t,\ t = \tau, \tau+1, \ldots$, and $\lambda_t = \lambda^\circ_t,\ t = \tau, \tau+1, \ldots$, and hence satisfies

$$c^{j*}_t + s^{j*}_t = (1-\theta_t - \lambda_t)[(1+r^*_t)s^{j*}_{t-1} + w^*_t],\ t = \tau, \tau+1, ..., \tag{44}$$

the first-order conditions

$$\beta_j(1-\theta^\circ_{t+1} - \lambda^\circ_{t+1})(1+r^*_{t+1})c^{j*}_t \leq c^{j*}_{t+1}\ (= \text{ if } s^{j*}_t > 0),\ t = \tau, \tau+1, ..., \tag{45}$$

and the transversality condition

$$\frac{\beta^t_j s^*_{t-1}}{c^*_t} \longrightarrow_{t\to\infty} 0. \tag{46}$$

Indeed, the validity of (40) follows directly from (44) and the choice of $\tilde{c}^j_\tau$ and $\tilde{s}^j_\tau$. The validity of (41) for $t = \tau+2, \tau+3, \ldots$ follows from (44) and the choice of $\tilde{c}^j_t$, $\tilde{s}^j_t$, $\tilde{s}^j_{t-1}$, $\tilde{r}_t$ and $\tilde{w}_t$ for $t = \tau+2, \tau+3, \dots$. As for the validity of (41) for $t = \tau + 1$, it is also readily checked. Indeed, since

$$1 + r^*_{\tau+1} = \nu_\tau^{1-\alpha}(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}(1+\tilde{r}_{\tau+1}),\ s^{j*}_\tau = \nu_\tau^{-1}\tilde{s}^j_\tau,$$
$$w^{j*}_{\tau+1} = \nu_\tau^{-\alpha}(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}\tilde{w}_{\tau+1},$$

by the choice of $\tilde{c}^j_{\tau+1}$ and $\tilde{s}^j_{\tau+1}$, we have

$$\begin{aligned}
\tilde{c}^j_{\tau+1} + \tilde{s}^j_{\tau+1} &= \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(c^{j*}_{\tau+1} + s^{j*}_{\tau+1}) \\
&= \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})[(1+r^*_{\tau+1})s^{j*}_\tau + w^*_{\tau+1}] \\
&= (1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})[\nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}\nu_\tau^{1-\alpha}(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}(1+\tilde{r}_{\tau+1})\nu_\tau^{-1}\tilde{s}^j_\tau \\
&\quad + \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}\nu_\tau^{-\alpha}(\lambda_\tau/\lambda^\circ_\tau)^{\alpha-1}\tilde{w}^j_{\tau+1}] \\
&= (1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})[(1+\tilde{r}_{\tau+1})\tilde{s}^j_\tau + \tilde{w}_{\tau+1}].
\end{aligned}$$

The validity of (42) for $t = \tau+1, \tau+2, \dots$ follows directly from (45) and the choice of $\tilde{c}^j_t$ and $1 + \tilde{r}_{t+1}$ for $t = \tau+1, \tau+2, \dots$. The validity of (42) for $t = \tau$ is also verified with ease. Indeed, by the choice of $\tilde{c}^j_\tau$, $\tilde{c}^j_{\tau+1}$ and $\tilde{r}_{\tau+1}$ and by (45),

$$\begin{aligned}
&(1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})(1+\tilde{r}_{\tau+1})\tilde{c}^j_\tau \\
&\quad = (1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})\nu_\tau^{\alpha-1}(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1+r^{j*}_{\tau+1})\nu_\tau c^{j*}_\tau \\
&\quad = \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})(1+r^{j*}_{\tau+1})c^{j*}_\tau \\
&\quad \le \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}c^{j*}_{\tau+1} = \tilde{c}^j_{\tau+1}.
\end{aligned}$$

Moreover, if $\tilde{s}^j_\tau > 0$, which is equivalent to $s^{j*}_\tau > 0$, then

$$\begin{aligned}
&(1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})(1+\tilde{r}_{\tau+1})\tilde{c}^j_\tau \\
&\quad = (1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})\nu_\tau^{\alpha-1}(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1+r^{j*}_{\tau+1})\nu_\tau c^{j*}_\tau \\
&\quad = \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}(1-\theta^\circ_{\tau+1}-\lambda^\circ_{\tau+1})(1+r^{j*}_{\tau+1})c^{j*}_\tau \\
&\quad = \nu^\alpha_\tau(\lambda_\tau/\lambda^\circ_\tau)^{1-\alpha}c^{j*}_{\tau+1} = \tilde{c}^j_{\tau+1}.
\end{aligned}$$

Finally, (43) follows directly from (46). □

C Appendix 3. Proof of Proposition 6.

Proof of Proposition 6. Suppose the equalities in (23) are satisfied. Then we have

$$\kappa_{med} = \frac{\delta_{med}(1-\beta_{med})}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})}, \ \chi_{med} = \frac{(1+\delta_{med})(1-\alpha)\beta_{med}}{1+\delta_{med}\beta_{med}}.$$

Therefore,

$$1-\kappa_{med} = \frac{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med}) - \delta_{med}(1-\beta_{med})}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})} = \frac{1-\beta_{med}+\alpha\beta_{med}+\delta_{med}\alpha\beta_{med}}{(1+\delta_{med})(1-\beta_{med}+\alpha\beta_{med})},$$

$$1-\chi_{med} = \frac{1+\delta_{med}\beta_{med}-(1+\delta_{med})(1-\alpha)\beta_{med}}{1+\delta_{med}\beta_{med}} = \frac{1-\beta_{med}+\alpha\beta_{med}+\delta_{med}\alpha\beta_{med}}{1+\delta_{med}\beta_{med}},$$

$$\kappa_{med}\chi_{med} = \frac{\delta_{med}(1-\beta_{med})(1-\alpha)\beta_{med}}{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})},$$

$$1-\kappa_{med}\chi_{med} = \frac{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med}) - \delta_{med}(1-\beta_{med})(1-\alpha)\beta_{med}}{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})} = \frac{1-\beta_{med}+\alpha\beta_{med}+\delta_{med}\alpha\beta_{med}}{(1-\beta_{med}+\alpha\beta_{med})(1+\delta_{med}\beta_{med})}.$$

It follows that

$$\theta^* = \frac{\kappa_{med}-\kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}} = \frac{\delta_{med}(1-\beta_{med})}{1+\delta_{med}},$$

$$\lambda^* = \frac{\chi_{med}-\kappa_{med}\chi_{med}}{1-\kappa_{med}\chi_{med}} = (1-\alpha)\beta_{med}. \ \square$$

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