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The existence of equilibrium paths in an AK-model with endogenous time preferences and borrowing constraints

Working paper Ec-01/13
Department of Economics

St. Petersburg
2013

УДК 330.35
ББК 65.012.2
В78

Европейский университет в Санкт-Петербурге
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Существование равновесных траекторий в АК-модели с эндогенными межвременными предпочтениями и ограничениями на заимствования
На английском языке

Borissov K. *The existence of equilibrium paths in an AK-model with endogenous time preferences and borrowing constraints.* – European University at St.Petersburg, Department of Economics. Working Paper Ec-01/13, 24 p.

Abstract:
An AK-model with borrowing constraints and endogenous time preferences is considered. It is assumed that the discount factor of a household is a continuous function of its relative income. We propose a definition of an equilibrium path and prove its existence.

Keywords: AK-model, endogenous time preferences
JEL Classification: D90, O41

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The existence of equilibrium paths in an AK-model with endogenous time preferences and borrowing constraints

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1. Introduction

In this note, we prove the existence of equilibrium paths in an AK-model of growth and distribution with endogenous time preferences and borrowing constraints. We assume that the discount factor of a household in this model is a continuous function of its *relative* income.[1]

The concept of endogenous time preferences was first proposed by Koopmans (1960) and Uzawa (1968) and was later extended and clarified by Epstein and Hynes (1983). Most of the earlier studies dealing with endogenous time preferences typically assume that the rich discount the future more heavily than the poor. Recent theoretical studies dealing with endogenous time preferences assume that it is the rich who are more patient (see, e.g., Das (2003), Stern (2006), Chang (2009), Hirose and Ikeda (2008), Schumacher (2006), Erol *et al*. (2011), Srtulik (2012)). In our model, the degree of impatience of a household also depends on its income, though we do not specify whether this dependence is increasing or decreasing.

Our approach differs substantially from the recursive utility approach accepted in most papers, where the discount rates are assumed to depend on variables endogenous to agents. The main

[1] An AK-model with exogenous but possibly different discount factors is a particular case of our model.

distinctive feature of our approach is the assumption that though time preferences are formed endogenously in the model as a whole, the households have no control over their discount factors, because the latter depend on a social variable, an economy-wide average income. Therefore, when making its decision on consumption and saving, each household takes its discount factor as exogenously given. In our model, households could be called time preference takers.

This paper is not the first attempt to model socially determined individual time preferences (see e.g. Drugeon (1998), Erol et al. (2011), Meng (2006) and Srtulik (2012)). Meng (2006) considers a model where "individual agents' time preference is largely determined by the surrounding environment – that is, by social forces that are viewed as entirely external and cannot be controlled by individual agents" (p. 2671). Erol *et al* (2011) explore a model with endogenous time preferences and consider optimal paths and competitive equilibria. They do not discuss differences between them. However, when considering optimal paths, they assume that the discount rate depends on variables taken as internal by the central planner, whereas when defining competitive equilibria, they assume that the discount factors are also dependent on social forces that are viewed as entirely external to the representative agent.

It should be said that most attempts at introducing individual time preferences that are socially determined are made in models with a representative agent[2], which is not entirely satisfactory as such frameworks ignore complications in relation to the distribution of wealth. We, however, consider a model with many consumers.

Also, it should be mentioned that if we adopt an approach where the discount rate depends on variables taken as internal by the model's agents, we, in fact, take preferences as fixed from the outset and, thus, given exogenously. To use the very term "endogenous

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[2] Borissov (2002) and Borissov and Lambrecht (2009) consider a model with many agents, but their consideration is constrained to balanced growth only.

time preferences” in such a framework is somewhat misleading. In our framework preferences are ‘more endogenous’ than that.

The rest of the paper is organized as follows.

Section 2 introduces main building blocks of our model. In Section 3, we articulate our main assumption about endogenous time preferences and give the definition of an equilibrium path. The existence of equilibrium paths is proved in Section 4.

2. Building blocks of the model

2.1 Technology

The economy consists of a fixed number B of firms indexed by b. The representative firm produces output using the production function

$$Y^b = F(K^b, AL^b),$$

where K^b denotes the individual firm’s capital stock, L^b denotes the individual firm’s employment of labor, A is the economy-wide stock of knowledge, so that AL^b measures the efficiency units of labor. Production has the usual neoclassical properties of positive, but diminishing, marginal physical products and constant returns to scale in K^b and AL^b.

Firms are competitive in all markets. Therefore, they all have the same capital-effective labor ratio, and the total output of the economy, $Y = \sum_b Y^b$, is given by the aggregate production function

$$Y = F(K, AL),$$

where $K = \sum_b K^b$ and $L = \sum_b L^b$, and, moreover, the wage rate, w, and the return to capital, $1+r$, are given by their marginal products:

$$w = \frac{\partial F(K, AL)}{\partial L}, \; 1 + r = \frac{\partial F(K, AL)}{\partial K}.$$

The total economy-wide level of employment is assumed to be constant over time and normalized to unity. Thus, in equilibrium,

$$\sum_b L^b = 1 .$$

The capital stock fully depreciates during one time period.

Our assumption about A is that it is proportional to the total stock of capital in the economy, namely:

$$A = \frac{K}{\bar{k}} ,$$

where $\bar{k}$ >0 is exogenously given. It follows that production exhibits constant returns to scale both in the accumulating factors, K^b and K, necessary for endogenous growth, and in the private factors, K^b and L^b, necessary for marginal product factor pricing in a competitive equilibrium. Moreover, the equilibrium wage rate, w, and the return to capital, $1+r$, are given as follows:

$$w = \bar{w}A = \bar{w}K / \bar{k} \ , \ \ 1 + r = 1 + \bar{r} \ ,$$

where

$$\bar{r} \equiv f'(\bar{k}) - 1 , \ \ \bar{w} \equiv f(\bar{k}) - f'(\bar{k})\bar{k} \ ,$$

and, in its turn, $f(k) \equiv F(k,1)$. Thus, the economy-wide aggregate production function can then be expressed as a linear function of the total capital stock, as in Romer (1986), namely:

$$Y = aK \ ,$$

where

$$a \equiv f(\bar{k}) / \bar{k} = 1 + \bar{r} + \bar{w} / \bar{k} \ . \tag{1}$$

2.2 Households

We assume that each consumer is endowed with one unit of labor.

Consider a household consisting of α consumers and, hence, endowed at each time with amount α of labor, constant over time. Given the initial level of its savings, $\hat{s}_{-1}\geq 0$, and the sequence of wage rates, $\{w_t\}_{t=0,1,\ldots}$, at time t=0 this household solves the following maximization problem:

$$\text{maximize } u(c_0)+\beta_1 u(c_1)+\beta_2\beta_1 u(c_2)+\ldots$$

subject to

$$c_t+s_t=(1+\bar{r})s_{t-1}+\alpha w_t,\ s_t\geq 0,\ t=0,1,\ldots,\ s_{-1}=\hat{s}_{-1},$$

where c_t and s_t are, respectively, its consumption and savings in period t, $\beta_t\in(0,1)$ is the factor by which the household discounts the time t utility at time t–1, and

$$u(c)=\begin{cases}(c^{1-\nu}-1)/(1-\nu), & \nu\neq 1\\ \ln c, & \nu=1\end{cases}.$$

It should be emphasized that in our model there are borrowing constraints, $s_t\geq 0$, t=0,1,…, so that future income cannot be discounted to the present. If the discount factor is constant over time, $\beta_t=\beta$, t=1,2,..., then the objective function of the household becomes $u(c_0)+\beta u(c_1)+\beta^2 u(c_2)+\ldots$.

2. *Main assumption*

There is a finite set $J=\{1,\ldots,N\}$ of households in our model. Each household $j\in J$ consists of α^j identical consumers. With no loss of generality, we normalize the total number of consumers to one: $\sum_{j\in J}\alpha^j=1$.

Our main assumption is that of the discounts factors of households being able to depend on their relative income. To specify this assumption, let $\varphi_j:\Re_+\to(0,1)$, $j\in J$, be continuous functions and the functions $\psi_j:\Re_+^N\setminus\{\mathbf{0}\}\to(0,1)$, $j\in J$, be defined by

$$\psi_j(z^1,\ldots,z^N)\equiv\varphi_j\left(\frac{z^j}{\sum_{i\in J}\alpha^i z^i}\right).$$

It is clear that the functions $\psi_j:\Re_+^N\setminus\{\mathbf{0}\}\to(0,1)$ are continuous and homogenous of degree 0.

We assume that at each time t the factors by which the households discount the time t+1 utilities, β_{t+1}^j, $j\in J$, are determined as follows:

$$\beta_{t+1}^j=\psi_j(\,z_t^1,\ldots,\,z_t^N\,),$$

where z_t^i is household's $i\in J$ per capita income at time t.

Remark 1. Nothing prevent us from assuming that

$$\varphi_j(x)=\beta^j\ \forall x>0,\ j\in J,$$

where β^j, $j\in J$, are exogenous. Therefore, an AK-model with possibly different but exogenously given discount factors is a particular case of our model.

We assume that

$$\hat{\beta}\equiv\max_j[\max\{\,\psi_j(x^1,\ldots,x^N)|x^1+\ldots+x^N=1\}]<(1+\overline{r}\,)^{\nu-1}.$$

A tuple $\{\hat{K}_0,(\hat{s}_{-1}^j)_{j\in J}\}$ is called a *non-degenerate initial state* if $\hat{K}_0>0$, $\hat{s}_{-1}^j\geq 0$, $j\in J$, and $\sum_{j\in J}\hat{s}_{-1}^j=\hat{K}_0$.

Suppose we are given a non-degenerate initial state $\{\hat{K}_0,(\hat{s}_{-1}^j)_{j\in J}\}$. *An equilibrium path* starting from this initial state is a sequence $\{K_t^*,\ (\beta_{t+1}^{j*})_{j\in J},\ (s_{t-1}^{j*})_{j\in J},\ (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots}$ satisfying the following conditions:

- for each $j\in J$, $(s_{t-1}^{j*},c_t^{j*})_{t=0,1,\ldots}$ is a solution to the problem

$$\max\{u(c_0^j)+\sum_{t=1}^{\infty}(\prod_{\tau=1}^{t}\beta_\tau^{j*})u(c_t^j)\ \mid\ 0\leq s_{-1}^j\leq\hat{s}_{-1}^j,$$

$$s_t^j + c_t^j \leq (1+\bar{r})\, s_{t-1}^j + \alpha^j w_t^*,\ \ s_t^j \geq 0,\ t=0,1,\dots\} \qquad (2)$$

at $w_t^* = \bar{w} K_t^* / \bar{k}$, $t=0,1,\dots$;

- $\beta_{t+1}^{j\,*} = \psi_j(z_t^{1*}, \dots, z_t^{N*}),\ j \in J,\ t=0,\dots$;
- $K_t^* = \sum_{j\in J} s_{t-1}^{j*},\ t=0,1,\dots,$

where

$$z_t^{i*} = \frac{\bar{r} s_{t-1}^{i*}}{\alpha^i} + \frac{\bar{w}}{\bar{k}} K_t^*,\ i \in J,\ t=0,\dots.$$

It is not difficult to check that equilibrium paths are time consistent and that on an equilibrium path $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (C_t^{j*})_{j\in J}\}_{t=0,1,\dots}$ the natural balance holds:

$$K_{t+1}^* + \sum_{j\in J} c_t^{j*} = a\, K_t^*,\ t=0,1,\dots.$$

4. Main result

Our aim is to prove the following existence theorem.

Theorem. *For any non-degenerate initial state* $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$, *there is an equilibrium path starting from that initial state.*

Proof. To prove the theorem, we first prove the existence of finite equilibrium paths and construct an infinite equilibrium path by means of the Cantor diagonal process.

A finite equilibrium path with horizon T starting from an initial state $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$ is a finite sequence $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\dots,T}$ such that

- for each $j\in J$, the sequence $(s_{t-1}^{j*}, c_t^{j*})_{t=0,1,\dots}$ is a solution to the problem

$$\max\{u(c_0^j)+\sum_{t=1}^{T}(\prod_{\tau=1}^{t}\beta_\tau^{j*})u(c_t^j)\mid 0\le s_{-1}^j\le \hat{s}_{-1}^j,$$
$$s_t^j+c_t^j\le(1+\bar{r})s_{t-1}^j+\alpha^j w_t^*,\ s_t^j\ge 0,$$
$$t=0,1,\dots,T-1,\ c_T^j\le(1+\bar{r})s_{T-1}^j+\alpha^j w_t^*\}, \qquad (3)$$

at $w_t^*=\bar{w}K_t^*/\bar{k}$, $t=0,1,\dots,T$;

- $\beta_{t+1}^{j*}=\psi_j(z_t^{1*},\dots,z_t^{N*})$, $j\in J$, $t=0,\dots,T$,

- $K_t^*=\sum_{j\in J}s_{t-1}^{j*}$, $t=0,1,\dots,T$,

where $z_t^{i*}=\dfrac{\bar{r}s_{t-1}^{i*}}{\alpha^i}+\dfrac{\bar{w}}{\bar{k}}K_t^*$, $i\in J$, $t=0,\dots,T$.

Lemma 1. *For any non-degenerate initial state* $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$ *and any* $T<\infty$ *there exists a finite equilibrium path with horizon T starting from that initial state.*

To prove this lemma, we will reduce the model to a generalized game. In order to specify a generalized game, it is necessary to specify the set of players, I, and for each $k\in I$, to describe 1) the strategy set X_k; 2) the strategy correspondence $\psi_k:\prod_{m\in I}X_m\to X_k$; and 3) the objective function $G_k:\prod_{m\in I}X_m\to\Re$.

A tuple $\boldsymbol{x}^*=(x_k^*)_{k\in I}$ is a *Nash equilibrium* in a generalized game if for all $k\in I$, x_k^* is a solution to the problem

$$\min_x\{G_k(x_1^*,\dots,x_{k-1}^*,x,x_{k+1}^*,\dots,x_{|I|}^*)\mid x\in\psi_k(\boldsymbol{x}^*)\}.$$

Lemma 2 (Shafer W. and Sonnenschein H. (1975)). *Suppose that for each* $k\in I$, X_k *is a convex compact subset of a finite* *dimensional space and* ψ_k *is an upper and lower semi-continuous correspondence with convex images and* $G_k(x_1,...,x_k,...,x_{|I|})$ *is continuous in all variables and convex in* x_k*. Then the generalized game has a Nash equilibrium.* □

The proof of the following lemma is trivial.

Lemma 3. *Suppose that* $F_r(x,y)$, $r=1,...,R$, *are continuous in x and y and concave in y functions defined on* $X\times Y$, *where X and Y are convex compact subsets of finite dimensional spaces. If there exists* $\hat{y}\in Y$ *such that* $F_r(x,\hat{y})<0$, $x\in X$, $r=1,...,R$, *then the correspondence* $x\to\bigcap_{r=1}^{R}\{y\,|\,F_r(x,y)\le 0\}$ *is upper and lower semi-continuous, and all sets* $\bigcap_{r=1}^{R}\{y\,|\,F_r(x,y)\le 0\}$ *are non-empty, convex and closed.* □

Proof of Lemma 1. The first-order necessary and sufficient conditions for problem (3) are as follows:

$s_{-1}^j=\hat{s}_{-1}^j$; $\quad s_t^j+c_t^j=(1+\bar{r})\,s_{t-1}^j+\alpha^j w_t^*$, $\quad t=0,1,\ldots,T$, $\quad$ where $s_T^j=0$;

$$\beta_{t+1}^{j\,*}(1+\bar{r})u'(c_{t+1}^j)\le u'(c_t^j), \tag{4}$$

$$s_t^j>0 \;\Rightarrow\; \beta_{t+1}^{j\,*}(1+\bar{r})u'(c_{t+1}^j)=u'(c_t^j),$$
$$t=0,1,\ldots,T-1. \tag{5}$$

It is clear that (4)-(5) can be rewritten as follows:

$$c_{t+1}^j\ge[\beta_{t+1}^{j\,*}(1+\bar{r})]^{1/\nu}c_t^j>0,$$

$$s_t^j>0 \;\Rightarrow\; c_{t+1}^j=[\beta_{t+1}^{j\,*}(1+\bar{r})]^{1/\nu}c_t^j>0,\ t=0,1,\ldots,T-1.$$

Therefore, for a given T, a sequence $\{K_t^*, (\beta_{t+1}^{j\,*})_{j\in J}, (s_{t-1}^{j\,*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$ is an equilibrium path with horizon T starting from a non-degenerate initial state $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$ if and only if

$$K_0{}^* = \hat{K}_0,\ K_t{}^* = \sum_{j\in J} s_{t-1}^{j*},\ t=1,\dots,T, \tag{6}$$

and for all $j\in J$,

$$s_{-1}^{j*} = \hat{s}_{-1}^{j};$$

$$s_t^{j*} + c_t^{j*} = (1+\bar{r})\, s_{t-1}^{j*} + \alpha^j\, \bar{w} K_t^* / \bar{k},\ t=0,1,\dots,T, \tag{7}$$

$$\beta_{t+1}^{j*} = \psi_j(z_t^{1*},\dots, z_t^{N*}),\ j\in J,\ t=0,\dots,T, \tag{8}$$

$$c_{t+1}^{j*} \ge [\beta_{t+1}^{j*}(1+\bar{r})]^{1/\nu} c_t^{j*} > 0, \tag{9}$$

$$s_t^{j*} > 0 \Rightarrow c_{t+1}^{j*} = [\beta_{t+1}^{j*}(1+\bar{r})]^{1/\nu} c_t^{j*} > 0,$$

$$t=0,1,\dots,T-1, \tag{10}$$

where $s_T^{j*}=0$ and $z_t^{i*} = \dfrac{\bar{r} s_{t-1}^{i*}}{\alpha^i} + \dfrac{\bar{w}}{\bar{k}} K_t^*$, $i\in J$, $t=0,\dots,T$.

Let

$$\xi \equiv \min_j\{\alpha^j\}\, \bar{w}/\bar{k},$$

$$\tilde{\beta} \equiv \min_j \varphi_j(0),$$

$$\tilde{g} \equiv [\tilde{\beta}(1+\bar{r})]^{1/\nu} - 1.$$

and let $\tilde{K}_0>0$ и $\lambda>0$ be such that

$$\tilde{K}_0 < \hat{K}_0,\ \lambda < \min\{\xi, a(1 - \frac{a}{a+(1+\tilde{g})})\}.$$

Let further

$$\tilde{K}_t \equiv \lambda^t \tilde{K}_0,\ \ \hat{K}_t \equiv a^t \hat{K}_0,\ t=0,1,\dots,T.$$

Consider the following generalized game with $(3T+2)|J|+T$ players. $3T+2$ players are connected with each $j\in J$, of which T players are responsible for determining s_t^j, $t=0,1,\dots,T-1$, $T+1$ players determine

β_{t+1}^{j}, t=0,1,...,T, and T+1 players are responsible for determining c_t^j, t=0,1,...,T. Also there are T players that determine K_t, t=1,...,T.

The player responsible for s_t^j, $j\in J$, t=0,1,...,T–1, solves the problem

$$\min_s\{(c_{t+1}^j-[\beta_{t+1}^j(1+\bar{r})]^{1/\nu}c_t^j)s \mid$$

$$\max\{0,\tilde{K}_{t+1}-\sum_{i\neq j}s_t^i\}\le s\le(1+\bar{r})s_{t-1}^j+\alpha^j\bar{w}K_t/\bar{k}\},\quad(11)$$

where $s_{-1}^j=\hat{s}_{-1}^j$. If $c_{t+1}^j>[\beta_{t+1}^j(1+\bar{r})]^{1/\nu}c_t^j$, then its solution is $s=\max\{0,\tilde{K}_{t+1}-\sum_{i\neq j}s_t^i\}$. If $c_{t+1}^j<[\beta_{t+1}^j(1+\bar{r})]^{1/\nu}c_t^j$, then its solution is $s=(1+\bar{r})s_{t-1}^j+\alpha^j\bar{w}K_t/\bar{k}$. If $c_{t+1}^j=[\beta_{t+1}^j(1+\bar{r})]^{1/\nu}c_t^j$, then the set of solutions is the segment

$$[\max\{0,\tilde{K}_{t+1}-\sum_{i\neq j}s_t^i\},(1+\bar{r})s_{t-1}^j+\alpha^j\bar{w}K_t/\bar{k}].$$

The player responsible for β_{t+1}^j, $j\in J$, t=0,1,...,T, solves the problem

$$\min_\beta\{|\beta-\psi_j(\bar{r}\frac{s_{t-1}^1}{\alpha^1}+\frac{\bar{w}}{\bar{k}}K_t,\ldots,\bar{r}\frac{s_{t-1}^N}{\alpha^N}+\frac{\bar{w}}{\bar{k}}K_t) \mid$$

$$\tilde{\beta}\le\beta\le\hat{\beta}\},\quad(12)$$

where $s_{-1}^j=\hat{s}_{-1}^j$, $j\in J$. The player responsible for c_t^j, $j\in J$, t=0,1,...,T, solves the problem

$$\min_c\{|c-\rho_t^j(s_{t-1}^j,s_t^j,K_t)| \mid 0\le c\le\hat{K}_{t+1}\},\quad(13)$$

where

$$\rho_t^j(s_{t-1}^j,s_t^j,K_t)\equiv(1+\bar{r})s_{t-1}^j+\alpha^j\bar{w}K_t/\bar{k}-s_t^j,$$

$K_0=\hat{K}_0$ and $s_T^j=0$. The player responsible for K_t, t=1,...,T, solves

$$\min_K\{|K-\sum_{j\in J}s_{t-1}^j| \mid \tilde{K}_t\le K\le\hat{K}_t\}.\quad(14)$$

The last three problems are of the form:

$$\min_x\{|x-\hat{x}|\mid a_1\le x\le a_2\}.$$

It is clear that if $\hat{x}<a_1$, then the solution is $x=a_1$, if $\hat{x}>a_2$, then the solution is $x=a_2$, and if $a_1\le\hat{x}\le a_2$, then the solution is $x=\hat{x}$.

The existence of a Nash equilibrium follows from Lemma 2. Its applicability follows from Lemma 3. To check this, it is sufficient to note that K_t appearing in (11) is feasible for problem (14) and, hence, satisfies $\tilde{K}_t\le K_t\le\hat{K}_t$, and that for any K_t satisfying these inequalities, the set of feasible values of s in problem (11) contains the interval $(\tilde{K}_{t+1},\xi\tilde{K}_t)$. This interval is nonempty, because, by the choice of λ, $\tilde{K}_{t+1}=\lambda\tilde{K}_t<\xi\tilde{K}_t$.

Let $\{(s_t^{j*})_{t=0,1,\dots,T-1,j\in J},(\beta_{t+1}^{j*})_{t=0,1,\dots,T-1,j\in J},(c_t^{j*})_{t=0,\dots,T,j\in J},(K_t^*)_{t=1,\dots,T}\}$ be a Nash equilibrium in the constructed generalized game and let $s_{-1}^{j*}=\hat{s}_{-1}^j$, $j\in J$, $K_0^*=\hat{K}_0$. We claim that $\{K_t^*,(\beta_{t+1}^{j*})_{j\in J},(s_{t-1}^{j*})_{j\in J},(c_t^{j*})_{j\in J}\}_{t=0,1,\dots,T}$ is a finite equilibrium path starting from $\{\hat{K}_0,(\hat{s}_{-1}^j)_{j\in J}\}$.

First, notice that (8) follows from the structure of problem (12) and the choice of $\tilde{\beta}$ and $\hat{\beta}$. Let us show that (6) and (7) are satisfied.

Since $\{K_0^*,(s_{-1}^{j*})_{j\in J}\}$ is a non-degenerate initial state, one has $\sum_{j\in J}s_{-1}^{j*}=K_0^*=\hat{K}_0$. Therefore, by (1) and the equality $\sum_{j\in J}\alpha^j=1$,

$$\sum_{j\in J}[(1+\bar{r})s_{-1}^{j*}+\alpha^j\bar{w}K_0^*/\bar{k}]$$

$$=(1+\bar{r})K_0^*+\bar{w}K_0^*/\bar{k}=aK_0^*\le a\hat{K}_0=\hat{K}_1.$$

Taking into account the constraints in problem (11), we obtain $s_0^{j*} \le (1+\bar{r})\, s_{-1}^{j*} + \alpha^j \bar{w} K_0^* / \bar{k}$. Therefore, $\sum_{j\in J} s_0^{j*} \le a K_0^* \le \hat{K}_1$. Repeating the argument, we get $\sum_{j\in J} s_t^{j*} \le \hat{K}_{t+1}$, t=0,1,...,T–1.

By taking a new look at the constraints in problem (11), we obtain $\tilde{K}_{t+1} \le \sum_{j\in J} s_t^{j*}$, t=0,1,…,T–1. It follows that $\sum_{j\in J} s_t^{j*} \in [\tilde{K}_{t+1}, \hat{K}_{t+1}]$, t=0,1,…,T–1, and, by the structure of problem (14), (6) is satisfied.

Consider the structure of problem (13). From (6), for all t=0,1,...,T–1 and $j\in J$, $0 \le \rho_t^j(s_{t-1}^{j*}, s_t^{j*}, K_t^*) \le \hat{K}_{t+1}$. Hence (7) is satisfied. Summing up (7) in $j\in J$, by (6) and (1), we get

$$\sum_{j\in J} s_t^{j*} + \sum_{j\in J} c_t^{j*} = a\, K_t^*,\ t=0,1,\dots,T. \tag{15}$$

Now we can prove by induction that

$$K_t^* = \sum_{j\in J} s_{t-1}^{j*} > \tilde{K}_t,\ t=0,1,\dots,T. \tag{16}$$

For t=0, (16) holds because $K_0^* = \hat{K}_0 > \tilde{K}_0$. Let us prove that if (16) is satisfied for $t=\tau-1\ge 0$, then (16) holds for $t=\tau$.

To do this, suppose that (16) holds for $t=\tau-1$, but does not hold for $t=\tau$ and, hence,

$$K_\tau^* = \sum_{j\in J} s_{\tau-1}^{j*} = \tilde{K}_\tau. \tag{17}$$

Therefore, $c_\tau^{j*} \ge [\beta_\tau^{j*}(1+\bar{r})]^{1/\nu} c_{\tau-1}^{j*}$, $j\in J$, because, otherwise, from the structure of problem (11) at $t=\tau-1$ and the choice of $\tilde{K}_\tau$ we would have for some $j\in J$:

$$s_{\tau-1}^{j*} = (1+\bar{r})\, s_{\tau-2}^{j*} + \alpha^j \bar{w} K_{\tau-1}^* / \bar{k} \ge \alpha^j \bar{w} K_{\tau-1}^* / \bar{k}$$

$$\ge \alpha^j \bar{w} \tilde{K}_{\tau-1} / \bar{k} > \tilde{K}_\tau,$$

which contradicts (17). By the choice of $\tilde{g}$, we have $\beta_\tau^{j\,*}(1+\bar{r})]^{1/\nu} \geq (1+\tilde{g})$, $j \in J$, and hence $c_\tau^{j*} \geq (1+\tilde{g})\, c_{\tau-1}^{j*}$, $j \in J$. Therefore, $\sum_{j\in J} c_\tau^{j*} \geq (1+\tilde{g}) \sum_{j\in J} c_{\tau-1}^{j*}$ and, by (15), $\sum_{j\in J} c_\tau^{j*} \leq aK_\tau^{\,*}$. Taking into account (6), (7) and (15), we obtain

$$(1+\tilde{g}) \sum_{j\in J} c_{\tau-1}^{j*} / a \leq \sum_{j\in J} c_\tau^{j*} / a \leq K_\tau^{\,*} = a\, K_{\tau-1}^{*} - \sum_{j\in J} c_{\tau-1}^{j*}.$$

It follows that $(1+\frac{1+\tilde{g}}{a}) \sum_{j\in J} c_{\tau-1}^{j*} \leq a\, K_{\tau-1}^{*}$, and hence, $\sum_{j\in J} c_{\tau-1}^{j*} \leq a \frac{a}{a+(1+\tilde{g})} K_{\tau-1}^{*}$. Since, by the choice of λ, $\lambda < a(1 - \frac{a}{a+(1+\tilde{g})})$, we get

$$K_\tau^{*} = a\, K_{\tau-1}^{*} - \sum_{j\in J} c_{\tau-1}^{j*} \geq \lambda\, K_{\tau-1}^{*} > \lambda\, \tilde{K}_{\tau-1} = \tilde{K}_\tau,$$

which contradicts (17). This contradiction proves that (16) is satisfied for $t=\tau$.

We claim that

$$c_t^{j*} > 0,\ j \in J,\ t = 0, 1, \ldots, T. \tag{18}$$

First, we prove (18) for $t=0$. To do this, suppose that $c_0^{j*}=0$ for some $j \in J$, and hence, by (7),

$$s_0^{j*} = s_0^{j*} + c_0^{j*} = (1+\bar{r})\, s_{-1}^{j*} + \alpha^j\, \bar{w}\, K_0^{\,*} / \bar{k} > 0. \tag{19}$$

Let us show that, in this case, $c_1^{j*}=0$. Suppose that $c_1^{j*}>0$. Therefore, $c_1^{j*} - [\beta_1^{j\,*}(1+\bar{r})]^{1/\nu} c_0^{j*} = c_1^{j*} > 0$. Taking a look at the structure of problem (11), we get $s_0^{j*} = \max\{0, \tilde{K}_1 - \sum_{i\neq j} s_t^{i*}\}$. It follows that $s_0^{j*}=0$, because otherwise we would have

$s_0^{j*} = \tilde{K}_1 - \sum_{i \neq j} s_t^{i*}$ and, hence, $\sum_{i \in J} s_0^{i*} = \tilde{K}_1$, which contradicts (16). Thus, $s_0^{j*} = c_0^{j*} + s_0^{j*}$=0, which contradicts (19). This contradiction shows that c_1^{j*}=0.

Therefore, by (6) and (19), $K_1^*>0$. Taking account of (7), we get $s_1^{j*}>0$. Repeating the argument, we conclude that $s_{t-1}^{j*}>0$ and $c_t^{j*}=0$ for all $t=1,\ldots,T$. In particular, $s_{T-1}^{j*}>0$, $c_T^{j*}=0$. At the same time, (7) holds for $t=T$. Thus, $K_t^*>0$ for all t=0,1,…,T. Therefore, $c_T^{j*}=(1+\bar{r})\, s_{T-1}^{j*} + \alpha^j\, \bar{w}\, K_T^*/\bar{k} > 0$. This contradiction shows that $c_0^{j*}>0$.

To prove (18) for all $t=1,\ldots,T$, it is sufficient to repeat the argument above. Finally, to complete the proof of Lemma 1, it is sufficient to notice that (9)-(10) follows from (7), (18), the structure of problem (11) and (16). □

Now we need three lemmas.

Lemma 4. *For any finite equilibrium path* $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$,

$$K_{t+1}^* \le (1+\hat{g})K_t^*,\ t=0,\ldots,T-1,$$

where

$$\hat{g} \equiv [\hat{\beta}(1+\bar{r})]^{1/\nu} - 1.$$

Proof. Consider a finite equilibrium path $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$ and suppose that, for some $\theta<T$, $K_{\theta+1}^*>(1+\hat{g})K_\theta^*$. Let

$$s_T^{j*} \equiv (1+\bar{r})\, s_{T-1}^{j*} + \alpha^j\, \bar{w}\, K_T^*/\bar{k} - c_T^{j*},\ j\in J,\ K_{T+1}^* \equiv \sum_{j\in J} s_T^{j*}.$$

It is clear that

$$K^*_{T+1} = s_T^{j*} = 0,\ j \in J. \tag{20}$$

Let

$$J(\theta) \equiv \{ j \in J \mid s_\theta^{j*} > 0 \}.$$

We have

$$\sum_{j \in J(\theta)} s_\theta^{j*} = \sum_{j \in J} s_\theta^{j*} = K^*_{\theta+1} > (1+\hat{g})K^*_\theta$$

$$= (1+\hat{g}) \sum_{j \in J} s_{\theta-1}^{j*} \geq (1+\hat{g}) \sum_{j \in J(\theta)} s_{\theta-1}^{j*}.$$

It follows that

$$\sum_{j\in J(\theta)} (s_{\theta+1}^{j*} + c_{\theta+1}^{j*}) = \sum_{j\in J(\theta)} ((1+\bar{r}) s_\theta^{j*} + \alpha^j \frac{\bar{w}}{\bar{k}} K^*_{\theta+1})$$

$$= (1+\bar{r}) K^*_{\theta+1} + \sum_{j\in J(\theta)} \alpha^j \frac{\bar{w}}{\bar{k}} K^*_{\theta+1}$$

$$> (1+\hat{g})[(1+\bar{r}) K^*_\theta + \sum_{j\in J(\theta)} \alpha^j \frac{\bar{w}}{\bar{k}} K^*_\theta]$$

$$\geq (1+\hat{g}) \sum_{j \in J(\theta)} [(1+\bar{r}) s_{\theta-1}^{j*} + \alpha^j \frac{\bar{w}}{\bar{k}} K^*_\theta]$$

$$= (1+\hat{g}) \sum_{j \in J(\theta)} (s_\theta^{j*} + c_\theta^{j*}).$$

Therefore, $\sum_{j\in J(\theta)} (s_{\theta+1}^{j*} + c_{\theta+1}^{j*}) > (1+\hat{g}) \sum_{j\in J(\theta)} (s_\theta^{j*} + c_\theta^{j*})$. At the same time, by (9)-(10) and the choice of $\hat{g}$, we have

$$c_{\theta+1}^{j*} = [\beta_{\theta+1}^{j*}(1+\bar{r})]^{1/\nu} c_\theta^{j*} \leq (1+\hat{g}) c_\theta^{j*},\ j \in J(\theta).$$

Thus, $K^*_{\theta+2} \geq \sum_{j\in J(\theta)} s_{\theta+1}^{j*} > (1+\hat{g}) \sum_{j\in J(\theta)} s_\theta^{j*} = (1+\hat{g}) K^*_{\theta+1}$. Repeating the argument, we get $K^*_{t+1} > (1+\hat{g})K^*_t$, $t=\theta+1,\ldots,T$. It follows that $K^*_{T+1} > 0$, which contradicts (20). This contradiction proves Lemma 4.

□

Lemma 5. *For any finite equilibrium path* $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$,

$$c_{t+1}^{j*} \leq (1+\hat{g})c_t^{j*},\ t=0,1,\ldots,T-1,\ j\in J.$$

Proof. If for some $j\in J$, $s_t^{j*}>0$, then, by (9)-(10) and the choice of $\hat{g}$, $c_{t+1}^{j*}=[\beta_{t+1}^{j*}(1+\bar{r})]^{1/\nu}c_t^{j*}\leq(1+\hat{g})c_t^{j*}$. If $s_t^{j*}=0$, then, by Lemma 4,

$$c_{t+1}^{j*}=\alpha^j\frac{\bar{w}}{\bar{k}}K_{t+1}^*-s_{t+1}^{j*}\leq\alpha^j\frac{\bar{w}}{\bar{k}}K_{t+1}^*\leq(1+\hat{g})\alpha^j\frac{\bar{w}}{\bar{k}}K_t^*$$

$$\leq(1+\hat{g})(\alpha^j\frac{\bar{w}}{\bar{k}}K_t^*+(1+\bar{r})s_{t-1}^{j*})=(1+\hat{g})c_t^{j*}.\ \square$$

Lemma 6. *For any finite equilibrium path* $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$ *starting from a non-degenerate initial state* $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$,

$$c_t^{j*}\geq\alpha^j\frac{\bar{w}}{\bar{k}}\frac{\bar{r}-\hat{g}}{1+\bar{r}}\tilde{K}_t,\ j\in J,\ t=0,1,\ldots. \tag{21}$$

and hence there are $\eta_j>0$, $j\in J$, *such that* $u'(c_0^{j*})\leq\eta^j$, $j\in J$, *for any finite equilibrium path* $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$ *starting from* $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$.

Proof. Let us show that for any finite equilibrium path $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots,T}$ starting from $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$, (21) holds for t=0.

Clearly,

$$c_0^{j*}+\frac{1}{1+\bar{r}}\,c_1^{j*}+\ldots+\frac{1}{(1+\bar{r})^T}\,c_T^{j*}$$

$$=(1+\bar{r}\,)\,\hat{s}_{-1}^{j}+\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_0^*+\frac{1}{1+\bar{r}}\,\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_1^*+\frac{1}{(1+\bar{r})^2}\,\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_2^*+$$

$$\ldots+\frac{1}{(1+\bar{r})^T}\,\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_T^*.$$

Therefore, taking into account Lemma 5 and the equality $s_{-1}^{j*}=\hat{s}_{-1}^{j}$, we obtain

$$\alpha^j\,\frac{\bar{w}}{\bar{k}}\,\tilde{K}_0\le\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_0^*\le(1+\bar{r}\,)\,s_{-1}^{j*}+\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_0^*$$

$$+\frac{1}{1+\bar{r}}\,\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_1^*+\frac{1}{(1+\bar{r})^2}\,\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_2^*+\ldots+\frac{1}{(1+\bar{r})^T}\,\alpha^j\,\frac{\bar{w}}{\bar{k}}\,K_T^*$$

$$=c_0^{j*}+\frac{1}{1+\bar{r}}\,c_1^{j*}+\ldots+\frac{1}{(1+\bar{r})^T}\,c_T^{j*}$$

$$\le c_0^{j*}+\frac{1+\hat{g}}{1+\bar{r}}\,c_0^{j*}+\ldots+\left(\frac{1+\hat{g}}{1+\bar{r}}\right)^t c_0^{j*}<\frac{1+\bar{r}}{\bar{r}-\hat{g}}\,c_0^{j*}.$$

This proves (21) for t=0. The same argument can be applied for all t.
□

Now we are ready to prove Theorem. The first-order conditions sufficient for optimality in problem (2) are

$$s_{-1}^{j}=\hat{s}_{-1}^{j};$$

$$s_t^j+c_t^j=(1+\bar{r}\,)\,s_{t-1}^{j}+\alpha^j w_t^*,\ t=0,1,\ldots;$$

$$\beta_{t+1}^{j*}(1+\bar{r}\,)u'(c_{t+1}^{j})\le u'(c_t^{j}),$$

$$s_t^j>0\ \Rightarrow\ \beta_{t+1}^{j*}(1+\bar{r}\,)u'(c_{t+1}^{j})=u'(c_t^{j}),\ t=0,1,\ldots;$$

$$\left(\prod\nolimits_{\tau=0}^{t}\beta_\tau^{j\,*}\right)u'(c_t^{j})(1+\bar{r}\,)\,s_{t-1}^{j}\to0\ \text{as}\ t\to\infty.$$

Therefore, a sequence $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots}$ is an equilibrium path if

$$s_{-1}^{j*} = \hat{s}_{-1}^{j}; \tag{22}$$

$$s_t^{j*} + c_t^{j*} = (1+\bar{r})\, s_{t-1}^{j*} + \alpha^j\, \bar{w} K_t^* / \bar{k},\ t=0,1,\ldots; \tag{23}$$

$$\beta_{t+1}^{j*} = \psi_j(z_t^{1*},\ldots, z_t^{N*}),\ j\in J,\ t=0,\ldots; \tag{24}$$

$$c_{t+1}^{j*} \geq [\beta_{t+1}^{j*}(1+\bar{r})]^{1/\nu} c_t^{j*} > 0,$$

$$s_t^{j*} > 0 \Rightarrow c_{t+1}^{j*} = [\beta_{t+1}^{j*}(1+\bar{r})]^{1/\nu} c_t^{j*} > 0,$$

$$t=0,1,\ldots; \tag{25}$$

$$\left(\prod_{\tau=1}^{t} \beta_\tau^{j*}\right) u'(c_t^{j*})(1+\bar{r})\, s_{t-1}^{j*} \to 0 \text{ as } t\to\infty, \tag{26}$$

where $z_t^{i*} = \dfrac{\bar{r} s_{t-1}^{i*}}{\alpha^i} + \dfrac{\bar{w}}{\bar{k}} K_t^*$, $i\in J$, $t=0,\ldots$.

Suppose that we are given a non-degenerate initial state $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$. Let us construct, for each $T=1,2,\ldots$, a finite equilibrium path starting from $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$, $\mathbf{P}_T=\{K_t(T), (\beta_{t+1}^j(T))_{j\in J}, (s_{t-1}^j(T))_{j\in J}, (c_t^j(T))_{j\in J}\}_{t=0,1,\ldots,T}$, and apply the Cantor diagonal process to the sequence $\{\mathbf{P}_T\}_{T=1,2,\ldots}$.

Namely, let us first take a cluster point of the sequence $\{K_0(T), (\beta_1^j(T))_{j\in J}, (s_0^j(T))_{j\in J}, (c_0^j(T))_{j\in J}\}_{T=1,2,\ldots}$, $\{K_0^*, (\beta_1^{j*})_{j\in J}, (s_0^{j*})_{j\in J}, (c_0^{j*})_{j\in J}\}$. Then consider a subsequence $\{T'\}_{T'=1,2,\ldots}$ of the sequence $\{T\}_{T=1,2,\ldots}$ such that $\{K_0(T'), (\beta_1^j(T'))_{j\in J}, (s_0^j(T'))_{j\in J}, (c_0^j(T'))_{j\in J}\}_{T'=1,2,\ldots}$ converges to $\{K_0^*, (\beta_1^{j*})_{j\in J}, (s_0^{j*})_{j\in J}, (c_0^{j*})_{j\in J}\}$ and take a cluster point of the sequence $\{K_1(T'), (\beta_2^j(T'))_{j\in J}, (s_1^j(T'))_{j\in J}, (c_1^j(T'))_{j\in J}\}_{T'=2,3,\ldots}$, $\{K_1^*, (\beta_2^{j*})_{j\in J}, (s_1^{j*})_{j\in J}, (c_1^{j*})_{j\in J}\}$.

Continuing this procedure *ad infinitum*, we construct an infinite sequence $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots}$. We claim that this sequence is an infinite equilibrium path starting from $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$.

It is clear that the sequence $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots}$ satisfies (22)-(25). To complete the proof it is sufficient to check that it satisfies the transversality condition (26). Let $\{K_t(T), (\beta_{t+1}^j(T))_{j\in J}, (s_{t-1}^j(T))_{j\in J}, (c_t^j(T))_{j\in J}\}_{t=0,1,\ldots,T}$ be a finite equilibrium path starting from $\{\hat{K}_0, (\hat{s}_{-1}^j)_{j\in J}\}$. By (5), we have

$$u'(c_t^j(T))s_{t-1}^j(T) \le \frac{1}{\beta_t^j(T)(1+\bar{r})}u'(c_{t-1}^j(T))s_{t-1}^j(T), \qquad t=0,1,\ldots,T-1.$$

Therefore, taking into account Lemma 4 and Lemma 6, for $t=1,\ldots,T$, we get

$$\left(\prod_{\tau=1}^{t}\beta_\tau^j(T)\right)u'(c_t^j(T))s_{t-1}^j(T) \le \frac{\prod_{\tau=1}^{t}\beta_\tau^j(T)}{\beta_t^j(T)(1+\bar{r})}u'(c_{t-1}^j(T))s_{t-1}^j(T)$$

$$\le \frac{\prod_{\tau=1}^{t}\beta_\tau^j(T)}{\beta_t^j(T)(1+\bar{r})}u'(c_{t-1}^j(T))K_t\ (T)$$

$$\le \left(\prod_{\tau=1}^{t-1}\beta_\tau^j(T)\right)\frac{1+\hat{g}}{1+\bar{r}}u'(c_{t-1}^j(T))K_{t-1}(T)$$

$$\le \ldots \le \left(\frac{1+\hat{g}}{1+\bar{r}}\right)^t u'(c_0^j(T))K_0\ (T) \le \left(\frac{1+\hat{g}}{1+\bar{r}}\right)^t \eta^j \hat{K}_0 .$$

Thus, for the sequence $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots}$,

$$\left(\prod_{\tau=1}^{t}\beta_\tau^{j*}\right)u'(c_t^{j*})s_{t-1}^{j*} \le \left(\frac{1+\hat{g}}{1+\bar{r}}\right)^t \eta^j \hat{K}_0,\ j\in J,\ t=1,2,\ldots.$$

Since $\hat{g} < \bar{r}$, $\{K_t^*, (\beta_{t+1}^{j*})_{j\in J}, (s_{t-1}^{j*})_{j\in J}, (c_t^{j*})_{j\in J}\}_{t=0,1,\ldots}$ satisfies (26). This completes the proof of the theorem. □

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